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Received May 12, 1972 (527)

STUDIA MATHEMATICA T. XLVIII. (1973)

Topological aspects of q-regular measures

by

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Abstract. Let $L(E, F)$ be the set of bounded linear operators from the Banach space E to the Banach space F. If m is a measure defined on a ring $\mathscr{C}$ of subsets of T with values in $L(E, F)$, for each y^* in the dual F^*, one defines a mesaures m_{y^*} from $\mathscr{C}$ into E^*. Also for each A in $\mathscr{C}$ one may define a semi-norm $p_{m,A}$ on F^* in terms of the q-variation of m_{y^*}. Topologies are defined on the unit sphere δ^* of F^* utilizing these semi-norms. We then investigate the relationships of these topologies to the properties of the measures. We consider when the topologies are Hausdorff and when they are compact. We then consider operators on $\mathscr{L}^p_E(\mu)$ $(1 < p \leqslant \infty)$ using the above topologies. For example, if U is a continuous operator from $\mathscr{L}^p_E(\mu)$ into F and if U is absolutely continuous with respect to μ then U is compact if and only if the associated topology makes δ^* compact. Additional results for continuous and compact operators U which are absolutely continuous with respect to μ are obtained.

1. Introduction. The recent definitive work by W. Orlicz in [6] generates additional interest in the relationship of topologies placed on the unit sphere σ^* of a dual space F^* to the measure theoretic properties. In particular, in [4] and [5] a topology associated with a measure is defined as follows.

Let $L(E, F)$ be the set of bounded linear operators from the Banach space E into the Banach space F and let $\mathscr{C}$ be a ring of subsets of a non empty set T. If m is a measure defined on $\mathscr{C}$ with values in $L(E, F)$, then for each A in $\mathscr{C}$ a semi-norm $p_{m,A}$ is defined on the dual F^* of F by

$$p_{m,A}(y^*) = \overline{m_{y^*}}(A)$$

where $\overline{m_{y^*}}$ denotes the variation of the measure m_{y^*} that maps $\mathscr{C}$ into the dual E^* and is defined by

$$m_{y^*}(A) = \langle m(A), y^*\rangle.$$

The collection P of all such semi-norms for A in $\mathscr{C}$ generates a topology in the usual way. This topology when restricted to σ^*, the unit sphere of F^*, turns out to be of interest. Also of interest is the topology generated by $p_{m,A}$ for A in $\mathscr{C}$ where m is now an element in the set $r(E, F)$ of finitely additive set functions from $\mathscr{C}$ into $L(E, F)$. Among the numerous results contained in [4] and [5] one main property seems to be central.

Namely, if the sphere σ^* is compact in the above topology the following statements are equivalent.

(1) The measure m_{y^*} is countably additive for y^* in σ^*.

(2) The measure m is variationally semi regular, that is, if the sequence $\{A_n\}_{n\in N}$ of sets monotonically decreases to $\emptyset$ then the sequence $\{\tilde{m}(A_n)\}_{n\in N}$ converges to 0, where $\tilde{m}$ is the semi-variation of m.

(3) The measure $\tilde{m}$ is norm countably additive.

For the space $C_0(H, E)$ of continuous functions defined on the locally compact space H and vanishing at infinity, operators are defined and studied in [5]. Among the main results is the characterization of compact operators on $C_0(H, E)$. An operator is shown to be compact if and only if the topology generated by $p_{m,A}$ for A in $\mathscr{C}$ is compact on σ^*. In this case m is the measure used to represent the operator as an integral. It is natural to study corresponding results for operators defined on $\mathscr{L}^p$ spaces. The q-semi variation of a measure seems to be the natural vehicle for such a study. An example of this may be found in the representation theorems for operators on $\mathscr{L}^p$ spaces contained in [1]. As a matter of fact the notion of q-semi variation has recently been generalized to φ-bounded variation and used for the study of $\mathscr{L}^\varphi$ spaces (see [7]).

In this article we will define a topology analogous to that above by replacing the variation $\overline{m_{y^*}}$ with the q-variation $(\overline{m_{y^*}})_q$ of the measure m_{y^*}. In particular, for A in $\mathscr{C}$ we will define the semi-norms $p_{m,A}$ by

$$p_{m,A}(y^*) = (\overline{m_{y^*}})_q(A).$$

As in [4] and [5] it will be of interest when σ^* is compact relative to this topology. However here the situation is different in that the above topology need not be Hausdorff. It also should be pointed out that in contrast to the countable additivity of $\overline{m_{y^*}}$, the q-variation $(\overline{m_{y^*}})_q$ is only countably subadditive. In [4] and [5] it was of interest to determine under what conditions m is countably additive. In the present situation countable additivity will follow from the fact that the q semivariation is finite (for $q \neq 1$) (see [1]). In this respect at the conclusion of this work, we will be able to state some additional ideas which will require further research.

In [6] Orlicz studied the properties of weakly absolutely continuous subadditive set functions. Some of the present results are applicable to the present situation when σ^* fails to be compact in contrast to the situation in [5] where compactness is always used.

The results of this article will be organized as follows. In Section 2, the main notations and definitions will be presented. The topology of σ^* will be studied, and the conditions under which the q semi-variation is right continuous will be established in Section 3. As pointed out earlier one of our hypothesis will be that σ^* is compact. If σ^* is not compact some conditions introduced in [6] by Orlicz will be used. Conditions for the topology to be Hausdorff will be defined and topologies corresponding to different values of q will be compared. In Section 4, operators on $\mathscr{L}^p_E(\mu)$ $(1 \leqslant p \leqslant \infty)$ spaces will be studied using the topology introduced in Section 3. If U is a continuous operator from $\mathscr{L}^p_E(\mu)$ into F with U absolutely continuous with respect to μ then U is shown to be compact if and only if the associated topology makes σ^* compact. It is then shown that the q-semi variation is right continuous if and only if there exists some sequence of open Baire sets converging to $\emptyset$ and the integral satisfies some continuity condition on the unit ball of $\mathscr{L}^p_E(\mu)$ (for $\frac{1}{p}+\frac{1}{q}=1$, and $p \neq \infty$). If U is a continuous and compact operator from $\mathscr{L}^\infty_E(\mu)$ into F with U absolutely continuous with respect to μ, it is then shown that the representative measure of U is countably additive. Finally if U is a continuous operator from $\mathscr{L}^p_E(\mu)$ into F $(p \neq \infty)$ and if

$$\langle U, y^*\rangle(f) = \langle U(f), y^*\rangle$$

for $f \in \mathscr{L}^p_E(\mu)$ then it is shown that whenever $\|\langle U, y^*\rangle_A\|_p$, for $y^* \in \sigma^*$, satisfies a Fatou condition and is dominated by a set function having the $\mathbf{O}_\mu$ property (see [6]), the representative measure of U has a right continuous q-semi-variation.

The book [1] by N. Dinculeanu on *Vector Measures* has generated much interest in this area of research. Frequent reference to it will be made throughout the paper.

2. Definitions and notations. As above $\mathscr{C}$ will denote a ring of subsets of the non-empty set T, and μ will denote a positive finite measure on $\mathscr{C}$. For the Banach spaces E and F, $L(E, F)$ will denote all bounded linear operators from E into F and σ^* will denote the unit sphere of the dual space F^* of F. By $\mathscr{L}^p_E(\mu)$ we will denote all E valued functions that are p-integrable with respect to μ (in the sense of [1]). If f belongs to $\mathscr{L}^p_E(\mu)$, then $N_p(f)$ will denote the p-norm of f. If U is a linear operator defined on $\mathscr{L}^p_E(\mu)$ we will write $U \ll \mu$ if $\|U_A\|_p = 0$ whenever $\mu(A) = 0$ (see [1]). The letter m will denote always a measure from $\mathscr{C}$ into $L(E, F)$.

As in [1], for $1 \leqslant q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$ the *q-semi variation* of the measure m is defined for A in $\mathscr{C}$ by

$$\tilde{m}_q(A) = \sup|\Sigma m(A_i)x_i|$$

where the supremum is taken over all disjoint sets A_i in $\mathscr{C}$ and x_i in E for i in a finite indexing set I and for which $N_p(\sum_{i\in I}\chi_{A_i}x_i) \leqslant 1$. For A in $\mathscr{C}$, χ_A represents the characteristic function of A. The q-variation $\overline{m}_q$

of the measure m is defined for A in $\mathscr{C}$ by $\overline{m}_q(A) = \sup \Sigma |m(A_i)|\, |x_i|$ where the supremum is taken in the same manner as the q-semi variation.

Two important properties of these definitions are

(1) $\tilde{m}_q(A) = \sup\{(\overline{m_{y^*}})_q(A)\colon y^* \text{ in } \sigma^*\}$.

(2) $\overline{m}_q = \tilde{m}_q$ if F is the field of scalars.

The set $r(E, F)$ will denote all finitely additive set functions from $\mathscr{C}$ into $L(E, F)$.

For the set $r(E, F)$ defined above, we will let r_q represent that sub-collection of set functions m in $r(E, F)$ whose q-variation, $\overline{m}_q$, is finite on $\mathscr{C}$. If $q \neq 1$, it is known that m is countably additive.

A sequence $\{A_n\}_{n\in N}$ of sets in $\mathscr{C}$ is said to be decreasing monotonically to $\emptyset$ if $\bigcap_{n=1}^{\infty} A_n = \emptyset$. In this case we will write $\{A_n\}_{n\in N}$ d.m. $\emptyset$.

A scalar valued set function $\mathscr{N}$ on $\mathscr{C}$ is said to be *right continuous* at the sequence $\{A_n\}_{n\in N}$ of sets in $\mathscr{C}$ if A_n d.m. $\emptyset$ implies that the sequence $\{\mathscr{N}(A_n)\}_{n\in N}$ converges to 0. The function $\mathscr{N}$ satisfies the $\mathbf{O}_\mu$ *property* (as in [6]) if for every sequence $\{B_n\}_{n\in N}$ of disjoint sets in $\mathscr{C}$, the sequence $\{\mathscr{N}(B_n)\}_{n\in N}$ converges to 0 (some authors have referred to this property as "strongly bounded").

It is shown in [6] that while every function of finite variation satisfies the $\mathbf{O}_\mu$ condition, the converse need not be true.

The scalar valued function η is said to satisfy the *Fatou property* if η is real valued and if $\liminf \eta(E_n) \geqslant \eta(E)$ whenever $E_n \subset E$ and the sequence $\{\mu(E - E_n)\}_{n\in N}$ converges to 0.

We finally recall that

(3) $(\overline{m}_q)(A) = \sup \left[\sum \frac{|m(A_i)|^q}{\mu(A_i)^{q-1}}\right]^{1/q}$ if $q \neq 1$ where the sup is taken over a finite family of disjoint sets A_i from $\mathscr{C}$ with $A_i \subset A \in \mathscr{C}$ and

(4) $(\overline{m}_\infty)(A) = \sup \frac{|m(B)|}{\mu(B)} = (\tilde{m}_\infty)(A)$ where the sup is taken over $B \in \mathscr{C}$, $B \subset A \in \mathscr{C}$. The convention that $\frac{0}{0}$ is interpreted as 0 is maintained.

In general all the notations and concepts pertaining to vector measures can be found in [1].

3. Topologies associated with m_q. For m in r_q and A in $\mathscr{C}$ we consider the functions $p_{m,A}$ defined on F^* by

$$p_{m,A}(y^*) = (\overline{m_{y^*}})_q(A).$$

In the unit sphere σ^* of F^* we consider the following two topologies.

We denote by $\delta_{m,q}$ the weakest topology on σ^* making all semi-norms (which we now prove) $p_{m,A}$ continuous. By δ_q we mean the topology on σ^* generated (in the usual way) by all the semi-norms $p_{m,A}$ for A in $\mathscr{C}$ and m in r_q.

LEMMA 1. *For every $1 \leqslant q \leqslant \infty$, $p_{m,A}$ is a semi-norm on F^*. Thus F^* is a locally convex space under the topology generated by $p_{m,A}$* (see [8]).

Proof. It is clear from the formula defining $(\overline{m_{y^*}})_q$ that $p_{m,A}(ay^*) = |a|\, p_{m,A}(y^*)$. Since

$$p_{m,A}(y_1^* + y_2^*) = \sup \left[\sum \frac{|m_{y_1}(A_i) + m_{y_2}(A_i)|^q}{\mu(A_i)^{q-1}}\right]^{1/q}$$

for $q \neq \infty$ (where the sup is taken over a finite sequence of disjoint sets A_i with $A_i \subset A$) it follows from the Minkowski inequality applied to the q-summable sequences

$$\{a_i\}_{i\in N} \text{ and } \{b_i\}_{i\in N} \quad \text{where } a_i = \frac{|m_{y_1}(A_i)|}{\mu(A_i)^{1-\frac{1}{q}}} \text{ and } b_i = \frac{|m_{y_2}(A_i)|}{\mu(A_i)^{1-\frac{1}{q}}},$$

that

$$p_{m,A}(y_1^* + y_2^*) \leqslant p_{m,A}(y_1^*) + p_{m,A}(y_2^*).$$

If $q = \infty$ the inequality follows immediately from the expression for $(\overline{m_{y^*}})_\infty$.

From [4] we are motivated to define the *boundary* of r_q to be all m in r_q such that whenever A is in $\mathscr{C}$ there exists some y^* in σ^* with $\tilde{m}_q(A) = (\overline{m_{y^*}})_q(A)$.

LEMMA 2. *If $(\sigma^*, \delta_{m,q})$ is a compact space then the boundary of r_q is r_q.*

Proof. There exists a sequence $\{y_n^*\}_{n\in N}$ in σ^* such that the sequence $\{(\overline{m_{y_n^*}})_q(A)\}_{n\in N}$ converges to $\tilde{m}_q(A)$. Without loss of generality we may assume (by compactness) that $\{y_n^*\}_{n\in N}$ converges to y^* in the $\delta_{m,q}$ topology (for some y^* in σ^*). Thus the sequence $\{|(m_{y^*})_q(A) - (\overline{m_{y_n^*}})_q(A)|\}_{n\in N}$ converges to 0 and $\tilde{m}_q(A) = (\overline{m_{y^*}})_q(A)$.

If σ^* is compact in the topology generated by the seminorm $p_{m,A}$ (for m and A fixed) then $\tilde{m}_q(A) = (\overline{m_{y^*}})_q(A)$. The proof follows the proof of Lemma 2.

Since right continuity of $\tilde{m}_q$ will be of importance for later results, the following theorem, which outlines some basic results in that direction, will be of interest.

THEOREM 1. *Let $\{A_n\}_{n\in N}$ be a sequence of sets in $\mathscr{C}$, decreasing monotonically to $\emptyset$. If σ^* is compact in the topology generated by p_{m,A_1}, then there exists a sequence $\{y_n^*\}_{n\in N}$ in σ^* such that $(\overline{m_{y_n^*}})_q(A_n) = \tilde{m}_q(A_n)$. Moreover if y^* is an accumulation point of $\{y_n^*\}_{n\in N}$ in the above topology, then the following statements hold.*

(a) *If* $(\overline{m_{y^*}})_q$ *is right continuous at* $\{A_n\}_{n\in N}$ *then* $\tilde{m}_q$ *is right continuous at* $\{A_n\}_{n\in N}$.

(b) *If* $q\neq 1$ *and* m *is in* r_q *then* m *is countably additive.*

(c) *If* $(\overline{m_{y^*}})_q$ *is right continuous at* $\{A_n\}_{n\in N}$ *and if* $\mu(A_n)>0$ *then* $\tilde{m}_r$ *is continuous at* $\{A_n\}_{n\in N}$ *for all* $1\leqslant r\leqslant q$.

(d) *If* m *is in* r_1 *and if* m_{y^*} *is countably additive for every* y^* *in* σ^* *then* $\tilde{m}_1$ *is right continuous at every sequence* $\{A_n\}_{n\in N}$ *d.m.* $\emptyset$.

(e) *If* $(\overline{m_{y^*}})_q$ *satisfies the* $\mathbf{O}_\mu$ *condition and the Fatou property for each increasing* $\{A_n\}_{n\in N}$ *then* $(\overline{m_{y^*}})_q$ *is right continuous at every sequence* $\{A_n\}_{n\in N}$ *d.m.* $\emptyset$.

If σ^* *is not necessarily compact in the topology* $\delta_{m,q}$, *then* $\tilde{m}_q$ *is still right continuous at every sequence* $\{A_n\}_{n\in N}$ *d.m.* $\emptyset$ *provided there exists some set function* λ *from* $\mathscr{C}$ *into* F *for which*

(f) $(\overline{m_{z^*}})_q\leqslant\lambda$ *for every* z^* *in* σ^*.

(g) λ *satisfies the* $\mathbf{O}_\mu$ *condition.*

(h) *Each* $(\overline{m_{z^*}})_q$ *satisfies the Fatou condition* (z^* *in* σ^*).

Proof. First we show statement (a). If $\tilde{m}_q$ is not right continuous at $\{A_n\}_{n\in N}$ we may assume that for some $\varepsilon>0$, $\tilde{m}_q(A_n)>\varepsilon$. Then $p_{m,A_n}(y_n^*)>\varepsilon$ (y_n^* exists by the note preceding the theorem). Thus $p_{m,A_1}(y_n^*-y^*)<\varepsilon/4$ for all $n\geqslant N$. Consequently $p_{m,A_n}(y_n^*-y^*)<\varepsilon/4$ for $n\geqslant N$. However by Lemma 1, $p_{m,A_n}(y^*)>\varepsilon/2$. This contradicts the hypothesis that $(\overline{m_{y^*}})_q$ is right continuous at $\{A_n\}_{n\in N}$.

Statement (b) is shown in [1]. Statement (c) follows from (a) and from the inequality

$$\mu(A)^{-1/r}\tilde{m}_r(A)\leqslant\mu(A)^{-1/q}\tilde{m}_q(A)$$

for A such that $\mu(A)>0$ (see [1]).

In statement (d), if m is in r_1, then $m\ll\mu$. Thus $\tilde{m}=\tilde{m}_1$. In [4] the stated property is shown to be true for $\tilde{m}$.

Statement (e) follows from Theorem 4 of [6] applied to $(\overline{m_{y^*}})_q$.

Finally the second part of the theorem follows from Theorem 7 of [6]. It is necessary to apply that theorem to the family $M=\{(\overline{m_{z^*}})_q\colon z^*\in\sigma^*\}$. In particular as needed there, if the sequence $\{(\overline{m_{z^*}})_q(A_n)\}_{n\in N}$ converges uniformly to 0 for z^* in σ^* then the sequence $\{\tilde{m}_q(A_n)\}_{n\in N}$ converges to 0. This completes the proof of the theorem.

Applying the results of [6] to the above family M would yield conditions under which the $(\overline{m_{z^*}})_q$ are uniformly absolutely continuous with respect to μ (in the $\varepsilon-\delta$ sense).

We can now obtain conditions equivalent to the space (σ^*,δ_q) being Hausdorff.

LEMMA 3. *The following conditions are equivalent.*

(1) *The space* (σ^*,δ_q) *is Hausdorff.*

(2) *The closure of the linear span of* $\bigcup_{m\in r_q}\bigcup_{A\in\mathscr{C}}m(A)\sigma$ *is* F *(for* σ *the unit sphere of* E*).*

(3) *The topology* δ_q *is stronger than the weak* topology.*

Proof. The proof follows a pattern similar to that in [4]. That (3) *implies* (1) is clear. Now assume that (1) is true and that (2) is false. Pick a non-zero z in σ^* such that for a finite indexing set I, $\langle\sum_{i\in I}s_i m(A_i)x_i, z\rangle=0$ (the s_i are scalars and the x_i belong to σ). Thus $m_z=0$ and $(\overline{m_z})_q=0$. Hence $p_{m,A}(z)=0$ for all m in r_q and A in $\mathscr{C}$. This contradicts (1).

Finally we show that (2) *implies* (3). Assume the net $\{z_a\}_{a\in A}$ converges to z in the topology δ_q. To show $\{z_a\}_{a\in A}$ converges to z in the weak* topology, let s_i, A_i, m_i be such that $\|y-\sum_{i=1}^k s_i m_i(A_i)x_i\|<\varepsilon/2$. For some a_0 and $a>a_0$ we have

$$\sum_{i=1}^k|s_i|N_p(X_{A_i}x_i)\,(\overline{m_i,z_a-z})_q(A)<\varepsilon.$$

So

$$\Big|\Big\langle\sum_{i=1}^k s_i m_i(A_i)x_i, z_a-z\Big\rangle\Big|=\Big|\Big\langle\sum_{i=1}^k s_i\int X_{A_i}x_i dm_i, z_a-z\Big\rangle\Big|$$
$$\leqslant\sum_{i=1}^k|s_i|N_p(X_{A_i}x_i)\,(\overline{m_i,z_a-z})_q(A_i)<\varepsilon.$$

It follows that $|\langle y, z_a-z\rangle|<2\varepsilon$.

THEOREM 2. (1) *If* (σ^*,δ_q) *is a Hausdorff space then* (σ^*,δ_q) *is compact if and only if* $(\sigma^*,\delta_q)=(\sigma^*,wk^*)$, *where* wk^* *represents the weak* topology for* σ^*.

(2) *If* (σ^*,δ_q) *and* (σ^*,δ_r) *are Hausdorff spaces then* (σ^*,δ_q) *and* (σ^*,δ_r) *are both compact if and only if* $\delta_q=\delta_r=wk^*$.

Proof. We show (1). If (σ^*,δ_q) is Hausdorff then the identity map from (σ^*,δ_q) onto (σ^*,wk^*) is continuous by Lemma 3. Since (σ^*,wk^*) is a Hausdorff space, the map is a homeomorphism. Of course statement (2) follows immediately from statement (1).

In contrast to the situation depicted in [4] one may have (σ^*,δ_q) as a non Hausdorff space. If μ is identically zero, then r_q reduces to zero. Thus statement (2) of Lemma 3 shows that (σ^*,δ_q) is non Hausdorff. The other extreme is to have μ purely atomic. Then (σ^*,δ_q) is always a Hausdorff space. In fact let $m_t(A)=0$ when $t\notin A$ and $m_t(A)=U\in L(E,F)$ when $t\in A$. If A_t is the atom containing t, $(\mu(A_t)>0$. If $B\subset A_t$, then $\mu(B) = 0$ if and only if $B = \emptyset$), then $(\tilde{m}_t)_q(A) = \dfrac{\|U\|}{\mu(A_i)^{1-\frac{1}{q}}}$ is finite. So m_t belongs to r_q. By statement (2) of Lemma 3, it follows that (σ^*, δ_q) is Hausdorff.

The preceding observations point out that there are many more countably additive measures than measures in r_q (for $q \neq 1$). In [4] some conditions were pointed out which were equivalent to the topology generated by $p_{m,A}$ (m finitely additive, fixed, and A in $\mathscr{C}$ also fixed). A brief look at the proof shows that this does not carry over to the present setting since the point mass in general is not in r_q. However we have the following result.

PROPOSITION 1. *Assume (σ^*, δ_q) is a Hausdorff space, then the following conditions are equivalent.*

(1) *The topology generated by $p_{m,A}$ (for m in r_q fixed and A in $\mathscr{C}$ also fixed) is Hausdorff.*

(2) *$r_q = \{n\colon n$ in r_q for which $(\overline{m_{y^*}})_q = 0$ implies $(\overline{n_{y^*}})_q = 0\}$.*

(3) *The topology generated by $p_{m,A}$ on σ^* is stronger than the wk* topology of σ^*.*

Proof. If (2) holds and (1) does not, there exists a non-zero y^* in σ^* such that $(\overline{m_{y^*}})_q = 0$. Thus for all n in r_q, $(\overline{n_{y^*}})_q = 0$. This contradicts the fact that (σ^*, δ_q) is a Hausdorff space. The rest of the proof follows the pattern of [4] and will not be reproduced here.

4. Linear operators on $\mathscr{L}^p_E$. In this section $\mathscr{C}_{\sigma,f}$ will denote the σ-ring of $\mu\sigma$-finite subsets of T (see [1]). Now if $1 \leqslant p < \infty$, if U is a continuous linear operator from $\mathscr{L}^p_E(\mu)$ into F with $U \ll \mu$ and if $T \in \mathscr{C}_{\sigma,f}$, then there exists a unique measure m from $\mathscr{C}$ into $L(E, F)$ with $\tilde{m}_q(T)$ finite and $U(f) = \int f\,dm$. If $p = \infty$, then there exists a finitely additive set function m from $\mathscr{C}$ into $L(R, X)$ with $\tilde{m}_\infty(T) < \infty$ such that $U(f) = \int f\,dm$ for all f in $\mathscr{L}^\infty_R(\mu)$ where R denotes the scalar field (see [1]).

THEOREM 3. (1) *Let $p \neq \infty$, $1/p + 1/q = 1$ and let $T \in \mathscr{C}_{\sigma,f}$. If U is a continuous linear operator from $\mathscr{L}^p_E(\mu)$ into F such that $U \ll \mu$, then U is a compact operator if and only if $(\sigma^*, \delta_{m,q})$ is a compact space.*

(2) *Let $p = \infty$. If U is a continuous linear operator from $\mathscr{L}^\infty_R(\mu)$ into F such that $U \ll \mu$, then U is a compact operator if and only if $(\sigma^*, \delta_{m,1})$ is a compact space.*

Proof. In showing (1), let us assume that U is compact and let $\{z^*_n\}_{n\in N}$ be a sequence in σ^*. Without loss of generality we may assume that the sequence converges to z^* in the weak* topology. Thus we need to show that the sequence converges to z^* in the $\delta_{m,q}$ topology. (Now the sequence $\{U^*(z^*_n)\}_{n\in N}$ converges to $U^*(z^*)$ in the norm (see [3]).)

Note that $(\overline{m_{y^*}})_q = (\tilde{m}_{y^*})_q$ since m_{y^*} has values in a dual space. Thus for A in $\mathscr{C}$, there exists a disjoint sequence of sets A_i in $\mathscr{C}$ with $A_i \subset A$, $i \in N$, such that if $\varepsilon > 0$ and if $N_p(\sum \chi_{A_i} \cdot x_i) \leqslant 1$ then,

$$(\overline{m_{z^*-z^*_n}})_q(A) \leqslant \left|\left\langle \sum m(A_i)x_i,\ z^* - z^*_n\right\rangle\right| + \varepsilon$$

Thus

$$\begin{aligned}(\overline{m_{z^*-z^*_n}})_q(A) &\leqslant \left|\left\langle U\left(\sum \chi_{A_i}\cdot x_i\right),\ z^* - z^*_n\right\rangle\right| + \varepsilon\\ &\leqslant N_p\left(\sum \chi_{A_i}\cdot x_i\right)\|U^*(z^* - z^*_n)\| + \varepsilon.\end{aligned}$$

Consequently the sequence $\{z^*_n\}_{n\in N}$ converges to z^* in the $\delta_{m,q}$ topology.

Now assume that $(\sigma^*, \delta_{m,q})$ is compact and let $\{z^*_n\}_{n\in N}$ be a sequence in σ^*. Without loss of generality we may assume (by compactness) that the sequence converges to z^* in the $\delta_{m,q}$ topology. If $f \in \mathscr{L}^p_E(\mu)$, then

$$\begin{aligned}|\langle f,\ U^*(z^*_n) - U^*(z^*)\rangle| &= |\langle U(f),\ z^*_n - z^*\rangle|\\ &= \left|\left\langle \int f\,dm,\ z^*_n - z^*\right\rangle\right|\\ &\leqslant N_p(f)\,(\overline{m_{z^*_n-z^*}})_q(T).\end{aligned}$$

Since the latter converges to zero for n in N, the sequence $\{U^*(z^*_n)\}_{n\in N}$ converges to $U^*(z^*)$ in the norm. Thus U^* is compact and by [3] U is compact. The proof of (2) is similar and will not be reproduced here.

For the next theorem let $\mathscr{C}$ denote the σ-ring generated by the compact G_δ subsets of the locally compact Hausdorff space T. Again if $1 \leqslant p \leqslant \infty$, $1/p + 1/q = 1$, and if $m \in r_q$ then it is shown in [1] that "the integral of $f \in \mathscr{L}^p_E(\mu)$ relative to m" is defined (and is denoted by $\int f\,dm$) provided that $\tilde{m}_q(A)$ is finite for all $A \in \mathscr{C}_{\sigma,f}$ (the variation of m however, need not be finite).

The next theorem establishes a relation between the continuity of the integral $\int f\,dm$ on the unit ball of $\mathscr{L}^p_E$ and the right continuity of $\tilde{m}_q$.

THEOREM 4. *Let $\mathscr{C}$ be as described above and let $p \neq \infty$, $1/p + 1/q = 1$. If m is a measure from $\mathscr{C}$ into $L(E, F)$ with $\tilde{m}_q$ finite on $\mathscr{C}_{\sigma,f}$, then the right continuity of $\tilde{m}_q$ is equivalent to the following two conditions taken simultaneously.*

(1) *For every sequence $\{A_n\}_{n\in N}$ of sets in $\mathscr{C}$ decreasing monotonically to $\emptyset$, there exists a sequence of open Baire sets U_n in T such that $A_n \subset U_n$, n in N, and the sequence $\{\tilde{m}_q(U_n)\}_{n\in N}$ converges to 0.*

(2) *The sequence $\{|\int f_n\,dm|\}_{n\in N}$ converges uniformly to 0 for every sequence $\{f_n\}_{n\in N}$ in $\mathscr{L}^p_E(\mu)$ with $N_p(f_n) \leqslant 1$ and $f_n(x) = 0$ for x in $T \setminus U_n$, n in N.*

Proof. Let us assume that $\tilde{m}_q$ is right continuous. As in the proof of a similar result given in [2], it can be shown (replacing the p quasi semi variation by $\tilde{m}_q$) that for every $A \in \mathscr{C}$ and $\varepsilon > 0$ there exists a compact

Baire set K and an open Baire set G with $K \subset A \subset G$ and $\tilde{m}_q(G-K) < \varepsilon$. Thus we obtain a sequence $\{U_n\}_{n\in N}$ of open Baire sets with the sequence $\{\tilde{m}_q(U_n)\}_{n\in N}$ converging to 0. For every f_n in $\mathscr{L}_E^p(\mu)$ satisfying (2)

$$\left|\int f_n\, dm\right| \leqslant N_p(f_n)\tilde{m}_q(U_n).$$

Of course the latter becomes arbitrarily small.

Conversely assume (1) and (2) hold. Let $\{A_n\}_{n\in N}$ d.m. $\emptyset$ and let U_n be as above. Going to a subsequence if necessary, let us assume $\tilde{m}_q(A_n) > \varepsilon$ for all n. Pick j large enough so that whenever the support of f is a subset of U_j and $N_p(f) \leqslant 1$, $\left|\int f\, dm\right| < \varepsilon/2$. There exists some $z \in \sigma^*$ and some finite set of disjoint subsets B_i of A_j such that $\left|\left\langle \sum_i m(B_i)x_i, z\right\rangle\right| > \varepsilon$ with $N_p\left(\sum_i \chi_{B_i}\cdot x_i\right) \leqslant 1$. If $f = \sum_i \chi_{B_i}\cdot x_i$ then $\left|\int f\, dm\right| < \varepsilon/2$ which contradicts $\left|\int f\, dm\right| > \varepsilon$. Consequently the sequence $\{\tilde{m}_q(A_n)\}_{n\in N}$ converges to 0.

We now study the case for $q = 1$. Also for $q \neq 1$ we may ask the question for what kind of operators on $\mathscr{L}_E^p(\mu)$ is the q-semi variation of the representative measure right continuous?

If U is a continuous (in the norm of $\mathscr{L}_E^p(\mu)$, $1 \leqslant p \leqslant \infty$) operator with $U \ll \mu$, from $\mathscr{L}_E^p(\mu)$ into F, then we introduce the operator $\langle U, y^*\rangle$ from $\mathscr{L}_E^p(\mu)$ into the scalar field R defined by

$$\langle U, y^*\rangle(f) = \langle U(f), y^*\rangle \quad (y^* \in \sigma^*).$$

THEOREM 5. (1) *If U is a continuous and compact operator from $\mathscr{L}_R^\infty(\mu)$ into F with $U \ll \mu$, then the representative measure of U is countably additive.*

(2) *Let U be a continuous operator from $\mathscr{L}_E^p(\mu)$ into F ($p \neq \infty$) with $U \ll \mu$ and let $T \in \mathscr{C}_{\sigma,f}$. If there exists a scalar valued set function λ satisfying the O_μ condition with $\|\langle U, y^*\rangle_A\|_p \leqslant \lambda(A)$ for all A in $\mathscr{C}$ and with $\liminf \|\langle U, y^*\rangle_{A_n}\|_p \geqslant \|\langle U, y^*\rangle_A\|_p$ for every A_n and A in $\mathscr{C}$ for which the sequence $\{\mu(A_n - A)\}_{n\in N}$ converges to 0, then the representative measure of U is q-variationally semiregular.*

Proof. First we show statement (1). Since U and y^* are continuous and since $\langle U, y^*\rangle$ has its range contained in R it follows that $\|\,|\langle U, y^*\rangle_A|\,\|_\infty = \|\langle U, y^*\rangle_A\|_\infty$ and that $\|\langle U, y^*\rangle_A\|_\infty \leqslant \|\langle U, y^*\rangle_T\| < \infty$. Now $U(f) = \int f\, dm$ where m is finitely additive (see [1]). It is easy to see that $\langle U, y^*\rangle(f) = \int f\, dm_{y^*}$. Thus $\|\,|\langle U, y^*\rangle_T|\,\|_\infty = \overline{m_{y^*}}(T)$ which is finite. By [1], m_{y^*} will be countably additive if and only if for every sequence $\{\varphi_n\}_{n\in N}$ in $\mathscr{L}^\infty(\mu)$ which is decreasing and converging to 0 a.e. implies that the sequence $\{\langle U, y^*\rangle(\varphi_n)\}_{n\in N}$ converges to 0. Since

$$|\langle U, y^*\rangle(\varphi_n)| \leqslant \|\varphi_n\|_\infty \overline{m_{y^*}}(T)$$

and the latter goes to 0 for $n \in N$, it follows that m_{y^*} is countably additive. Since U is assumed compact it follows that U^* is also. We now show that σ^* is compact in the topology generated by $p_{m,T}$. Let $\{y_a\}_{a\in A}$ be a net in σ^*. Without loss of generality we may assume $\{y_a^*\}_{a\in A}$ converges to y^* in the weak* topology. By compactness of U, $U(y_n^*)$ converges to $U(y^*)$ in the norm of $\left(\mathscr{L}_R^\infty(\mu)\right)^*$. Since m_{y^*} has values in a dual space (in the scalar field here) there exists a family A_i of disjoint sets of $\mathscr{C}$ and scalars α_i, $|\alpha_i| \leqslant 1$ such that

$$\begin{aligned} m_{y_n^* - y^*}(T) &< \left|\left\langle \sum_i m(A_i)\alpha_i, y_n^* - y^*\right\rangle\right| + \varepsilon \\ &= \left|\left\langle \int \sum \chi_{A_i}\cdot \alpha_i\, dm, y_n^* - y^*\right\rangle\right| + \varepsilon \\ &= \left|\left\langle \sum \chi_{A_i}\cdot \alpha_i, U^*(y_n^* - y^*)\right\rangle\right| + \varepsilon \\ &\leqslant \|U^*(y_n^* - y^*)\| + \varepsilon. \end{aligned}$$

Since the latter becomes arbitrarily small for n in N, it follows that σ^* is compact in the topology generated by $p_{m,T}$. Using the compactness of σ^* and the fact that m_{y^*} is countably additive, it is easy to give an argument by contradiction to show that m is countably additive.

To show (2) we know from [1] since $T \in \mathscr{C}_{\sigma,f}$ that $U(f) = \int f\, dm$ where $\tilde{m}_q(T)$ is finite. It is easy to check that

$$\|\,|\langle U, y^*\rangle_A|\,\|_p = \|\langle U, y^*\rangle_A\|_p = (\overline{m_{y^*}})_q(A).$$

It then follows from the last part of Theorem 1 that m has a right continuous q-semi variation.

5. Some concluding remarks. It would be interesting to further study these topological spaces associated with these measures. The topological spaces under consideration, as has been seen, need not be metrizable in fact they need not even be Hausdorff. It would be interesting to consider the requirement that (σ^*, δ_m) or (σ^*, δ_{m_q}) be paracompact, metacompact or any of the other "compactness type" conditions. What is the effect of these conditions on the corresponding operator defined on $\mathscr{L}_E^p(\mu)$? The compact operators are then a subclass of the class of operators so obtained. Let us emphasize again that to go beyond the more restricted setting of compactness, we found essential the results of Orlicz in [6].

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Received May 27, 1972 (538)

STUDIA MATHEMATICA T. XLVIII. (1973)

Uniformly non-$l^{(1)}$ and B-convex Banach spaces*

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Abstract. Banach space is uniformly non-$l^{(1)}$ if and only if it is B-convex if and only if $l^{(1)}$ is not finitely representable in it. If all B-convex Banach spaces are reflexive, then B-convexity is equivalent to super-reflexivity. The non-reflexive space J which is isometrically isomorphic to J^{**} is not only not B-convex, but posseses a property which is sufficient but not necessary for non-B-convexity (c_0 is finitely representable in J).

It has long been known that a Banach space is reflexive if it is uniformly non-square. It is not known whether a Banach space is reflexive if it is uniformly non-$l^{(1)}$. It is shown that if this conjecture is correct, then a Banach space is super-reflexive if and only if it is uniformly non-$l^{(1)}$. The space J that is nonreflexive and isometric to J^{**} might have been a prime candidate for a counterexample to this conjecture, but it is shown that both c_0 and $l^{(1)}$ are finitely representable in J. It also is shown that, if $n \geqslant 2$ and every uniformly non-$l_n^{(1)}$ Banach space is reflexive, then every uniformly non-$l_n^{(1)}$ space is super-reflexive.

DEFINITION 1. For $n \geqslant 2$ and $\varepsilon > 0$, a normed linear space being *(n, ε)-convex* means that there does not exist a subset $\{x_1, \ldots, x_n\}$ of the unit ball such that, for all choices of signs,

$$\|x_1 \pm x_2 \pm \ldots \pm x_n\| > n(1-\varepsilon).$$

For $n \geqslant 2$, a *uniformly non-$l_n^{(1)}$* normed linear space is a normed linear space that is (n, ε)-convex for some $\varepsilon > 0$. A *B-convex* normed linear space is a normed linear space that is uniformly non-$l_n^{(1)}$ for some $n \geqslant 2$.

A B-convex Banach space is known to be reflexive if it has an *unconditional basis* (see [3], Theorem III.6, p. 142 or [9], Theorem 2.2), or (in the real case) if it can be endowed with a partial order under which it becomes a normed Riesz space (equivalently, a normed linear vector lattice; see [5]). Beck proved that a Banach space is B-convex if and only if a certain law of large numbers is valid for random variables with ranges in the space [1], which implies that B-convexity is isomorphically

* This research was supported in part by NSF Grant GP-28578.