a definition of topological sum of two topological spaces, we refer to Dugundji [3].

LEMMA 6. *If X, Y are compact Hausdorff spaces then $C(X, \boldsymbol{R}^n)$ is isometric with $C(Y, \boldsymbol{R}^n)$ if and only if the sum of n copies of X is homeomorphic with the sum of n copies of Y.*

Proof. It is verified that $C(X, \boldsymbol{R}^n)$ is isometric with $C(Y, \boldsymbol{R}^n)$ if and only if $C(X\times n, \boldsymbol{R})$ is isometric with $C(Y\times n, \boldsymbol{R})$. Hence from Banach-Stone theorem $X\times n$ is homeomorphic with $Y\times n$ i.e. the topological sum of n copies of X is homeomorphic with the topological sum of n copies of Y.

It is known that for each integer $n\geqslant 2$, there are non-homeomorphic compact metric spaces X, Y such that $X\times n$ is homeomorphic with $Y\times n$, Hanf [4]. A concrete description of such spaces X, Y, $n=2$ is provided in Sundaresan [10]. More generally there exist compact metric spaces X, Y such that $X\times k\neq Y\times k$ for $k=1,2,\ldots,n-1$, and $X\times n = Y\times n$, Kroonenberg [6].

It follows from Lemma 6 and preceeding remarks that there are non-homeomorphic compact metric spaces X, Y such that $C(X, \boldsymbol{R}^n)$ is isometric with $C(Y, \boldsymbol{R}^n)$ for $n\neq 2$. This justifies the additional hypothesis on the isometry in the preceeding theorem.

Bibliography

[1] S. Banach, *Théorie des Opérations Lineaires*, Warsaw 1932.
[2] M. M. Day, *Normed Linear Spaces*, Berlin 1962.
[3] J. Dugundji, *Topology*, Boston 1966.
[4] W. Hanf, *On some fundamental problems concerning isomorphism of Boolean algebras*, Math. Scand. 5 (1957), pp. 205–217.
[5] M. Jerison, *The space of bounded maps into a Banach space*, Ann. of Math. 52 (1950), pp. 309–327.
[6] N. Kroonenberg, *On an example of Hanf*, Research Report, Mathematisch Centrum, Amsterdam 1970.
[7] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), pp. 70–84.
[8] I. Singer, *Sur la meilleure approximation des fonctions abstractes continues á valeurs dans un espace de Banach*, Revue De Mathématiques Pures Et Appliquées, Tome 11, 1957, pp. 245–262.
[9] K. Sundaresan, *Some geometric properties of the unit cell in spaces $C(X, B)$*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. XIX (11) (1971), pp. 1007–1012.
[10] — *Spaces of continuous functions into $\boldsymbol{R}^2$*, Research Report, Department of Mathematics, Carnegie-Mellon University, 1971.

Received March 1, 1972 (496)

STUDIA MATHEMATICA T. XLVIII. (1973)

Decompositions of set functions

by

L. DREWNOWSKI (Poznań)

Abstract. Let $\mathscr{R}$ be a ring of sets. With each set $E\in\mathscr{R}$ a collection of classes $\mathscr{D}\subset\mathscr{R}$, consisting of disjoint sets, is associated in such a way that the set $\mathfrak{S}$ of all resulting pairs $(E, \mathscr{D})$ satisfies certain very natural conditions. The $\mathfrak{S}$ is then called an additivity on $\mathscr{R}$ (Section 2). Notions of $\mathfrak{S}$-additive and $\mathfrak{S}$-singular group valued set functions are next introduced and investigated to some degree; when specifying $\mathfrak{S}$ one obtains, e.g., notions of σ-additive and purely finitely additive or η-continuous and η-singular functions. For a very important class of the so called exhaustive (= strongly bounded) set functions a decomposition theorem (3.11) is proved, whose special cases are the Hewitt–Yosida and Lebesgue decompositions for group valued functions. Analogons of general and special decompositions are established also for some nonadditive functions (sumbeasures) and for Fréchet–Nikodym topologies on $\mathscr{R}$ (Section 4). By the way a theorem is given (2.14′) which contains the Vitali–Hahn–Saks, Nikodym and Brooks–Jewett theorems.

Introduction. Let $\mathscr{R}$ be a ring of sets and let μ, η be additive real-valued set functions on $\mathscr{R}$ with μ bounded and $\eta\geqslant 0$. We say that μ is η-continuous and write $\mu\ll\eta$ if, given $\varepsilon>0$, there is $\delta>0$ such that $|\mu(E)|\leqslant\varepsilon$ whenever $\eta(E)\leqslant\delta$, $E\in\mathscr{R}$. At first sight it is not seen at all that the properties "μ is countably additive" and "μ is η-continuous" have much in common. However, it can be proved ([10]; [7], II) that $\mu\ll\eta$ iff $\mu(E_n)\to 0$ provided $E_n\searrow$ and $\eta(E_n)\to 0$, $(E_n)\subset\mathscr{R}$. The latter condition can be equivalently formulated as follows: if (E_n) is a disjoint sequence of sets in $\mathscr{R}$, $E\in\mathscr{R}$, $\bigcup_{n=1}^{\infty}E_n\subset E$ and $\eta\big(E\setminus\bigcup_{k=1}^{n}E_k\big)\to 0$, then $\mu(E)=\sum_{n=1}^{\infty}\mu(E_n)$; the resemblance with the definition of countably additivity is striking. This observation was first made and employed by W. Orlicz in his study of absolute continuity of vector valued set functions [10]; it motivates the general notion of $\mathfrak{S}$-additivity introduced in Section 2. Also, it had suggested a quite natural conjecture that it should been possible to obtain the well known Hewitt–Yosida and Lebesgue decompositions of additive set function in a unified fashion. This is realized in the present paper for exhaustive additive set functions with values in an arbitrary abelian complete topological group G. The method we use seems to be new even in the scalar case: lattice methods are excluded — instead of them completeness of G is exploited, assumption of boundedness of a set function is replaced by its exhaustivity — a property the importance of which was only recently brought to light [2], [5], [7], [9], and the relation $\ll$ plays the leading role in definitions, theorems and proofs. For example, our definition of a purely finitely additive set function $\mu\colon \mathscr{R}\to G$ sounds as follows: μ is p.f.a. if the conditions $\lambda\colon \mathscr{R}\to G$ is σ-additive, $\lambda\ll\mu$, imply $\lambda=0$.

0. Basic notions, terminology and notations. Everywhere in the sequel $\mathscr{R}$ is a ring of sets (δ- or σ-ring if explicitely stated). If $\mathscr{A},\mathscr{B}\subset\mathscr{R}$ then $\mathscr{A}\mathring{\cap}\mathscr{B}:=\{A\cap B\colon A\in\mathscr{A}, B\in\mathscr{B}\}$; in the case $\mathscr{B}=\{B\}$ we write $\mathscr{A}\mathring{\cap}B$ instead of $\mathscr{A}\mathring{\cap}\{B\}$. The operations $\mathring{\cup}$, $\mathring{\triangle}$, $\mathring{\dot-}$ (difference) are defined similarly. $\mathscr{D}$ is the generic notation for a class of pairwise disjoint sets from $\mathscr{R}$ and $\Delta=\Delta(\mathscr{R})$ denotes the set of all such classes. Δ_f is the set of all finite classes $\mathscr{D}\in\Delta$, Δ_c the set of all at most countable $\mathscr{D}$ in Δ. If $\mathscr{D}_1,\mathscr{D}_2\in\Delta$ then $\mathscr{D}_1\leqslant\mathscr{D}_2$ means that for every $D_2\in\mathscr{D}_2$ there exists $D_1\in\mathscr{D}_1$ such that $D_2\subset D_1$. It is clear that $\leqslant$ partially orders Δ, and more, that Δ is directed by $\leqslant$, for $\mathscr{D}_i\leqslant\mathscr{D}_1\mathring{\cap}\mathscr{D}_2$ for arbitrary $\mathscr{D}_i\in\Delta$, $i=1,2$.

If A is a set then the class of all finite subsets F of A is denoted by $\mathfrak{f}(A)$.

Let G be a topological abelian group. Then $a(R;G)$ denotes the group of all additive set functions $\mu\colon\mathscr{R}\to G$ and $ca(\mathscr{R};G)$, $ea(\mathscr{R};G)$ are its subgroups consisting of all countably additive and all exhausting μ, respectively. Recall that $\mu\in a(\mathscr{R};G)$ is said to be *exhaustive* [7] (= strongly bounded [11], [2], [5]) if $\mu(E_n)\to 0$ for each disjoint sequence $(E_n)\subset\mathscr{R}$.

If G is a weakly sequentially complete locally convex linear space or a Banach space which contains no subspace isomorphic to c_0, or if $\mathscr{R}$ is a σ-ring and G is separable normed linear space, then $\mu\in a(\mathscr{R};G)$ is exhaustive iff it is bounded [5].

If $\mathscr{R}$ is a σ-ring then $ca(\mathscr{R};G)\subset ea(\mathscr{R};G)$; in general this is no longer true if $\mathscr{R}$ is merely a ring ([7] II).

As concerns the definitions and results cited below, the reader is refered to [7].

A topology Γ on $\mathscr{R}$ is called a *Fréchet-Nikodym topology* (shortly: FN-topology) if $\mathscr{R}$ (with the symmetric difference $E\triangle F=(E\setminus F)\cup(F\setminus E)$ as addition) is a topological group under Γ and if, moreover, the operation of intersection $(E,F)\mapsto E\cap F$ is uniformly continuous on $\mathscr{R}$. Thus $(\mathscr{R};\triangle,\cap)$ equipped with Γ is a topological ring with the uniformly continuous multiplication $\cap$. If η is a *submeasure* on $\mathscr{R}$ (i.e., $\eta\colon\mathscr{R}\to[0,\infty]$ and $\eta(\emptyset)=0$, $A\subset B\Rightarrow\eta(A)\leqslant\eta(B)$, $\eta(A\cup B)\leqslant\eta(A)+\eta(B)$), then $\Gamma(\eta)$ is the FN-topology on $\mathscr{R}$ determined by η, that is by the Fréchet–Nikodym ecart $(A,B)\mapsto\eta(A\triangle B)$. Every FN-topology Γ on $\mathscr{R}$ is generated

by a family $(\eta_t\colon t\in T)$ of submeasures on $\mathscr{R}$ and, conversely, with each such a family a unique FN-topology, denoted $\Gamma(\eta_t\colon t\in T)$, is associated.

If $\mu\in a(\mathscr{R};G)$ then $\mu\ll\Gamma$ means that μ is *Γ-continuous*, which is equivalent with Γ-continuity of μ at $\emptyset$ as well as with uniform Γ-continuity of μ on $\mathscr{R}$. For each $\mu\in a(\mathscr{R};G)$ there exists the coarsest FN-topology, $\Gamma(\mu)$, with respect to which μ is continuous. If $\mathscr{B}$ is a base of neighbourhoods of 0 in G then the classes $\mathscr{U}_U=\{E\in\mathscr{R}\colon \mu(F)\in U \text{ for each } F\subset E, F\in\mathscr{R}\}$, $U\in\mathscr{B}$, constitute a base of neighbourhoods of $\emptyset$ in $(\mathscr{R},\Gamma(\mu))$. Another way to describe $\Gamma(\mu)$ is the following: Let a family $(|\cdot|_t\colon t\in T)$ of quasi-norms on G determine the topology of G. Then $\Gamma(\mu)=\Gamma(\overline{\mu}^t\colon t\in T)$, where the submeasure $\overline{\mu}^t$ on $\mathscr{R}$, called the *submeasure majorant for* μ with respect to $|\cdot|_t$, is defined by the formula $\overline{\mu}^t(E)=\sup\{|\mu(F)|_t\colon F\subset E, F\in\mathscr{R}\}$.

If μ, ν are topological groups valued additive set functions then μ is *ν-continuous* means that $\mu\ll\Gamma(\nu)$ (equivalently, $\Gamma(\mu)\subset\Gamma(\nu)$), and we write $\mu\ll\nu$. If one (or both) of μ, ν is a submeasure then ν-continuity of μ is defined in the same way. μ and ν are called *equivalent*, $\mu\sim\nu$, if $\mu\ll\nu\ll\mu$, i.e., $\Gamma(\mu)=\Gamma(\nu)$.

An FN-topology Γ on $\mathscr{R}$ is said to be *exhaustive* (*order continuous*) if each infinite disjoint sequence $(E_n)\subset\mathscr{R}$ is Γ-convergent to $\emptyset$ (respectively, if $E_n\searrow\emptyset$ implies $E_n\xrightarrow{\Gamma}\emptyset$); similarly for submeasures.

If $\mu\in ea(\mathscr{R};G)$ or $\mu\in ca(\mathscr{R};G)$ then $\Gamma(\mu)$ is exhaustive or order continuous, respectively.

Let H be the set of all submeasures on $\mathscr{R}$ with the order relation $\leqslant$ defined in the usual way: $\eta_1\leqslant\eta_2$ iff $\eta_1(E)\leqslant\eta_2(E)$ for each $E\in\mathscr{R}$. $(H,\leqslant)$ is a complete lattice: if $\emptyset\neq M\subset H$ then the supremum $\bigvee M$ and the infinum $\bigwedge M$ of M in H exist and are defined by the formulas:

$$\bigvee M(E)=\sup\{\mu(E)\colon \mu\in M\},$$
$$\bigwedge M(E)=\inf\{\mu_1(E_1)+\ldots+\mu_n(E_n)\},\qquad E\in\mathscr{R},$$

where the infinum is taken over all finite (disjoint) decompositions $E_1,\ldots,E_n$ ($E_i\in\mathscr{R}$) of E and all sequences $\mu_1,\ldots,\mu_n$ ($\mu_i\in M$).

In particular, $\mu_1\wedge\mu_2(E)=\inf\{\mu_1(F)+\mu_2(E\setminus F)\colon F\in\mathscr{R}, F\subset E\}$. Evidently, if $\mu_1,\mu_2\in H$ then $\mu_1\vee\mu_2\sim\mu_1+\mu_2$.

Note that if $\lambda,\eta\in H$ and $\lambda\ll\eta$ then $\lambda\sim\lambda\wedge\eta\leqslant\eta$ and $\lambda\leqslant\lambda\vee\eta\sim\eta$.

The set of all FN-topologies on $\mathscr{R}$ is a complete lattice under order relation $\subset$; the trivial topology $\{\emptyset,\mathscr{R}\}$, in what follows denoted often by 0, and the discrete topology $\mathfrak{P}(\mathscr{R})$, are the least and greatest FN-topologies on $\mathscr{R}$, respectively.

It is easy to verify that if $\Gamma_i=\Gamma(\eta_t\colon t\in T_i)$, $i=1,\ldots,n$, then

$$\Gamma_1\vee\ldots\vee\Gamma_n=\Gamma(\eta_{t_1}\vee\ldots\vee\eta_{t_n}\colon (t_1,\ldots,t_n)\in T_1\times\ldots\times T_n)$$
$$=\Gamma(\eta_{t_1}+\ldots+\eta_{t_n}\colon (t_1,\ldots,t_n)\in T_1\times\ldots\times T_n)$$

and

$$\Gamma_1 \wedge \ldots \wedge \Gamma_n = \Gamma(\eta_{t_1} \wedge \ldots \wedge \eta_{t_n}\colon (t_1, \ldots, t_n) \in T_1 \times \ldots \times T_n).$$

1. Preparatory lemmas. In all of this section, with the exception of Lemma 1.5, G is a complete Hausdorff topological abelian group. As concerns summability in topological groups we refer to [3].

1.1. Lemma. *An additive set function $\mu\colon \mathscr{R} \to G$ is exhaustive iff for each $\mathscr{D} \in \Delta$ the family $(\mu(D)\colon D \in \mathscr{D})$ is summable in G.*

Proof. Let $\mu \in ea(\mathscr{R}; G)$, $\mathscr{D} \in \Delta$. We show that $(\mu(D)\colon D \in \mathscr{D})$ satisfies the Cauchy condition for summability. Otherwise there is a neighbourhood U of 0 in G such that for each $\mathscr{D}' \in \mathfrak{f}(\mathscr{D})$ there exists $\mathscr{D}'' \in \mathfrak{f}(\mathscr{D} \setminus \mathscr{D}')$ with $\mu(\bigcup \mathscr{D}'') \notin U$. It is seen that an infinite sequence $(\mathscr{D}_n) \subset \mathfrak{f}(\mathscr{D})$ can be found such that $\mathscr{D}_n \cap \mathscr{D}_m = \emptyset$ if $n \neq m$ and $\mu(\bigcup \mathscr{D}_n) \notin U$, $n = 1, 2, \ldots$, contrary to the assumption that $\mu \in ea(\mathscr{R}; G)$. The converse implication is trivial.

The sum $\sum_{D \in \mathscr{D}} \mu(D)$ of a family $(\mu(D)\colon D \in \mathscr{D})$, where $\mathscr{D} \in \Delta$, will be frequently in the sequel denoted by $\mu(\mathscr{D})$.

Remark. A family $(x_\iota\colon \iota \in I)$ of elements of G is summable iff the additive set function $\xi\colon \mathfrak{f}(I) \to G$ defined by means of the formula $\xi(J) = \sum_{\iota \in J} x_\iota$, $J \in \mathfrak{f}(I)$, is exhaustive.

1.2. Lemma. *Let $\mu \in ea(\mathscr{R}; G)$, $(D_\iota)_{\iota \in I} \in \Delta$ (thus $\iota \neq \iota' \Rightarrow D_\iota \cap D_{\iota'} = \emptyset$). Then for each closed neighbourhood U of 0 in G there exists $I_0 \in \mathfrak{f}(I)$ such that if $J \subset I \setminus I_0$ and for each $\iota \in J$, $\mathscr{D}_\iota \in \Delta$ and $\bigcup \mathscr{D}_\iota \subset D_\iota$, then*

$$\sum_{\iota \in J} \mu(\mathscr{D}_\iota) \in U.$$

The particular case: $U = \{x \in G\colon |x| \leqslant \varepsilon\}$, where $|\cdot|$ is a (continuous) quasi-norm on G and $\varepsilon > 0$.

Proof. Suppose that it is not so for some U. Thus, given $K \in \mathfrak{f}(I)$ there is $J \subset I \setminus K$ and a family $(\mathscr{D}_\iota)_{\iota \in J}$, where $\mathscr{D}_\iota \in \Delta$ and $\bigcup \mathscr{D}_\iota \subset D_\iota$, such that $\sum_{\iota \in J} \mu(D_\iota) \notin U$. Then we can find $J' \in \mathfrak{f}(J)$ with the property that $\sum_{\iota \in J'} \mu(\mathscr{D}_\iota) \notin U$ and next, for each $\iota \in J'$, a class $\mathscr{D}'_\iota \in \mathfrak{f}(\mathscr{D}_\iota)$ in such a way that $\sum_{\iota \in J'} \mu(\mathscr{D}'_\iota) = \mu(\bigcup_{\iota \in J'} \bigcup \mathscr{D}'_\iota) \notin U$.

Now it is evident that starting from an arbitrary $K \in \mathfrak{f}(I)$, repeating the above argument to the set $K' = K \cup J'$, etc., we obtain a contradiction.

1.3. Lemma. *If $\mu \in ea(\mathscr{R}; G)$, $\mathscr{D} \in \Delta$ and $\bigcup \mathscr{D} \subset E$, where $E \in \mathscr{R}$, then for each continuous quasi-norm $|\cdot|$ on G we have*

$$|\mu(\mathscr{D})| \leqslant \bar{\mu}(E),$$

$$|\mu(E \mathbin{\dot{-}} \mathscr{D})| \leqslant \bar{\mu}(E).$$

1.4. Lemma. *If $\mu \in ea(\mathscr{R}; G)$, $\emptyset \neq \Delta_0 \subset \Delta$ and Δ_0 is directed by $\leqslant$, then $\lim_{\mathscr{D} \in \Delta_0} \mu(\mathscr{D})$ exists.*

Proof. We are going to prove that $(\mu(\mathscr{D})\colon \mathscr{D} \in \Delta_0)$ is a Cauchy net in G. Let $|\cdot|$ be a continuous quasi-norm on G and suppose that $(\mu(\mathscr{D})\colon \mathscr{D} \in \Delta_0)$ does not satisfy the Cauchy condition with respect to $|\cdot|$. Then there exists a sequence $(\mathscr{D}_n) \subset \Delta_0$ and a number $\varepsilon > 0$ such that

$$\mathscr{D}_1 \leqslant \mathscr{D}_2 \leqslant \ldots$$

and

$$|\mu(\mathscr{D}_{n+1}) - \mu(\mathscr{D}_n)| > 3\varepsilon, \quad n = 1, 2, \ldots$$

Applying 1.2 to the class $\mathscr{D}_1$ we find $\mathscr{D}'_1 \in \mathfrak{f}(\mathscr{D}_1)$ such that

$$\Big|\sum (\mu(\mathscr{D}_n \mathbin{\mathring{\cap}} D)\colon D \in \mathscr{D}''_1)\Big| \leqslant \varepsilon/2, \quad n = 1, 2, \ldots,$$

where $\mathscr{D}''_1 = \mathscr{D}_1 \setminus \mathscr{D}'_1$.

Let $E^1 = \bigcup \mathscr{D}'_1$, $\mathscr{D}^1_n = E^1 \mathbin{\mathring{\cap}} \mathscr{D}_n$ $(n = 1, 2, \ldots)$. Since $\mu(\mathscr{D}^1_1) = \mu(E^1)$ and $\mu(\mathscr{D}_n) = \sum_{D \in \mathscr{D}_1} \mu(\mathscr{D}_n \mathbin{\mathring{\cap}} D) = \mu(\mathscr{D}^1_n) + \sum_{D \in \mathscr{D}''_1} \mu(\mathscr{D}_n \mathbin{\mathring{\cap}} D)$, we have

$$|\mu(\mathscr{D}^1_{n+1}) - \mu(\mathscr{D}^1_n)| > 2\varepsilon, \quad n = 1, 2, \ldots$$

Now, again by Lemma 1.2, there is $\mathscr{D}'_2 \in \mathfrak{f}(\mathscr{D}^1_2)$ such that

$$\Big|\sum (\mu(\mathscr{D}^1_n \mathbin{\mathring{\cap}} D)\colon D \in \mathscr{D}''_2)\Big| \leqslant \varepsilon/2^2, \quad n = 2, 3, \ldots,$$

where $\mathscr{D}''_2 = \mathscr{D}^1_2 \setminus \mathscr{D}'_2$. Let $E^2 = \bigcup \mathscr{D}'_2$, $\mathscr{D}^2_n = \mathscr{D}^1_n \mathbin{\mathring{\cap}} E^2$ $(n = 2, 3, \ldots)$. We have $\mu(\mathscr{D}^2_2) = \mu(E^2)$, $E^2 \subset E^1$,

$$|\mu(\mathscr{D}^2_{n+1}) - \mu(\mathscr{D}^2_n)| > \varepsilon + \varepsilon/2, \quad n = 2, 3, \ldots$$

and

$$|\mu(E^1 \setminus E^2)| = |\mu(\mathscr{D}^1_1) - \mu(\mathscr{D}^2_2)| > \varepsilon + (3/4)\varepsilon.$$

Continuing we find a sequence $(E^n)_{n \in \mathbf{N}}$ in $\mathscr{R}$ such that $E^n \downarrow$ and $|\mu(E^n \setminus E^{n+1})| > \varepsilon$ for $n = 1, 2, \ldots$ A contradiction.

Lemma 1.5. *Let G, H be arbitrary commutative topological groups and let $\mu\colon \mathscr{R} \to G$, $\lambda\colon \mathscr{R} \to H$ be additive set functions. Then $\lambda \ll \mu$ iff for each continuous quasi-norm $\|\cdot\|$ on H there is a continuous quasi-norm $|\cdot|$ on G such that*

$$\|\lambda(:)\| \ll \bar{\mu}(:),$$

where $\bar{\mu}$ denotes the submeasure majorant for μ with respect to $|\cdot|$.

Proof. Sufficiency of the condition is obvious.

Necessity. According to ([7] I; 2.8) there exists a sequence $(|\cdot|_n)$ of quasi-norms on G such that

$$\|\lambda(:)\| \ll \eta = \sum_{n=1}^{\infty} 2^{-n} (\bar{\mu}^1 + \ldots + \bar{\mu}^n)(1 + \bar{\mu}^1 + \ldots + \bar{\mu}^n)^{-1},$$

where $\overline{\mu}^i$ is the submeasure majorant for μ with respect to $|\cdot|_i$. Let

$$|\cdot| = \sum_{n=1}^{\infty} 2^{-n}(|\cdot|_1+\ldots+|\cdot|_n)(1+|\cdot|_1+\ldots+|\cdot|_n)^{-1}.$$

Clearly, $\overline{\mu} \leqslant \eta$, where $\overline{\mu}$ is with respect to $|\cdot|$. Since $2^{-n}(\overline{\mu}^1+\ldots+\overline{\mu}^n)\times$ $\times(1+\overline{\mu}^1+\ldots+\overline{\mu}^n)^{-1} \ll \overline{\mu}$ $(n = 1, 2, \ldots)$, we have $\eta \ll \overline{\mu}$. Thus $\overline{\mu} \ll \eta \ll \overline{\mu}$ and hence $\|\lambda(:)\| \ll \overline{\mu}(:)$.

Remark. The above lemma, modified in the obvious way, is valid also if λ is a submeasure on $\mathscr{R}$.

2. $\mathfrak{S}$-additivity and $\mathfrak{S}$-singularity. Given a set $\mathfrak{S} \subset \mathscr{R}\times\varDelta(\mathscr{R})$, let us denote

$$\mathfrak{S}[E] = \{\mathscr{D}\in\varDelta\colon (E, \mathscr{D})\in\mathfrak{S}\}, \qquad E\in\mathscr{R}$$

and

$$\varDelta_{\mathfrak{S}} = \bigcup_{E\in\mathscr{R}} \mathfrak{S}[E].$$

2.1. Definition. An *additivity* on $\mathscr{R}$ is a set $\mathfrak{S} \subset \mathscr{R}\times\varDelta(\mathscr{R})$ such that the following conditions are satisfied:

(a 1) $\varDelta_f \subset \varDelta_{\mathfrak{S}}$ and $\bigcup_{E\in\mathscr{R}} \{E\}\times\mathfrak{S}[E] = \mathfrak{S}$.

(a 2) If $E\in\mathscr{R}$, $\mathscr{D}\in\mathfrak{S}[E]$, then $\bigcup\mathscr{D}\subset E$.

(a 3) If $E\in\mathscr{D}$ and $\mathscr{D}_1, \mathscr{D}_2\in\mathfrak{S}[E]$ then $\mathscr{D}_1 \mathring{\cap} \mathscr{D}_2 \in \mathfrak{S}[E]$.

(a 4) If $E, F\in\mathscr{R}$, $F\subset E$ and $\mathscr{D}\in\mathfrak{S}[E]$ then $\mathscr{D}\mathring{\cap} F\in\mathfrak{S}[F]$.

(a 5) If $E_1, E_2\in\mathscr{R}$, $E_1\cap E_2 = \emptyset$ and $\mathscr{D}_i\in\mathfrak{S}[E_i]$ $(i = 1, 2)$ then $\mathscr{D}_1\cup\mathscr{D}_2\in\mathfrak{S}[E_1\cup E_2]$.

(a 6) If $E\in\mathscr{R}$, $\mathscr{D}\in\mathfrak{S}[E]$, and each $D\in\mathscr{D}$ is the union of two disjoint sets $D_1, D_2\in\mathscr{R}$, then $\mathscr{D}^* = \{D_i\colon D\in\mathscr{D},\ i = 1, 2\}\in\mathfrak{S}[E]$.

It follows from (a 1) and (a 3) that for each $E\in\mathscr{R}$ the class $\mathfrak{S}[E]$ is nonempty and is directed by the order relation $\leqslant$ defined in Section 0.

2.2. Examples of additivities. 1) $\mathfrak{S}_f = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta_f$ and $\bigcup\mathscr{D} = E\}$.

2) $\mathfrak{S}_c = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta_c$ and $\bigcup\mathscr{D} = E\}$. More generally: let $\mathfrak{m}$ be an infinite cardinal; then $\mathfrak{S}_{\mathfrak{m}} = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta,\ \operatorname{card}(\mathscr{D})\leqslant\mathfrak{m}$ and $\bigcup\mathscr{D} = E\}$ is also an additivity on $\mathscr{R}$.

3) $\mathfrak{S}_a = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta$ and $\bigcup\mathscr{D} = E\}$.

4) Let $\mathscr{N}$ be an ideal of $\mathscr{R}$. Then $\mathfrak{S}_f(\mathscr{N}) = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta_f$, $\bigcup\mathscr{D}\subset E$ and $E\setminus\bigcup\mathscr{D}\in\mathscr{N}\}$ and $\mathfrak{S}_c(\mathscr{N})$, $\mathfrak{S}_{\mathfrak{m}}(\mathscr{N})$, $\mathfrak{S}_a(\mathscr{N})$ — defined in a similar fashion—are additivities on $\mathscr{R}$.

5) Let Γ be an FN-topology on $\mathscr{R}$. Given a class $\mathscr{E}\subset\mathscr{R}$, call a set $E\in\mathscr{R}$ to be a *Γ-union* of $\mathscr{E}$, $E = (\Gamma)\bigcup\mathscr{E}$, if for every neighbourhood $\mathscr{U}$ of $\emptyset$ in $(\mathscr{R}, \Gamma)$ there exists $\mathscr{E}_{\mathscr{U}}\in\mathfrak{f}(\mathscr{E})$ such that $E\triangle\bigcup\mathscr{E}'\in\mathscr{U}$ whenever $\mathscr{E}_{\mathscr{U}}\subset\mathscr{E}'$ $\in\mathfrak{f}(\mathscr{E})$. (Thus $E = (\Gamma)\lim\big(\bigcup\mathscr{E}'\colon \mathscr{E}'\in\mathfrak{f}(\mathscr{E})\big)$, i.e., E is the limit in $(\mathscr{R}, \Gamma)$ of the net $\big(\bigcup\mathscr{E}'\colon \mathscr{E}'\in\mathfrak{f}(\mathscr{E})\big)$. Then $\mathfrak{S}_c(\Gamma) = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta_c,\ \bigcup\mathscr{D}\subset E$ and $(\Gamma)\bigcup\mathscr{D} = E\}$ as well as $\mathfrak{S}_a(\Gamma)$, similarly defined, and also $\mathfrak{S}_{uc}(\Gamma)$, $\mathfrak{S}_{ua}(\Gamma)$, in the definitions of which we require that $\bigcup\mathscr{D} = E$, are additivities on $\mathscr{R}$. Special cases of these additivities are obtained if $\Gamma = \Gamma(\eta)$, where η is a submeasure on $\mathscr{R}$. The *η-additivity* on $\mathscr{R}$ is, by definition, $\mathfrak{S}(\eta) = \mathfrak{S}_c\big(\Gamma(\eta)\big)$. Note that if $\mathscr{D}\in\varDelta_c$, $(D_n\colon n\in\mathbf{N})$ is an enumeration of $\mathscr{D}$ $(D_n\cap D_m = \emptyset$ if $n\neq m)$ and $\bigcup\mathscr{D}\subset E$, then $(\eta)\bigcup\mathscr{D} = E$ iff $\eta\big(E\setminus\bigcup_{k=1}^{n} D_k\big)\to 0$ as $n\to\infty$.

6) Let G be a topological group, and let $\Phi\subset ea(\mathscr{R}; G)$. Then the additivity generated by Φ is, by definition, $\mathfrak{S}(\Phi) = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta$, $\bigcup\mathscr{D}\subset E$ and for each $\mu\in\Phi$ and $F\subset E$, $F\in\mathscr{R}$, the family $\big(\mu(D\cap F)\colon D\in\mathscr{D}\big)$ is summable in G and $\Sigma\big(\mu(D\cap F)\colon D\in\mathscr{D}\big) = \mu(F)\}$.

2.3. Let $\mathscr{S}$ be the set of all additivities on $\mathscr{R}$; $\mathscr{S}$ is (partially) ordered by the set inclusion $\subset$. Evidently, $\mathfrak{S}_f$ and $\mathfrak{S}_0 = \{(E, \mathscr{D})\colon E\in\mathscr{R},\ \mathscr{D}\in\varDelta$, $\cup\mathscr{D}\subset E\}$ are the least and the greatest elements of $\mathscr{S}$, respectively. It is also clear that if $\mathfrak{S}_1, \mathfrak{S}_2\in\mathscr{S}$ then the supremum and the infimum of $\{\mathfrak{S}_1, \mathfrak{S}_2\}$ in $(\mathscr{S}, \subset)$ exist and are, respectively,

$$\mathfrak{S}_1\vee\mathfrak{S}_2 = \bigcup_{E\in\mathscr{R}} \{E\}\times\{\mathscr{D}_1\cap\mathscr{D}_2\colon \mathscr{D}_i\in\mathfrak{S}_i[E],\ i = 1, 2\}$$

and

$$\mathfrak{S}_1\wedge\mathfrak{S}_2 = \mathfrak{S}_1\cap\mathfrak{S}_2.$$

If $(\mathfrak{S}_\iota\colon \iota\in I)$ is a family of additivities then $\bigwedge_{\iota\in I}\mathfrak{S}_\iota$ exists and $= \bigcap_{\iota\in I}\mathfrak{S}_\iota$. Therefore $(\mathscr{S}, \subset)$ is a complete lattice.

In the remaining part of this section G, H are commutative topological groups and $\mathfrak{S}$ is an additivity on $\mathscr{R}$.

2.4. Definitions. A set function $\mu\colon \mathscr{R}\to G$ is said to be *$\mathfrak{S}$-additive* if for each $E\in\mathscr{R}$ and $\mathscr{D}\in\mathfrak{S}[E]$ the family $\big(\mu(D)\colon D\in\mathscr{D}\big)$ is summable in G and

$$\sum_{D\in\mathscr{D}}\mu(D) = \mu(E)$$

or, equivalently,

$$\lim_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} \mu(E\setminus\bigcup\mathscr{D}') = 0,$$

where $\mathfrak{f}(\mathscr{D})$ is obviously considered with the directing relation $\subset$. An FN-topology Γ on $\mathscr{R}$ will be called *$\mathfrak{S}$-continuous* if

$$(\Gamma)\lim_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} (E\setminus\bigcup\mathscr{D}') = \emptyset$$

for each $E\in\mathscr{R}$, $\mathscr{D}\in\mathfrak{S}[E]$. In particular, a submeasure η on $\mathscr{R}$ is $\mathfrak{S}$-continuous if

$$\lim_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} \eta(E\setminus\bigcup\mathscr{D}') = 0, \qquad E\in\mathscr{R},\ \mathscr{D}\in\mathfrak{S}[E].$$

2.5. We explain the sense of $\mathfrak{S}$-additivity in the case of additivities given in 2.3. Let $\mu\colon \mathscr{R} \to G$ (G Hausdorff).

The $\mathfrak{S}_\mathfrak{f}$-, $\mathfrak{S}_c$-, $\mathfrak{S}_\mathfrak{m}$- and $\mathfrak{S}_a$-additivity of μ is simply the finite, countable, $\mathfrak{m}$—and complete (total) additivity, respectively.

The $\mathfrak{S}_\mathfrak{f}(\mathscr{N})$-, $\mathfrak{S}_c(\mathscr{N})$-, $\mathfrak{S}_\mathfrak{m}(\mathscr{N})$- and $\mathfrak{S}_a(\mathscr{N})$*-additivity* means the $\mathfrak{S}_\mathfrak{f}$-, etc., -additivity, respectively, together with the property $E \in \mathscr{N} \Rightarrow \mu(E) = 0$. Notice that $\mathfrak{S}_\mathfrak{z}(\mathscr{N}) = \mathfrak{S}_\mathfrak{z} \vee \mathfrak{S}_\mathfrak{f}(\mathscr{N})$.

If η is a submeasure on $\mathscr{R}$ then $\mathfrak{S}(\eta)$- or, shortly, η-additivity of μ means simply that

$$(E_n) \subset \mathscr{R},\ E_n \searrow \text{ and } \eta(E_n) \searrow 0 \Rightarrow \mu(E_n) \to 0;$$

if μ is a submeasure then the same condition is equivalent to $\mathfrak{S}(\eta)$-continuity of μ. The notion of η-additivity has its origin in [10].

It is obvious that $\mu \ll \Gamma$ implies $\mathfrak{S}_c(\Gamma)$-, etc., -additivity (or -continuity) of μ. And also, if Γ is an $\mathfrak{S}$-continuous FN-topology on $\mathscr{R}$ then $\mu \ll \Gamma$ implies the $\mathfrak{S}$-additivity (or $\mathfrak{S}$-continuity) of μ.

Notice that $\mathfrak{S}_c$-continuity of an FN-topology or submeasure is nothing else as its order continuity.

If Γ is associated with a family $(\eta_t\colon t \in T)$ of submeasures then Γ is $\mathfrak{S}$-continuous iff each η_t is $\mathfrak{S}$-continuous (comp. with ([7] III; 8.1). Hence, given an additivity $\mathfrak{S}$ on $\mathscr{R}$, there exists the strongest $\mathfrak{S}$-continuous FN-topology $\Gamma(\mathfrak{S})$ such that an FN-topology Γ is $\mathfrak{S}$-continuous iff $\Gamma \subset \Gamma(\mathfrak{S})$.

2.6. THEOREM. *If $\mu\colon \mathscr{R} \to G$ is $\mathfrak{S}$-additive then the topology $\Gamma(\mu)$ is $\mathfrak{S}$-continuous.*

Proof. First, let $|\cdot|$ be a continuous quasi-norm on G, and let $\bar{\mu}$ be the corresponding submeasure majorant for μ. Obviously, $\mu\colon \mathscr{R} \to (G, |\cdot|)$ is $\mathfrak{S}$-continuous. We claim that $\bar{\mu}$ is $\mathfrak{S}$-continuous. Otherwise there exist $E \in \mathscr{R}$, $\mathfrak{D} \in \mathfrak{S}[E]$ and a number $\varepsilon > 0$ such that for each $\mathfrak{D}' \in \mathfrak{f}(\mathfrak{D})$

$$\bar{\mu}(E \setminus \bigcup \mathfrak{D}') > \varepsilon;$$

hence

$$|\mu(F_{\mathfrak{D}'})| > \varepsilon$$

for some $F_{\mathfrak{D}'} \subset E \setminus \bigcup \mathfrak{D}'$, $F_{\mathfrak{D}'} \in \mathscr{R}$.

Let us fix some $\mathfrak{D}' \in \mathfrak{f}(\mathfrak{D})$, Since $\mathfrak{D} \overset{\circ}{\cap} F_{\mathfrak{D}'} \in \mathfrak{S}[F_{\mathfrak{D}'}]$ and $\mu(F_{\mathfrak{D}'}) = \Sigma\big(\mu(D \cap F_{\mathfrak{D}'})\colon D \in \mathfrak{D}\big)$, there exists $\mathfrak{D}'' \in \mathfrak{f}(\mathfrak{D} \setminus \mathfrak{D}')$ such that

$$|\mu(F_{\mathfrak{D}'} \cap \bigcup \mathfrak{D}'')| > \varepsilon.$$

For $\mathfrak{D}' \cup \mathfrak{D}''$ we can repeat this argument. It is therefore evident, that an infinite disjoint sequence $(\mathfrak{D}_n) \subset \mathfrak{f}(\mathfrak{D})$ and a sequence $(F_n) \subset \mathscr{R}$ can be found such that

$$F_n \subset \bigcup \mathfrak{D}_n,\ |\mu(F_n)| > \varepsilon, \quad n = 1, 2, \ldots$$

If $\mathfrak{D}^* = \bigcup_{n=1}^{\infty} (\mathfrak{D}_n \overset{\circ}{\cap} F_n) \cup \bigcup_{n=1}^{\infty} (\mathfrak{D}_n \dot{-} F_n) \cup (\mathfrak{D} \setminus \bigcup_{n=1}^{\infty} \mathfrak{D}_n)$ $(\in \mathfrak{S}[E])$ then $\mu(E) = \sum_{D \in \mathfrak{D}^*} \mu(D)$, hence $\mu(F_n) = \sum_{D \in \mathfrak{D}_n \overset{\circ}{\cap} F_n} \mu(D) \to 0$. We have obtained a contradiction. The assertion of the Theorem now follows, for $\Gamma(\mu)$ is generated by the family of all such $\bar{\mu}$'s.

2.7. COROLLARY. *If $\mu \in a(\mathscr{R}; G)$ is σ-additive then $\Gamma(\mu)$ is order continuous* ([7] III; 8.4).

2.8. COROLLARY. *Let $\mu \in a(\mathscr{R}; G)$ and $\lambda \in a(\mathscr{R}; H)$, or let λ be a submeasure on $\mathscr{R}$. Then:*

(a) *μ is $\mathfrak{S}$-continuous iff $\mu \ll \Gamma(\mathfrak{S})$.*

(b) *If μ is $\mathfrak{S}$-additive, $\lambda \ll \mu$ then λ is $\mathfrak{S}$-additive* (*or $\mathfrak{S}$-continuous*), *too.*

2.9. THEOREM. *Let η be an arbitrary submeasure on $\mathscr{R}$, and μ be an exhaustive submeasure or $\mu \in ea(\mathscr{R}; G)$. Then the $\mathfrak{S}$-additivity* (*or $\mathfrak{S}$-continuity*) *of μ is equivalent with its η-continuity in each of the following cases:*

(a) $\mathfrak{S} = \mathfrak{S}_c\big(\Gamma(\eta)\big)$;

(b) *$\mathscr{R}$ is a δ-ring and* $\mathfrak{S} = \mathfrak{S}_{uc}\big(\Gamma(\eta)\big) \vee \mathfrak{S}_f(\mathscr{N}_\eta)$;

(c) *$\mathscr{R}$ is a δ-ring and* $\mathfrak{S} = \mathfrak{S}_c(\mathscr{N}_\eta)$,

where $\mathscr{N}_\eta = \{E \in \mathscr{R}\colon \eta(E) = 0\}$.

(*Note that* $\mathfrak{S}_c(\mathscr{N}_\eta) \subset \mathfrak{S}_{uc}\big(\Gamma(\eta)\big) \vee \mathfrak{S}_f(\mathscr{N}_\eta) \subset \mathfrak{S}_c\big(\Gamma(\eta)\big)$.

The meaning of the $\mathfrak{S}$-additivity (or $\mathfrak{S}$-continuity) of μ in cases (a), (b), (c) is the following:

(a) $\big(E_n \searrow, \eta(E_n) \searrow 0\big) \Rightarrow \mu(E_n) \to 0$.

(b) $\big(E_n \searrow \emptyset, \eta(E_n) \searrow 0\big) \Rightarrow \mu(E_n) \to 0$, and $\eta(E) = 0 \Rightarrow \mu(E) = 0$.

(c) μ is σ-additive (or order continuous) and

$$\eta(E) = 0 \Rightarrow \mu(E) = 0.$$

Proof. In the case μ is a submeasure the Theorem is a corollary to ([7], II; 6.1). So let $\mu \in ea(\mathscr{R}; G)$. Obviously, $\mu \ll \eta$ implies that μ is $\mathfrak{S}$-additive. The inverse implication is less trivial.

(a) We assume that μ is $\mathfrak{S}_c(\eta)$-additive, i.e.,

$$\big(E_n \searrow, \eta(E_n) \searrow 0\big) \Rightarrow \mu(E_n) \to 0.$$

In order to prove that $\mu \ll \eta$ it suffices to show that if $|\cdot|$ is a quasi-norm on G and $\bar{\mu}$ the corresponding submeasure majorant for μ, then

$$(*) \qquad \big(E_n \searrow, \eta(E_n) \searrow 0\big) \Rightarrow \bar{\mu}(E_n) \searrow 0.$$

The implication $(*)$ being proved, we shall have $\bar{\mu} \ll \eta$ by ([7], II; 6.1 (a)). If $(*)$ is not valid then there exists a sequence $E_n \searrow$, $\eta(E_n) \searrow 0$ such that $\bar{\mu}(E_n) > \varepsilon > 0$, $n = 1, 2, \ldots$ Hence for every $n \in \mathbf{N}$ we can find a set $F_n \subset E_n$ with $|\mu(F_n)| > \varepsilon$. If n is fixed then $F_n \cap E_m \searrow$ and $\eta(F_n \cap E_m) \searrow 0$ $(m \to \infty)$, so $|\mu(F_n \cap F_m)| \to 0$ $(m \to \infty)$. Since $|\mu(F_n \setminus E_m)| \geqslant |\mu(F_n)| -$

$-|\mu(F_n \setminus E_m)|$, we have $|\mu(F_n \setminus E_m)| > \varepsilon$ for sufficiently large m. Now it is clear that a sequence $1 = n_1 < n_2 < \ldots$ can be found such that

$$|\mu(F_{n_k} \setminus E_{n_{k+1}})| > \varepsilon, \quad k = 1, 2, \ldots$$

This however is impossible, for the sets $F_{n_k} \setminus E_{n_{k+1}}$ are disjoint and $\mu \in ea(\mathscr{R}; G)$. Similary, $\mu \ll \eta$ in the case (b) follows from ([7], II; 6.1 (b)) and in (c) from ([7], II; 6.1 (c)) (see [7], III; 8.5).

2.10. Proposition. *Let G be complete and Hausdorff and let T be a family of continuous additive mappings $f\colon G \to H$, total on G, i.e., if $x \in G$ and $f(x) = 0$ for each $f \in T$ then $x = 0$. Then a function $\mu \in ea(\mathscr{R}; G)$ is $\mathfrak{S}$-additive iff $f \circ \mu$ is $\mathfrak{S}$-additive for each $f \in T$.*

Proof. Indeed, if $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, then $\mu(\mathscr{D})$ exists by 1.1. Hence, assuming that all $f \circ \mu$ are $\mathfrak{S}$-additive, we have

$$f \circ \mu(E) = f \circ \mu(\mathscr{D}), f \in T, \quad \text{so } \mu(E) = \mu(\mathscr{D}).$$

As a consequence of Theorem 2.9 and the preceeding proposition we have

2.11. Corollary. *Under the assumptions of 2.10, if $\mu \in ea(\mathscr{R}; G)$ and η is a submeasure on $\mathscr{R}$, then $\mu \ll \eta$ iff $f \circ \mu \ll \eta$ for each $f \in T$.*

One immediately obtains special cases of 2.10 and 2.11 when G is a locally convex linear space and T a total set in its conjugate.

2.12. Definition. A family $(\mu_t\colon t \in T)$ of G-valued set functions (or submeasures) is said to be *uniformly $\mathfrak{S}$-additive* (*$\mathfrak{S}$-continuous*) if for each $E \in \mathscr{R}$ and $\mathscr{D} \in \mathfrak{S}[E]$, $\lim_{\mathscr{D}' \in \mathfrak{f}(\mathscr{D})} \mu_t(E \setminus \bigcup \mathscr{D}') = 0$ uniformly for $t \in T$.

2.13. Theorem. *Let $(\mu_t\colon t \in T)$ be a family of uniformly exhaustive G-valued additive set functions (or submeasures). If each of μ_t is $\mathfrak{S}$-additive ($\mathfrak{S}$-continuous) then the family $(\mu_t\colon t \in T)$ is uniformly $\mathfrak{S}$-additive (uniformly $\mathfrak{S}$-continuous).*

Proof. To prove the uniform $\mathfrak{S}$-additivity of the family $(\mu_t\colon t \in T)$ if suffices to show that, given a quasi-norm $|\cdot|$ on G, the corresponding family of submeasure majorants $(\overline{\mu}_t\colon t \in T)$ is uniformly $\mathfrak{S}$-continuous. The family $(\overline{\mu}_t\colon t \in T)$ is uniformly exhaustive ([7], II; 4.2) and each $\overline{\mu}_t$ is $\mathfrak{S}$-continuous, by 2.5. Therefore we can assume that μ_t are submeasures, and have to verify that $\mu(E) = \sup_{t \in T} \mu_t(E)$, $E \in \mathscr{R}$, is an $\mathfrak{S}$-continuous submeasure. Clearly μ is exhaustive. Suppose μ is not $\mathfrak{S}$-continuous. Thus there exist $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$ and $\varepsilon > 0$ such that

$$\mu(E \setminus \bigcup \mathscr{D}') > \varepsilon$$

for each $\mathscr{D}' \in \mathfrak{f}(\mathscr{D})$. As in the proof of 2.6 one easily obtains a sequence $(t_n) \subset T$ and a disjoint sequence $(\mathscr{D}_n) \subset \mathfrak{f}(\mathscr{D})$ such that $\mu_{t_n}(\bigcup \mathscr{D}_n) > \varepsilon$, $n = 1, 2, \ldots$ A contradiction.

The above Theorem and a theorem of Brooks and Jewett [2], especially the version of this result presented in ([9]; 4.3) imply the following

2.14. Corollary. *If $\mathscr{R}$ is a σ-ring and a sequence $(\mu_n) \subset ea(\mathscr{R}; G)$ is such that each μ_n is $\mathfrak{S}$-additive and the limit*

$$\mu(E) = \lim_{n \to \infty} \mu_n(E)$$

exists for each $E \in \mathscr{R}$, then μ, μ_n, $n \in \mathbf{N}$, are uniformly exhaustive and uniformly $\mathfrak{S}$-additive.

Particular cases: (a) $\mathfrak{S} = \mathfrak{S}_f$. (b) $\mathfrak{S} = \mathfrak{S}_c$ (The Nikodym theorem ([7], III; 8.6), ([9]; 3.1). (b′) $\mathfrak{S} = \mathfrak{S}_m$, $\mathfrak{m} \geq \aleph_0$ (The generalized Nikodym theorem). (c) $\mathfrak{S} = \mathfrak{S}(\eta)$, η is a submeasure on $\mathscr{R}$ (The Vitali–Hahn–Saks theorem [2]; Th. 3, [9]; 4.4): If $\mu_n \ll \eta$, $n = 1, 2, \ldots$, then μ, μ_n are uniformly η-continuous.). (c) is a consequence of 2.9 and the foregoing proposition.

All these cases and the Corollary 2.14 itself are included in the following general theorem of Vitali–Hahn–Saks type.

2.14′. Theorem. *Let Γ be an FN-topology on a σ-ring $\mathscr{R}$ and let (μ_n) be a sequence of exhaustive additive G-valued set functions on $\mathscr{R}$ such that each μ_n is Γ-continuous and the limit*

$$\mu(E) = \lim \mu_n(E)$$

exists for each $E \in \mathscr{R}$. Then μ, μ_n ($n \in \mathbf{N}$) are uniformly exhaustive and uniformly Γ-continuous.

Proof. Let $\|\cdot\|$ be a continuous quasi-norm on G. Then for each $n \in N$ there is a Γ-continuous submeasure η_n on $\mathscr{R}$ such that $\|\mu_n(:)\| \ll \eta_n$ (comp. with 1.5 and [7], I; 2.8). It is obvious that for each $n \in N$ we have $\|\mu_n(:)\| \ll \eta = \sum_{n=1}^{\infty} 2^{-n}\min(1, \eta_n)$ and η is a Γ-continuous submeasure on $\mathscr{R}$. By the usual Vitali–Hahn–Saks theorem (case (c) above), the set functions $\mu_n\colon \mathscr{R} \to (G, \|\cdot\|)$, $n \in \mathbf{N}$, are uniformly η-continuous, hence uniformly Γ-continuous. Q.E.D.

Setting $\Gamma = \Gamma(\mathfrak{S})$ one obtains from 2.14′ the Corollary 2.14.

2.14″. Remark. In a similar fashion it can be proved that if (μ_n) is a sequence of uniformly exhaustive additive G-valued functions on a ring $\mathscr{R}$, each μ_n being Γ-continuous, then μ_n are uniformly Γ-continuous (comp. the proofs of 2.9 and ([7], II; 6.1)).

2.15. Proposition. *Suppose that $\mathfrak{S}$ and an FN-topology Γ on $\mathscr{R}$ are related in such a way that the $\mathfrak{S}$-additivity of an (arbitrary) group valued set function μ defined on $\mathscr{R}$, implies $\mu \ll \Gamma$. Then the uniform $\mathfrak{S}$-additivity of a family $(\mu_t\colon t \in T) \subset a(\mathscr{R}; G)$ implies its uniform Γ-continuity.*

Similarly for submeasures.

Proof. Let $F(T;G)$ be the group of all functions $f\colon T \to G$, equipped with the topology of uniform convergence on T. The uniform $\mathfrak{S}$-additivity of $(\mu_t\colon t \in T)$ is equivalent with the $\mathfrak{S}$-additivity of the set function $\mu\colon \mathcal{R} \to F(T;G)$ defined by the formula $\mu(E) = (\mu_t(E)\colon t \in T)$, and this in turn implies $\mu \ll \Gamma$, i.e., the uniform Γ-continuity of $(\mu_t\colon t \in T)$.

In the case μ_t are submeasures we assume the uniform $\mathfrak{S}$-continuity of $(\mu_t\colon t \in T)$ and consider the submeasure $\mu = \bigvee_{t\in T} \mu_t$. It is $\mathfrak{S}$-continuous, hence μ_t are uniformly Γ-continuous.

In the next Theorem one easily recognizes the result of Diestel [5], mentioned in Section 0 (proved here in a slightly different way than the original one), and the well known theorems of Pettis on "weak" measures.

2.16. Theorem. *Let $\mathcal{R}$ be a σ-ring and X a Banach space. Let $\mu \in a(\mathcal{R}; X)$ and let η be a submeasure on $\mathcal{R}$.*

(a) *If μ is bounded and separable valued (i.e., the set $\mu[\mathcal{R}]$ is separable), then μ is exhaustive.*

(b) *If μ is weakly σ-additive, i.e., $x^*\circ\mu$ is σ-additive for each $x^* \in X^*$, then μ is σ-additive.*

(c) *If μ is bounded and separable valued or σ-additive, then the weak η-continuity of μ implies η-continuity of μ.*

Proof. (a) We can assume that X is separable. Since μ is bounded, $x^*\circ\mu$ is exhaustive for each $x^* \in X^*$ ([7], II; 4.14). If μ is not exhaustive then there exist a disjoint sequence $(E_n) \subset \mathcal{R}$ and a number $\varepsilon > 0$ such that $\|\mu(E_n)\| > \varepsilon$, $n = 1, 2, \ldots$ Let a sequence $(x_n^*) \subset \{x^*\colon \|x^*\| \leqslant 1\}$ be so chosen that $|x_n^*(\mu(E_n))| > \varepsilon$, $n = 1, 2, \ldots$ Since X is separable, we can assume that the sequence (x_n^*) is pointwise convergent on X to some $x_0^* \in X^*$. Contrary to 2.14, the sequence $(x_n^*\circ\mu)$ is not uniformly exhaustive, for $|x_n^*\circ\mu(E_n)| > \varepsilon$.

(b) In view of 2.10, we need prove that μ is exhaustive. It is clear that μ is bounded. Let (E_n) be a disjoint sequence in $\mathcal{R}$, and let $\mathcal{E}$ be the σ-ring generated by (E_n). Then $\mu[\mathcal{E}]$ is separable, hence $\mu(E_n) \to 0$, by (a).

(c) follows from (a), (b) and 2.11.

Remarks. 1) The Theorem is obviously valid without the assumption that X is complete. Besides, (a) and (c) remain true if "μ is separable valued" is replaced by "X is weakly sequentially complete (or $\not\supset c_0$)" (see Section 0).

2) The following observation is a corollary to 2.14: If $\mathcal{R}$ is a σ-ring, X a normed linear space, (μ_n) a sequence of bounded additive set functions on $\mathcal{R}$ to X and $\mu_0(E) = \lim \mu_n(E)$ exists for each $E \in \mathcal{R}$, then μ_n $(n = 0, 1, \ldots)$ are uniformly bounded, i.e.,

$$\sup\{\|\mu_n(E)\|\colon E \in \mathcal{R},\ n = 0, 1, \ldots\} < \infty.$$

To see this, consider μ_n as set functions from $\mathcal{R}$ into $(X, \sigma(X, X^*))$ and apply 2.14 ($\mathfrak{S} = \mathfrak{S}_f$), ([7], II; 4.5, 4.8) and the fact that $\sigma(X, X^*)$-boundedness implies boundedness in the norm topology of X.

We close this section with the definition of $\mathfrak{S}$-singularity and some remarks on it.

2.17. Definition. An FN-topology Γ on $\mathcal{R}$ is said to be $\mathfrak{S}$-*singular* if the only $\mathfrak{S}$-continuous FN-topology coarser than Γ is the trivial one, $0 = \{\emptyset, \mathcal{R}\}$, in other words if $\Gamma \wedge \Gamma(\mathfrak{S}) = 0$.

An additive group-valued set function or submeasure μ on $\mathcal{R}$ will be called *G-singular* if the topology $\Gamma(\mu)$ is G-singular.

Probably more adequate name for the defined property of Γ or μ is *extremely non G-continuous or non-G-additive*; we shall use it sometimes.

The following facts are quite obvious.

1) Γ is $\mathfrak{S}$-singular iff each submeasure both $\mathfrak{S}$-continuous and Γ-continuous vanishes on $\mathcal{R}$.

2) If Γ is $\mathfrak{S}$-singular and $\Gamma_1 \subset \Gamma$ then Γ_1 is $\mathfrak{S}$-singular.

3) If μ is $\mathfrak{S}$-singular and $\mu_1 \ll \mu$ then μ_1 is $\mathfrak{S}$-singular.

4) If μ is $\mathfrak{S}$-singular then the relation $\lambda \ll \mu$ for an $\mathfrak{S}$-additive set function taking valued in a Hausdorff group or an $\mathfrak{S}$-continuous submeasure λ is possible only if $\lambda = 0$. The converse statement is true if μ is a submeasure or $\mu \in ea(\mathcal{R}; G)$, where G is complete (see 3.4).

5) Suppose that η is a submeasure on $\mathcal{R}$. Then η is $\mathfrak{S}$-singular iff any $\mathfrak{S}$-continuous submeasure λ on $\mathcal{R}$ such that $\lambda \leqslant \eta$ is identically zero. "Only if" follows from 1). "If": Let λ be an $\mathfrak{S}$-continuous submeasure on $\mathcal{R}$ such that $\lambda \ll \eta$. Then $\lambda \sim \lambda\wedge\eta \leqslant \eta$ and, obviously, $\lambda\wedge\eta$ is $\mathfrak{S}$-continuous. Therefore $\lambda\wedge\eta = 0$ and hence $\lambda = 0$.

6) Suppose that G_1, G_2 are subgroups of a topological group G, and let an additive set function μ maps $\mathcal{R}$ into G_1 as well as into G_2. Then the properties of μ "to be $\mathfrak{S}$-additive" and "to be $\mathfrak{S}$-singular" do not depend on the choice of the G_i we wish consider as a range of μ. Thus, in particular, these properties preserve if one replaces G by its completion $\tilde{G}$. It was just the reason we did not take the condition from 4) as a definition of $\mathfrak{S}$-singularity of μ.

If $\mathfrak{S} = \mathfrak{S}_c$ then, following Hewitt and Yosida [8], an additive $\mathfrak{S}_c$-singular set function μ may be called *purely finitely additive*. Let us note, however, that this is not an appropriate term for the extremely non-σ-additive part in the general Hewitt–Yosida decomposition established in 3.12 (a)—rather "purely exhaustive" would be the better, the more so as an analogon of H–Y. decomposition holds also for submeasures (see 4.3, 1).

If η is submeasure on $\mathscr{R}$ then instead of "$\mathfrak{S}(\eta)$-singular" we shall say "*η-singular*", in accordance with the standard use of this term. Thus, e.g., a submeasure λ is η-singular iff any submeasure $\nu \ll \lambda$ and simultaneously $\ll \eta$ is equal to zero, and iff, by 5), $\lambda \wedge \eta = 0$.

3. Decompositions of additive exhaustive set functions. Throughout this section G, H are topological commutative groups with G Hausdorff and complete (unless otherwise is explicitely stated), and $\mathfrak{S}$ is a fixed additivity on a ring $\mathscr{R}$. Furthermore, with only a few exceptions, we shall deal with additive exhaustive set functions only.

Let $\mu \in ea(\mathscr{R}; G)$. By 1.4 for each $E \in \mathscr{R}$ there exists the limit

$$\mu'(E) = \lim_{\mathscr{D} \in \mathfrak{S}[E]} \mu(\mathscr{D}).$$

It is clear that $\mu' \in a(\mathscr{R}; G)$. From 1.3 it follows that for each quasi-norm $|\cdot|$ on G we have $|\mu'(E)| \leqslant \bar{\mu}(E)$, $E \in \mathscr{R}$. Hence, $\bar{\mu}$ being an exhaustive submeasure, $\mu' \in ea(\mathscr{R}; G)$ and $\mu' \ll \mu$. The mapping $\mu \mapsto \mu'$ of $ea(\mathscr{R}; G)$ into $ea(\mathscr{R}; G)$ just defined will be denoted S (or $S_{\mathfrak{s}}$ if needed). Thus $S\mu(E) = \lim_{\mathscr{D} \in \mathfrak{S}[E]} \mu(\mathscr{D})$, $E \in \mathscr{R}$, and $S\mu \ll \mu$. Evidently $S\colon ea(\mathscr{R}; G) \to ea(\mathscr{R}; G)$ is additive (linear if G is a topological linear space).

3.1. LEMMA. *For each $\mu \in ea(\mathscr{R}; G)$ the function $S\mu$ is $\mathfrak{S}$-additive and μ-continuous.*

Proof. Let $|\cdot|$ be a quasi-norm on G. Let $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, and let $\varepsilon > 0$. According to Lemma 1.2 there is $\mathscr{D}_0 \in \mathfrak{f}(\mathscr{D})$ such that if $\mathscr{D}_D \in \mathfrak{S}[D]$, $D \in \mathscr{D}$, then

$$(\cdot) \qquad \Big|\sum_{D \in \mathscr{D}'} \mu(\mathscr{D}_D)\Big| \leqslant \varepsilon/2$$

for every $\mathscr{D}' \subset \mathscr{D} \setminus \mathscr{D}_0$. Let $\mathscr{D}^* \in \mathfrak{S}[E]$, $\mathscr{D} \leqslant \mathscr{D}^*$, be such that

$$|S\mu(E) - \mu(\mathscr{D}^*)| \leqslant \varepsilon/2.$$

Since $\mu(\mathscr{D}^*) = \sum_{D \in \mathscr{D}} (\mathscr{D}^* \mathring{\cap} D)$, for each $\mathscr{D}_1 \in \mathfrak{f}(\mathscr{D})$, $\mathscr{D}_0 \subset \mathscr{D}_1$, we have

$$\Big|S\mu(E) - \sum_{D \in \mathscr{D}_1} \mu(\mathscr{D}^* \mathring{\cap} D)\Big| = \Big|\{S\mu(E) - \mu(\mathscr{D}^*)\} + \Big\{\mu(\mathscr{D}^*) - \sum_{D \in \mathscr{D}_1} \mu(\mathscr{D}^* \mathring{\cap} D)\Big\}\Big|$$
$$\leqslant \tfrac{1}{2}\varepsilon + \Big|\sum_{D \in \mathscr{D} \setminus \mathscr{D}_1} \mu(\mathscr{D}^* \mathring{\cap} D)\Big| \leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

by $(\cdot)$. Hence

$$\Big|S\mu(E) - \sum_{D \in \mathscr{D}_1} S\mu(D)\Big| \leqslant \varepsilon.$$

Consequently,

$$S\mu(E) = \sum_{D \in \mathscr{D}} S\mu(D).$$

Let us note that if $\mu \in ea(\mathscr{R}; G)$ then μ is $\mathfrak{S}$-additive iff $S\mu = \mu$. Therefore S maps $ea(\mathscr{R}; G)$ onto the subgroup of $ea(\mathscr{R}; G)$ consisting of $\mathfrak{S}$-additive functions, and $S \circ S = S$ (i.e., S is a projection).

3.2 THEOREM. *Let $\lambda \in ea(\mathscr{R}; H)$, $\mu \in ea(\mathscr{R}; G)$. If $\lambda \ll \mu$ and λ is $\mathfrak{S}$-additive then*

$$\lambda \ll S\mu.$$

The same is true if λ is μ-continuous and $\mathfrak{S}$-continuous submeasure on $\mathscr{R}$.

Proof. Let $\|\cdot\|$ be a quasi-norm on H. Since $\lambda \ll \mu$, in virtue of Lemma 1.5 there is a quasi-norm $|\cdot|$ on G such that $\|\lambda(:)\| \ll \bar{\mu}$. Take $\varepsilon > 0$ and let $\delta > 0$ be such that

$$(*) \qquad \bar{\mu}(E) \leqslant 2\delta \Rightarrow \|\lambda(E)\| \leqslant \varepsilon, \qquad E \in \mathscr{R}.$$

We claim that

$$\overline{S\mu}(E) < \delta \Rightarrow \|\lambda(E)\| \leqslant \varepsilon, \qquad E \in \mathscr{R}.$$

Suppose that a set $E \in \mathscr{R}$ is such that

$$\overline{S\mu}(E) < \delta \text{ but } \|\lambda(E)\| \geqslant \varepsilon + \gamma, \text{ where } \gamma > 0.$$

In view of $(*)$ it must be $\bar{\mu}(E) > 2\delta$ and hence $|\mu(F)| > 2\delta$ for some $F \in \mathscr{R}$, $F \subset E$.

Let $\mathscr{D} \in \mathfrak{S}[E]$ be such that $|\mu(\mathscr{D})| < \delta$ and $|\mu(\mathscr{D} \mathring{\cap} F)| < \delta$. Since λ is $\mathfrak{S}$-additive, there exists $\mathscr{D}^1 \in \mathfrak{f}(\mathscr{D})$ for which

$$\|\lambda(E \setminus \bigcup \mathscr{D}^1)\| < \gamma/2$$

as well as

$$|\mu(\mathscr{D}^1)| < \delta \quad \text{and} \quad |\mu(\mathscr{D}^1 \mathring{\cap} F)| < \delta.$$

Let us denote $\bigcup \mathscr{D}^1 = E^1$, $F^1 = F \setminus E^1$. We have $|\mu(E^1)| < \delta$, $|\mu(F \cap E^1| < \delta$ and $\|\lambda(E \setminus E^1)\| < \gamma/2$. Hence $|\mu(F^1)| \geqslant |\mu(F)| - |\mu(F \cap E^1)| > \delta$ and $\varepsilon + \gamma \leqslant \|\lambda(E)\| \leqslant \|\lambda(E \setminus E^1)\| + \|\lambda(E^1)\| < \frac{1}{2}\gamma + \|\lambda(E^1)\|$, so $\|\lambda(E^1)\| > \varepsilon$. Thus, assuming that $\overline{S\mu}(E) < \delta$ and $\|\lambda(E)\| > \varepsilon$, we have found two disjoint sets E^1, $F^1 \subset E$ such that $\|\lambda(E^1)\| > \varepsilon$ and $|\mu(F^1)| > \delta$. Repeating, since $\overline{S\mu}(E^1) < \delta$, there are disjoint subsets E^2, F^2 of E^1 for which $\|\lambda(E^2)\| > \varepsilon$ and $|\mu(F^2)| > \delta$. Continuing in this manner we get a contradiction, for $\mu \in ea(\mathscr{R}; G)$. The proof in the case λ is a submeasure is almost unchanged.

3.3. COROLLARY. *If G, H are complete and $\mu \in ea(\mathscr{R}; G)$, $\lambda \in ea(\mathscr{R}; H)$, then $\lambda \ll \mu$ implies $S\lambda \ll S\mu$.*

Indeed, $\lambda \ll \mu \Rightarrow S\lambda \ll \lambda \ll \mu \Rightarrow S\lambda \ll S\mu$, by 3.1, 3.2.

3.4. PROPOSITION. *A function $\nu \in ea(\mathscr{R}; G)$ is $\mathfrak{S}$-singular iff*

(a) *$(\lambda \in ea(\mathscr{R}; G)$, $\lambda \ll \nu$, λ is $\mathfrak{S}$-additive$) \Rightarrow \lambda = 0$, and iff*

(b) $S\nu = 0$.

Proof. Obviously, $\mathfrak{S}$-singularity of $\nu \Rightarrow$ (a), and (a) $\Rightarrow$ (b), by 3.1. Now assume that (b) holds. If η is an $\mathfrak{S}$-continuous submeasure on $\mathscr{R}$ and $\eta \ll \nu$, then $\eta \ll S\nu$, by 3.2. Hence $\eta = 0$ and, in a consequence, $\Gamma(\nu)$ is $\mathfrak{S}$-singular, i.e., ν is $\mathfrak{S}$-singular.

3.5. Corollary. *Let T be a total set of continuous additive mappings $f\colon G \to H$. If $\nu \in ea(\mathscr{R}; G)$ then ν is $\mathfrak{S}$-singular iff $f\circ\nu$ is $\mathfrak{S}$-singular for each $f \in T$.*

Proof. If ν is $\mathfrak{S}$-singular then $S\nu = 0$, hence $S(f\circ\nu) = 0$ for each $f \in T$, i.e., $f\circ\nu$ are $\mathfrak{S}$-singular. The converse is proved similarly (we can assume that H is complete).

3.6. Corollary. *Let X be a separated locally convex linear space. If $\nu \in ea(\mathscr{R}; X)$ then ν is $\mathfrak{S}$-singular iff $x^*\circ\nu$ is $\mathfrak{S}$-singular for each $x^* \in X^*$.*

This follows immediately from 3.5, for we can assume now that X is complete.

3.7. Corollary. *For each $\mu \in ea(\mathscr{R}; G)$ the function $\mu - S\mu$ is $\mathfrak{S}$-singular and μ-continuous.*

Proof. $\mu - S\mu$ is $\mathfrak{S}$-singular, for $S(\mu - S\mu) = 0$. Its μ-continuity follows from 1.3: $|\mu(E \dot{-} \mathscr{D})| \leqslant \overline{\mu}(E) \Rightarrow \overline{\mu - S\mu} \leqslant \overline{\mu}$.

Notation: $S'\mu = \mu - S\mu$.

3.8. Theorem. *Let $\mu \in ea(\mathscr{R}; G)$, $\lambda \in ea(\mathscr{R}; H)$. If $\lambda \ll \mu$ and λ is $\mathfrak{S}$-singular then*

$$\lambda \ll S'\mu .$$

Proof. We begin as in the proof of 3.2 and claim that

$$\overline{S'\mu}(E) < \delta \Rightarrow \|\lambda(E)\| \leqslant \varepsilon, \qquad E \in \mathscr{R}.$$

Suppose that for a set E we have

$$\overline{S'\mu}(E) < \delta \quad \text{and} \quad \|\lambda(E)\| \geqslant \varepsilon + \gamma, \quad \gamma > 0.$$

Then we can find $F \subset E$ such that $|\mu(F)| > 2\delta$. We can assume that H is complete. By the definition of $S'\mu$ and the equality $S\lambda = 0$, there is $\mathscr{D} \in \mathfrak{S}[F]$ such that

$$|\mu(F) - \mu(\mathscr{D})| < \delta \quad \text{and} \quad \|\lambda(\mathscr{D})\| < \gamma/2.$$

Hence for some $\mathscr{D}^1 \in \mathfrak{f}(\mathscr{D})$, denoting $\bigcup \mathscr{D}^1 = F^1$ and $E \setminus F^1 = E^1$, we have $|\mu(F \setminus F^1)| < \delta$ and $\|\lambda(F^1)\| < \gamma/2$. Consequently, $|\mu(F^1)| > \delta$ and $\|\lambda(E^1)\| > \varepsilon$, and similarly as in 3.2 this easily leads to a contradiction.

3.9. Corollary. *If G, H are complete, $\mu \in ea(\mathscr{R}; G)$, $\lambda \in ea(\mathscr{R}; H)$ and $\lambda \ll \mu$, then $S'\lambda \ll S'\mu$.*

3.10. Theorem. *If G, H are complete and $\mu \in ea(\mathscr{R}; G)$, $\lambda \in ea(\mathscr{R}; H)$, then*

$$\lambda \ll \mu \quad \text{iff} \quad S\lambda \ll S\mu \quad \text{and} \quad S'\lambda \ll S'\mu .$$

Proof. "If": Since $S\lambda \ll \mu$, $S'\lambda \ll \mu$, we have $\lambda \ll \mu$. "Only if" follows from 3.3 and 3.9.

3.11. Main Theorem. *Each $\mu \in ea(\mathscr{R}; G)$ can be uniquely written in the form*

$$\mu = \lambda + \nu,$$

where $\lambda, \nu \in ea(\mathscr{R}; G)$, λ is $\mathfrak{S}$-additive and ν is $\mathfrak{S}$-singular. These functions are namely $\lambda = S\mu$, $\nu = S'\mu$, both μ-continuous.

Proof. If λ, ν are as stated then $S\mu - \lambda = \nu - S'\mu$. Since $S\mu - \lambda$ is $\mathfrak{S}$-additive, $S\mu - \lambda = S(S\mu - \lambda) = S(\nu - S'\mu) = S\nu - SS'\mu = 0$. Hence $\lambda = S\mu$ and $\nu = S'\mu$.

3.12. Particular cases in the Theorem 3.11.

(a) $\mathfrak{S} = \mathfrak{S}_c \Rightarrow$ The Hewitt–Yosida decomposition: λ is σ-additive, ν is extremely non-σ-additive ($=$ purely finitely additive).

(b) $\mathfrak{S} = \mathfrak{S}(\eta) \Rightarrow$ The Lebesgue decomposition: λ is η-continuous, ν is η-singular, where η is a submeasure on $\mathscr{R}$. Here η can also be a quasi-normed group valued additive set function, for $\mathfrak{S}_c(\Gamma(\eta)) = \mathfrak{S}(\overline{\eta})$, $\overline{\eta}$ being the submeasure majorant for η with respect to the given quasi-norm.

It should be strongly marked that the functions λ and ν in 3.11 are not merely $\mathfrak{S}$-additive and $\mathfrak{S}$-singular, respectively, but, also *exhaustive*.

3.13. Remarks. 1) The Hewitt–Yosida and Lebesgue decompositions of additive exhaustive set functions with values in a Banach space, almost in the form stated above, were obtained by J. J. Uhl [13] (see also [5]). The only difference is in defining the notions of pure finite additivity and η-singularity: Uhl reduces them to scalar case via compositions of the given vector function with continuous linear functionals. This, however, in virtue of 3.6, is equivalent with our definitions of these notions. (While preparing this paper, the authors acquaintance with [13] (at that time in print) was based on the preprint of [5].)

As concerns the earlier related results of Brooks [1], they were yet unsatisfactory: his Theorem 1 (H–Y. decomposition) involves X^{**} and Theorem 2 (L. decomposition) is too restrictive, though its hypotheses correspond to those made in the classical scalar case ([6]; III. 4.14). The latter was generalized by Darst ([4]; Theorem 3.1) to scalar valued finitely additive set functions on an algebra of sets.

As was (implicitely) mentioned in the Introduction, the original definition of a purely finitely additive set function and the proof of the H–Y. decomposition were based on lattice properties of $ba(\mathscr{R}; \boldsymbol{R})$ ($= ea(\mathscr{R}; \boldsymbol{R})$) and $ca(\mathscr{R}; \boldsymbol{R})$ [8] (see also ([12]; § 17.3) and [14], IV. 2). Evidently, by 2.17 Remark 5, the definition we use here is equivalent with the original one if $G = \boldsymbol{R}$ (or $G = \boldsymbol{C}$).

If $\mu \in ba^+(\mathscr{R}; R)$ then the unique σ-additive part of μ can be defined by the formula $\lambda(E) = \inf\{\sum_{n=1}^{\infty} \mu(E_n) \colon (E_n) \in \mathfrak{S}_c[E]\}$, $E \in \mathscr{R}$. It is quite natural that the question of what a procedure could replace that of taking infimum if we had no order on G, quickly led to the observation that $\sum \mu(E_n)$ form a Cauchy net if we take more and more finer decompositions of E. This was just the origin of the present paper.

2) The results established in this section remain valid without the assumption that G or/and H are complete—it suffices to require the existence of $S\mu$ or/and $S\lambda$. Let us note also that it may happen for a function $\mu \in ea(\mathscr{R}; G)$, where G is *not* complete, that for some $\mathscr{D} \in \mathfrak{S}[E]$ the values $\mu(\mathscr{D})$ belong to the completion $\tilde{G}$ of G but not to G, and yet $S\mu(E)$, evaluated in $\tilde{G}$, is in G. Therefore the decomposition of μ into an $\mathfrak{S}$-additive part λ and an $\mathfrak{S}$-singular part ν, if exists, is unique, and both λ and ν are then μ-continuous.

3) Let $\mathscr{T}_1$, $\mathscr{T}_2$ be two Hausdorff topologies on G compatible with its group structure such that $\mathscr{T}_1 \subset \mathscr{T}_2$ and $(G, \mathscr{T}_2)$ is complete. Further, let $\mu \in ea(\mathscr{R}; (G, \mathscr{T}_2))$. Then, obviously, $\mu \in ea(\mathscr{R}; (G, \mathscr{T}_1))$, too. It is evident that the decompositions of μ exist under $\mathscr{T}_1$ and $\mathscr{T}_2$, and are identical. Consequently, if $\mu \colon \mathscr{R} \to (G, \mathscr{T}_1)$ is $\mathfrak{S}$-additive and $\mu \colon \mathscr{R} \to (G, \mathscr{T}_2)$ is exhaustive, then $\mu \colon \mathscr{R} \to (G, \mathscr{T}_2)$ is $\mathfrak{S}$-additive. (This follows also from 2.10.)

4) For each $t \in T$ let G_t be a complete topological group and let $\mu_t \in ea(\mathscr{R}; G_t)$. Then $\prod_{t\in T} \mu_t \colon \mathscr{R} \to \prod_{t\in T} G_t$ is additive and exhaustive. It is clear that $S(\prod_{t\in T} \mu_t) = \prod_{t\in T} (S\mu_t)$.

5) Suppose that $\mathfrak{S}_1$, $\mathfrak{S}_2$ are additivities on $\mathscr{R}$ and let S_1, S_2 be the corresponding operators in $ea(\mathscr{R}; G)$.

a) If $\mathfrak{S}_1 \subset \mathfrak{S}_2$ then $S_2\mu \ll S_1\mu$ and $S_1'\mu \ll S_2'\mu$ for each $\mu \in ea(\mathscr{R}; G)$. Indeed, since $\mathfrak{S}_2$-additivity implies $\mathfrak{S}_1$-additivity and $S_2\mu$ is $\mathfrak{S}_2$-additive, we have $S_2\mu \ll S_1\mu$ by 3.2. Similarly, $\mathfrak{S}_1$-singularity implies $\mathfrak{S}_2$-singularity and therefore $S_1'\mu \ll S_2'\mu$ by 3.7 and 3.8.

b) $\mathfrak{S}_1$, $\mathfrak{S}_2$ being arbitrary, the operators S_1, S_1', S_2, S_2' are pairwise commutative and $S_0 = S_1S_2$, where S_0 corresponds to the additivity $\mathfrak{S}_0 = \mathfrak{S}_1 \vee \mathfrak{S}_2$.

Let $\mu \in ea(\mathscr{R}; G)$. Then $\mu = S_1\mu + S_1'\mu$ and next $S_1S_2\mu = S_1S_2S_1\mu + S_1S_2S_1'\mu$. But $S_2S_1\mu$ is $\mathfrak{S}_1$-additive and $S_2S_1'\mu$ is $\mathfrak{S}_1$-singular, for $S_2S_1\mu \ll S_1\mu$ and $S_2S_1'\mu \ll S_1'\mu$, by 3.1 and 3.7. Hence $S_1S_2S_1\mu = S_2S_1\mu$ and $S_1S_2S_1'\mu = 0$. Consequently, $S_1S_2\mu = S_2S_1\mu$. The remaining cases are treated similarly. In order to prove that $S_0 = S_1S_2$ let us first observe that $\mathfrak{S}_0$-additivity of a function $\lambda \in ea(\mathscr{R} \colon G)$ is equivalent to its join $\mathfrak{S}_1$- and $\mathfrak{S}_2$-additivity. For, if λ is $\mathfrak{S}_1$- and $\mathfrak{S}_2$-additive, $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}_0[E]$,

then there are $\mathscr{D}_1 \in \mathfrak{S}_1[E]$, $\mathscr{D}_2 \in \mathfrak{S}_2[E]$ such that $\mathscr{D} = \mathscr{D}_1 \mathring{\cap} \mathscr{D}_2$, and we have

$$\lambda(E) = \sum_{D_1 \in \mathscr{D}_1} \lambda(D_1) = \sum_{D_1 \in \mathscr{D}_1} \sum_{D_2 \in \mathscr{D}_2} \lambda(D_1 \cap D_2) = \sum_{D \in \mathscr{D}} \lambda(D).$$

Thus, if $\lambda \in ea(\mathscr{R}; G)$ is $\mathfrak{S}_0$-additive then $S_0\lambda = S_1S_2\lambda = \lambda$. Hence, given $\mu \in ea(\mathscr{R}; G)$, we have $S_0\mu = S_1S_2S_0\mu = S_0S_1S_2\mu = S_1S_2\mu$.

3.14. Proposition. *Suppose that a family $(\mu_t \colon t \in T)$ of additive set functions from $\mathscr{R}$ to G is uniformly exhausting. Then for each $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, $\mu_t(\mathscr{D}) = \sum(\mu_t(D) \colon D \in \mathscr{D})$ and $\lim_{\mathscr{D} \in \mathfrak{S}[E]} \mu_t(\mathscr{D}) = S\mu_t(E)$ uniformly in t. Hence the family $(S\mu_t \colon t \in T)$ is uniformly exhausting and uniformly $\mathfrak{S}$-additive.*

3.15. Theorem. *Let $\mathscr{R}$ be a σ-ring and let $\mu_n \in ea(\mathscr{R}; G)$, $n = 1, 2, \ldots$ Suppose that for each $E \in \mathscr{R}$ there exists*

$$\mu_0(E) = \lim_n \mu_n(E).$$

Then the family $(\mu_n \colon n = 0, 1, 2, \ldots)$ is uniformly exhausting as well as $(S\mu_n \colon n = 0, 1, 2, \ldots)$, and

$$S\mu_0(E) = \lim S\mu_n(E),$$
$$S'\mu_0(E) = \lim S'\mu_n(E)$$

for each $E \in \mathscr{R}$.

Proof. The assertion follows from the Brooks–Jewett theorem and the preceding proposition (or consider the mapping $\mu = (\mu_1, \mu_2, \ldots, \mu_0) \colon \mathscr{R} \to c(G)$, where $c(G)$ is the group of convergent sequences of elements of G (together with their limits) equipped with the topology of uniform convergence on $\mathbf{N} \cup \{0\}$).

3.16. Corollary. *If $\mathscr{R}$ and μ_n, $n = 1, 2, \ldots$, are as in the Theorem above, and all μ_n are $\mathfrak{S}$-additive (resp. $\mathfrak{S}$-singular), then μ_0 is $\mathfrak{S}$-additive (resp. $\mathfrak{S}$-singular).*

3.17. **Final remark.** All results established in this section remain valid in a little more general situation when instead of exhaustivity (or uniform exhaustivity) of set functions under considerations one assumes (and suitable modifies assertions) that these properties hold only locally on $\mathscr{R}$, i.e. on each algebra $\mathscr{R}_E := \{F \colon F \subset E, F \in \mathscr{R}\}$, $E \in \mathscr{R}$. Then, moreover, a σ-ring in 3.15, 3.16 can be replaced by a δ-ring.

Thus, e.g., under these circumstances $S\mu$ is constructed in the following way: For each $E \in \mathscr{R}$ let $S_E\mu_E$ be the $\mathfrak{S}$-additive part of $\mu_E = \mu|\mathscr{R}_E$, and then put $S\mu(E) := S_E\mu_E(E)$.

4. Decompositions of exhaustive submeasures and FN-topologies.

Let $\mathscr{R}$ be a ring of sets, $\mathfrak{S}$ an additivity on $\mathscr{R}$, $es(\mathscr{R})$ the set of all exhaustive submeasures on $\mathscr{R}$. For each $\eta \in es(\mathscr{R})$ we define two submeasures $S\eta$ and

$S'\eta$ by means of the formulas:

$$S\eta(E) = \inf_{\mathscr{D}\in\mathfrak{S}[E]} \sup\{\eta(\bigcup\mathscr{D}')\colon \mathscr{D}'\in\mathfrak{f}(\mathscr{D})\} = \lim_{\mathscr{D}\in\mathfrak{S}[E]}\ \lim_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} \eta(\bigcup\mathscr{D}').$$

$$S'\eta(E) = \sup_{\mathscr{D}\in\mathfrak{S}[E]} \inf\{\eta(E\setminus\bigcup\mathscr{D}')\colon \mathscr{D}'\in\mathfrak{f}(\mathscr{D})\}, \qquad E\in\mathscr{R}.$$

4.1. Theorem. *Let $\eta\in es(\mathscr{R})$. Then $S\eta,\ S'\eta\leqslant\eta$, hence the submeasures $S\eta$ and $S'\eta$ are η-continuous and exhaustive. Moreover:*

(a) *The submeasure $S\eta$ is $\mathfrak{S}$-continuous and if λ is an $\mathfrak{S}$-continuous submeasure on $\mathscr{R}$ such that $\lambda\ll\eta$, then $\lambda\ll S\eta$.*

(b) *The submeasure $S'\eta$ is $\mathfrak{S}$-singular and if ν is an $\mathfrak{S}$-singular submeasure on $\mathscr{R}$ that $\nu\ll\eta$, then $\nu\ll S'\eta$.*

(c) *$\eta\sim S\eta+S'\eta\ (\sim S\eta\vee S'\eta)$ and if λ is an $\mathfrak{S}$-continuous and ν an $\mathfrak{S}$-singular submeasure on $\mathscr{R}$ such that $\eta\sim\lambda+\nu$, then $\lambda\sim S\eta$, $\nu\sim S'\eta$.*

Proof. (a) We have to prove that if $E\in\mathscr{R}$, $\mathscr{D}\in\mathfrak{S}[E]$, then

$$\lim_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} S\eta(E\setminus\bigcup\mathscr{D}') = 0.$$

Suppose, on the contrary, that

$$\inf_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} S\eta(E\setminus\bigcup\mathscr{D}') > \varepsilon > 0.$$

Since $S\eta(E)>\varepsilon$ and $\mathscr{D}\in\mathfrak{S}[E]$, there exists $\mathscr{D}_1\in\mathfrak{f}(\mathscr{D})$ such that $\eta(\bigcup\mathscr{D}_1)>\varepsilon$. But $S\eta(E\setminus\bigcup\mathscr{D}_1)>\varepsilon$ and $\mathscr{D}\setminus\mathscr{D}_1\in\mathfrak{S}[E\setminus\bigcup\mathscr{D}_1]$, so we can find $\mathscr{D}_2\in\mathfrak{f}(\mathscr{D}\setminus\mathscr{D}_1)$ such that $\eta(\bigcup\mathscr{D}_2)>\varepsilon$. Now $S\eta(E\setminus\bigcup(\mathscr{D}_1\cup\mathscr{D}_2))>\varepsilon$, etc. Continuing in this manner we obtain a contradiction with the assumption that η is exhaustive.

We proceed to the proof of the second assertion in (a). Let us take an arbitrary $\varepsilon>0$ and choose $\delta>0$ such that

$$(+)\qquad \eta(E)<\delta \Rightarrow \lambda(E)\leqslant\varepsilon.$$

We claim that

$$S\eta(E)<\delta \Rightarrow \lambda(E)\leqslant\varepsilon, \qquad E\in\mathscr{R}.$$

Otherwise there is $E\in\mathscr{R}$ with $S\eta(E)<\delta$ and $\lambda(E)\geqslant\varepsilon+\gamma$, $\gamma>0$. Let $\mathscr{D}\in\mathfrak{S}[E]$ be such that

$$\sup_{\mathscr{D}'\in\mathfrak{f}(\mathscr{D})} \eta(\bigcup\mathscr{D}')<\delta.$$

Since λ is $\mathfrak{S}$-continuous, we can find $\mathscr{D}'\in\mathfrak{f}(\mathscr{D})$ for which

$$\lambda(E\setminus\bigcup\mathscr{D}')<\gamma/2.$$

Hence $\lambda(\bigcup\mathscr{D}')>\varepsilon+\gamma/2$ and $\eta(\bigcup\mathscr{D}')<\delta$, a contradiction with $(+)$.

(b) Let λ be $\mathfrak{S}$-continuous and $\lambda\ll S'\eta$. Suppose that for some $E\in\mathscr{R}$ we have $\lambda(E)>\varepsilon>0$. Let $\delta>0$ be such that $\lambda(F)\leqslant\varepsilon$ whenever $S'\eta(F)\leqslant\delta$. Thus $S'\eta(E)>\delta$, and there exists $\mathscr{D}\in\mathfrak{S}[E]$ such that $\eta(E\setminus\bigcup\mathscr{D}')>\delta$ for each $\mathscr{D}'\in\mathfrak{f}(\mathscr{D})$. Since λ is $\mathfrak{S}$-continuous, we can find $\mathscr{D}'\in\mathfrak{f}(\mathscr{D})$ with $\lambda(\bigcup\mathscr{D}')>\varepsilon$. Hence $S'\eta(\bigcup\mathscr{D}')>\delta$. Let us denote $E_1=\bigcup\mathscr{D}'$, $A_1=E\setminus\bigcup\mathscr{D}'$. We have $\eta(A_1)>\delta$, $S'\eta(E_1)>\delta$, $\lambda(E_1)>\varepsilon$, and $E_1\cap A_1=\emptyset$. Applying the same argument to the set E_1 we shall find a set $E_2\subset E_1$ such that $\eta(A_2)>\delta$, $S'\eta(E_2)>\delta$, $\lambda(E_2)>\varepsilon$, where $A_2=E_1\setminus E_2$, etc. Thus we see that there exists a disjoint sequence of sets $A_n\in\mathscr{R}$ such that $\eta(A_n)>\delta$, $n=1,2,\ldots$ This however is impossible, for η is exhaustive.

We are going now to prove the second statement. Given $\varepsilon>0$, let $\delta>0$ be such that

$$\eta(E)\leqslant\delta \Rightarrow \nu(E)\leqslant\varepsilon.$$

Suppose that for some $E\in\mathscr{R}$ we have $S'\eta(E)<\delta$ and $\nu(E)\geqslant\varepsilon+\gamma$, where $0<\gamma<\infty$. Since $S\nu=0$, there is $\mathscr{D}\in\mathfrak{S}[E]$ with

$$\sup\{\nu(\bigcup\mathscr{D}')\colon \mathscr{D}'\in\mathfrak{f}(\mathscr{D})\}<\gamma/2, \qquad \inf\{\eta(E\setminus\bigcup\mathscr{D}')\colon \mathscr{D}'\in\mathfrak{f}(\mathscr{D})\}<\delta.$$

Hence for some $\mathscr{D}'\in\mathfrak{f}(\mathscr{D})$ we have $\nu(F)<\gamma/2$, $\eta(E\setminus F)<\delta$, where $F=\bigcup\mathscr{D}'$. Therefore $\nu(E\setminus F)>\varepsilon+\gamma/2$ and $\eta(E\setminus F)<\delta$, a contradiction.

(c) We know already that $S\eta$ is $\mathfrak{S}$-continuous and $S'\eta$ is $\mathfrak{S}$-singular. Since $\frac{1}{2}(S\eta+S'\eta)\leqslant\eta\leqslant S\eta+S'\eta$, we have also $S\eta+S'\eta\sim\eta$. Let λ, ν be as stated in (c). Since $\lambda\ll\eta$ and $\nu\ll\eta$, we have $\lambda\ll S\eta$ and $\nu\ll S'\eta$, by (a) and (b). On the other hand $S\eta$, $S'\eta\ll\lambda+\nu$ imply $S\eta\ll S(\lambda+\nu)=S\lambda+S\nu=S\lambda$ and $S'\eta\ll S'(\lambda+\nu)=S'\lambda+S'\nu=S'\nu$. Therefore $\lambda\sim S\eta$, $\nu\sim S\eta$.

The next result characterizes $S\eta$ and $S'\eta$ in terms of order relations $\leqslant$ among submeasures.

4.2. Proposition. *If $\eta\in es(\mathscr{R})$ then*

(a) *$S\eta=\bigvee\{\lambda\colon \lambda$ is an $\mathfrak{S}$-continuous submeasure on $\mathscr{R}$ and $\lambda\leqslant\eta\}$,*

(b) *$S'\eta=\bigvee\{\lambda\colon \lambda$ is $\mathfrak{S}$-singular submeasure on $\mathscr{R}$ and $\lambda\leqslant\eta\}$.*

Proof. Suppose that a submeasure λ is $\mathfrak{S}$-continuous and $\lambda\leqslant\eta$. Let $E\in\mathscr{R}$, $\mathscr{D}\in\mathfrak{S}[E]$. If $\mathscr{D}'\in\mathfrak{f}(\mathscr{D})$ then $\lambda(\bigcup\mathscr{D}')\leqslant\eta(\bigcup\mathscr{D}')$. Since λ is $\mathfrak{S}$-continuous, we have $\lambda(E)\leqslant\sup\{\eta(\bigcup\mathscr{D}')\colon \mathscr{D}'\in\mathfrak{f}(\mathscr{D})\}$. Hence $\lambda(E)\leqslant S\eta(E)$, and (a) follows from the fact that $S\eta$ itself is $\mathfrak{S}$-continuous and $\leqslant\eta$. The proof of (b) is quite similar.

4.3. Special cases in 4.1. Let $\eta\in es(\mathscr{R})$.

1) The Hewitt–Yosida decomposition of η: Putting $\mathfrak{S}=\mathfrak{S}_c$ we see that $S\eta$ is an order continuous submeasure and $S'\eta$ a "purely exhaustive" submeasure, i.e., if λ is an order continuous submeasure on $\mathscr{R}$ such that $\lambda\leqslant S'\eta$ (or even $\ll S'\eta$) then $\lambda=0$.

2) The LEBESGUE DECOMPOSITION of η: Let μ be an arbitrary submeasure on $\mathscr{R}$ and let $\mathfrak{S} = \mathfrak{S}(\mu)$. Then 4.1 gives us a decomposition of η into a μ-continuous part $S\eta$ and a μ-singular part $S'\eta$.

If μ, ν are submeasures on $\mathscr{R}$ then we write $\nu \perp \mu$ to denote that ν is μ-singular, i.e., $\nu \wedge \mu = 0$, by the final remarks in Section 2. Thus $\nu \perp \mu$ iff $\mu \perp \nu$. From the formula for $\nu \wedge \mu$ it follows that $\nu \perp \mu$ iff for each $E \in \mathscr{R}$ and $\varepsilon > 0$ there exists a set $F \subset E$ such that $\mu(F) < \varepsilon$ and $\nu(E \setminus F) < \varepsilon$. This characterization of μ-singularity can be simplified if some additional assumptions on $\mathscr{R}$, μ, ν are made.

4.4. PROPOSITION. (a) *If $\mathscr{R}$ is a field or if $\nu \in es(\mathscr{R})$ then $\nu \perp \mu$ iff for each $\varepsilon > 0$ there is a set $F_\varepsilon \in \mathscr{R}$ such that $\mu(F_\varepsilon) < \varepsilon$ and $\nu(A) < \varepsilon$ whenever $A \cap F_\varepsilon = \emptyset$.*

(b) *If $\mathscr{R}$ is a σ-field and ν is σ-subadditive, or if $\mathscr{R}$ is a σ-ring and ν is order continuous, then $\nu \perp \mu$ iff for each $\varepsilon > 0$ there is a set $E_\varepsilon \in \mathscr{R}$ such that $\mu(E_\varepsilon) = 0$ and $\nu(A) < \varepsilon$ whenever $A \cap E_\varepsilon = \emptyset$.*

(c) *If $\mathscr{R}$ is a σ-field, ν is σ-subadditive and $\mathscr{N}_\mu = \{E \in \mathscr{R}: \mu(E) = 0\}$ is a σ-ideal in $\mathscr{R}$ (or vice versa), or if $\mathscr{R}$ is a σ-ring, ν is order continuous and $\mathscr{N}_\mu$ is a σ-ideal in $\mathscr{R}$, then $\nu \perp \mu$ iff there exists a set $E_0 \in \mathscr{R}$ such that $\mu(E_0) = 0$ and $\nu(A) = 0$ whenever $A \cap E_0 = \emptyset$.*

Proof. The "if" parts of (a)–(c) are obvious. We shall therefore assume in what follows that $\nu \perp \mu$. (a) In the case $\mathscr{R}$ is a field it is nothing to prove. So let $\nu \in es(\mathscr{R})$, and take $\varepsilon > 0$. By using the method of exhaustion (cf. [7], II; 4.7) we can find a set $E \in \mathscr{R}$ such that $\nu(A) < \varepsilon/2$ if $A \cap E = \emptyset$. Then any set $F_\varepsilon \subset E$ with $\mu(F_\varepsilon) < \varepsilon/2$ and $\nu(E \setminus F_\varepsilon) < \varepsilon/2$ has the required property.

(b) It suffices to put $E_\varepsilon = \bigcap_{n=1}^{\infty} F_{\varepsilon 2^{-n}}$. (If $\mathscr{R}$ is a σ-ring and ν is order continuous then ν is σ-subadditive and exhaustive ([6], II).

(c) The set $E_0 = \bigcup_{n=1}^{\infty} E_{1/n}$ has the required property.

4.5 Remarks. 1) Suppose $\mathfrak{S}$ has the following property:
(*) If $(E, \mathscr{D}) \in \mathfrak{S}$, $D_n \in \mathscr{D}$ $(n = 1, 2, \ldots)$ and $1 = m_0 < m_1 < \ldots$ then $\mathscr{D}^* = (\mathscr{D} \setminus \{D_n: n \in \mathbf{N}\}) \cup \{D_{m_k} \cup \ldots \cup D_{m_{k+1}-1}: k = 0, 1, \ldots\} \in \mathfrak{S}[E]$.
(For example: $\mathfrak{S}_c$, $\mathfrak{S}(\mu)$.) Let $\eta \in es(\mathscr{R})$ and let $S\eta$ be the submeasure defined as at the beginning of this section. Then

$$(+) \qquad S\eta(E) = \inf\Big\{\sum_{D \in \mathscr{D}} \eta(D): \mathscr{D} \in \mathfrak{S}[E]\Big\}, \quad E \in \mathscr{R}.$$

In order to prove this equality, let $\lambda(E)$ denote the value of its right side. It is obvious that $S\eta(E) \leqslant \lambda(E)$. Conversely, if $\mathscr{D} \in \mathfrak{S}[E]$ then the class $\{D \in \mathscr{D}: \eta(D) > 0\}$ is at most countable, for η is exhaustive; let $D_1, D_2, \ldots$ be an enumeration of this class. Given $\varepsilon > 0$, let a sequence

$1 = m_0 < m_1 < \ldots$ be such that $\eta(D_k^*) < \varepsilon/2^k$ $(k \geqslant 1)$, where $D_k^* = D_{m_k} \cup \ldots \cup D_{m_{k+1}-1}$ for $k \geqslant 0$ (cf. [7], II; 4.1). Define $\mathscr{D}^*$ as in (*) above. Then

$$\sum_{D^* \in \mathscr{D}^*} \eta(D^*) \leqslant \eta(D_0^*) + \varepsilon \leqslant \varepsilon + \sup\{\eta(\bigcup \mathscr{D}'): \mathscr{D}' \in \mathfrak{f}(\mathscr{D})\}.$$

Therefore $\lambda(E) \leqslant S\eta(E)$.

2) It is clear that the equality (+) holds always if η is additive. Then also $S\eta$ and $S'\eta$ are additive and $\eta = S\eta + S'\eta$. Hence if $\eta \in ca(\mathscr{R}; \mathbf{R})$, $\eta \geqslant 0$, then the decomposition $\eta = S\eta + S'\eta$ is identical with that in Theorem 3.11. In this case it can be supposed in 4.2 (a) that λs are additive, and thus we obtain a definition of $S\eta$ of the kind used in [8]. Then for arbitrary $\eta \in ca(\mathscr{R}; \mathbf{R})$, $S\eta$ could be defined as $S\eta^+ - S\eta^-$, where $\eta^+(E) = (\eta \vee 0)(E) = \sup\{\eta(F): F \subset E, F \in \mathscr{R}\}$, $\eta^- = (-\eta)^+$.

3) If $\eta \in es(\mathscr{R})$ then the decomposition $\eta \sim S\eta + S'\eta$ with respect to $\mathfrak{S}$ is identical with *the* Lebesgue decomposition (4.3 (b)) of η with respect to $\mu = S\eta$. In fact, since $S\eta \ll \mu$ and $S'\eta \perp \mu$, $S\eta + S'\eta$ is *a* Lebesgue decomposition of η with respect to μ. Let S_μ, S'_μ be the operators on $es(\mathscr{R})$ associated with $\mathfrak{S}(\mu)$. In virtue of 4.1 (c), applied to $\mathfrak{S} = \mathfrak{S}(\mu)$, we have $S\eta \sim S_\mu\eta$ and $S'\eta \sim S'_\mu\eta$. Hence $S_\mu\eta$ is $\mathfrak{S}$-continuous and $S'_\mu\eta$ is $\mathfrak{S}$-singular. In view of 4.2 it must be $S_\mu\eta = S\eta$, $S'_\mu\eta = S'\eta$. (A similar remark applies if $\eta \in ca(\mathscr{R}; G)$, where G is a complete normed abelian group $(\mu = S\eta)$.)

4) If $\eta \in es(\mathscr{R})$ or $\eta \in ca(\mathscr{R}; G)$, where G is as above, and $\eta \sim \lambda + \nu$ (resp. $\eta = \lambda + \nu$, $\lambda, \nu \in ca(\mathscr{R}; G)$) with ν being λ-singular (resp. $\bar{\lambda}$-singular) then $\lambda + \nu$ is a (resp. the) Lebesgue decomposition of η with respect to the submeasure λ (resp. $\bar{\lambda}$). In particular, if $\eta \in ca(\mathscr{R}; \mathbf{R})$ $(= \mathrm{ba}(\mathscr{R}; \mathbf{R}))$ then the Jordan decomposition $\eta = \eta^+ + (-\eta^+)$ is the Lebesgue decomposition of η with respect to η^+. Therefore, if $\mathscr{R}$ is a σ-ring and $\eta \in ca(\mathscr{R}; \mathbf{R})$, we derive from 4.4 (c) the existence of a set $E_0 \in \mathscr{R}$ such that $\eta^+(E_0) = 0$ and $\eta^-(A) = 0$ whenever $A \cap E_0 = \emptyset$, i.e., the Hahn decomposition of η.

A decomposition of a submeasure η into an $\mathfrak{S}$-continuous part λ and an $\mathfrak{S}$-singular part ν is not unique; precisely, it is unique "up to equivalent submeasures". But $\Gamma(\eta) = \Gamma(\lambda) \vee \Gamma(\nu)$, and it is easy to see that the topologies on the right side are determined uniquely. This observation is generalized in the following theorem.

4.6 THEOREM. *Let Γ be an exhaustive FN-topology on $\mathscr{R}$. Then there exists a unique pair Γ_1, Γ_2 of FN-topologies on $\mathscr{R}$ such that Γ_1 is $\mathfrak{S}$-continuous, Γ_2 is $\mathfrak{S}$-singular and $\Gamma = \Gamma_1 \vee \Gamma_2$.*

Proof. Let $(\eta_t: t \in T)$ be the family of all Γ-continuous submeasures on $\mathscr{R}$. Then $\Gamma = \Gamma(\eta_t: t \in T)$ and each η_t is exhaustive. Put $\Gamma_1 = \Gamma(S\eta_t: t \in T)$, $\Gamma_2 = \Gamma(S'\eta_t: t \in T)$. It is clear that Γ_1 is $\mathfrak{S}$-continuous and Γ_2 is $\mathfrak{S}$-singular. Since $\Gamma_1, \Gamma_2 \subset \Gamma$ we have $\Gamma_1 \vee \Gamma_2 = \Gamma(S\eta_s + S'\eta_t: (s, t)$

$\in T\times T) \subset \Gamma$. But for each $t \in T$ we have $\eta_t \sim S\eta_t + S'\eta_t$, so $\Gamma \subset \Gamma_1 \vee \Gamma_2$. On the other hand, if $\Gamma = \Gamma^1 \vee \Gamma^2$ where Γ^1 is $\mathfrak{S}$-continuous and Γ^2 is $\mathfrak{S}$-singular, then each Γ^i-continuous submeasure must be Γ_i-continuous. Consequently, $\Gamma^i \subset \Gamma_i$, $i = 1, 2$. Conversely, let $\eta \ll \Gamma_i$. Then there exist sequences (λ_n), (ν_n) of submeasures Γ^1- and Γ^2-continuous, respectively, each of them bounded by 1, such that $\eta \ll \gamma = \sum_{n=1}^{\infty} 2^{-n}(\lambda_n + \nu_n)$ (see [7], I; 2.8). Since $S\gamma = \sum 2^{-n}\lambda_n \ll \Gamma^1$, $S'\gamma = \sum 2^{-n}\nu_n \ll \Gamma^2$, we have $\eta \ll S\gamma \ll \Gamma^1$ (if $i = 1$) or $\eta \ll S'\gamma \ll \Gamma^2$ (if $i = 2$). Hence $\Gamma_i \subset \Gamma^i$, $i = 1, 2$.

4.7. THEOREM. *Suppose that G is a complete topological abelian group and let $\mu \in ea(\mathscr{R}; G)$. Let $|\cdot|$ be a continuous quasi-norm on G and $\overline{\mu}$ the corresponding to it submeasure majorant for μ. Then:*

(a) $$\overline{S\mu} = S\overline{\mu}.$$

(b) $$\overline{S'\mu} = S'\overline{\mu}.$$

(c) *$\Gamma_1 = \Gamma(S\mu)$ and $\Gamma_2 = \Gamma(S'\mu)$ form the decomposition of $\Gamma = \Gamma(\mu)$ described in* 4.6.

Proof. (a) Let $(E, \mathscr{D}) \in \mathfrak{S}$. If $\mathscr{D}' \in \mathfrak{f}(\mathscr{D})$ then $|\mu(F \cap \bigcup \mathscr{D}')| \leqslant \overline{\mu}(\bigcup \mathscr{D}')$, hence

$$|\mu(F \mathring{\cap} \mathscr{D})| \leqslant \sup\{\overline{\mu}(\bigcup \mathscr{D}'):\ \mathscr{D}' \in \mathfrak{f}(\mathscr{D})\}$$

for each $F \subset E$, $F \subset \mathscr{R}$. Since the family $(\mu_F\colon F \subset E, F \in \mathscr{R})$, where $\mu_F(A) = \mu(A \cap F)$, $A \in \mathscr{R}$, is uniformly exhaustive, we have

$$\lim_{\mathscr{D} \in \mathfrak{S}[E]} |\mu(F \mathring{\cap} \mathscr{D})| = |S\mu(F)|$$

uniformly for $F \subset E$. Therefore

$$|S\mu(F)| \leqslant S\overline{\mu}(E), \qquad F \subset E$$

and

(+) $$\overline{S\mu}(E) \leqslant S\overline{\mu}(E).$$

If $\overline{S\mu}(E) < \alpha'' < \alpha' < \alpha \leqslant S\overline{\mu}(E)$ then $|S\mu(F)| < \alpha''$ for each $F \subset E$. Therefore we can find $\mathscr{D}_0 \in \mathfrak{S}[E]$ such that $\mathscr{D}_0 \leqslant \mathscr{D} \in \mathfrak{S}[E]$ implies $|\mu(F \mathring{\cap} \mathscr{D})| < \alpha'$ for each $F \subset E$. Let us fix $\mathscr{D} \in \mathfrak{S}[E]$ with $\mathscr{D}_0 \leqslant \mathscr{D}$. There exists $\mathscr{D}^* \in \mathfrak{f}(\mathscr{D})$ such that $\mathscr{D}^* \subset \mathscr{D}' \in \mathfrak{f}(\mathscr{D})$ implies $|\mu(F \cap \bigcup \mathscr{D}')| < \alpha$, $F \subset E$. Hence $\overline{\mu}(\bigcup \mathscr{D}') \leqslant \alpha$. In a consequence $S\overline{\mu}(E) \leqslant \alpha$. Thus we see that the strict inequality in (+) is impossible.

The proof of (b) is similar.

(c) $\Gamma(S\mu)$ is $\mathfrak{S}$-continuous by 2.6, $\Gamma(S'\mu)$ is $\mathfrak{S}$-singular by definition 2.17, so we have only to show that $\Gamma(\mu) = \Gamma(S\mu) \vee \Gamma(S'\mu)$. This, however, is easily derived from (a) and (b).

4.8. Remark. Let us recall that we have denoted by $\Gamma(\mathfrak{S})$ the strongest $\mathfrak{S}$-continuous FN-topology on $\mathscr{R}$. It is evident that the Theorem 4.6 can be formulated as follows: If Γ is an exhaustive FN-topology on $\mathscr{R}$ then there exists a unique pair Γ_1, Γ_2 of FN-topologies on $\mathscr{R}$ such that $\Gamma_1 \subset \Gamma(\mathfrak{S})$, $\Gamma_2 \wedge \Gamma(\mathfrak{S}) = 0$ and $\Gamma = \Gamma_1 \vee \Gamma_2$. If η is a submeasure on $\mathscr{R}$ then $\Gamma(\eta) = \Gamma(\mathfrak{S}(\eta))$ by 2.9. It follows that every exhaustive FN-topology Γ can be uniquely decomposed into the "η-continuous part" Γ_1 $(\Gamma_1 \subset \Gamma(\eta))$ and the "η-singular part" Γ_2 $(\Gamma_2 \wedge \Gamma(\eta) = 0)$. Obviously this is an analogon of Lebesgue decomposition. Let us note however that the following question concerning the general Lebesgue decomposition is still open: Suppose that Γ_0 is an arbitrary FN-topology on $\mathscr{R}$ and let $\mu \in ea(\mathscr{R}; G)$ (let Γ be an exhaustive FN-topology on $\mathscr{R}$). Is μ (resp. Γ) decomposable into a Γ_0-continuous and a Γ_0-singular parts?

We know only, by the results of this paper, that it is the case if $\Gamma_0 = \Gamma(\mathfrak{S})$, in particular if $\Gamma = \Gamma(\eta)$.

Added in proof (3. 6. 1973). Decompositions of exhaustive additive set functions are considered also in the recent papers of Tim Traynor: *Decomposition of group-valued additive set functions*, Ann. Inst. Fourier, Grenoble, 22 (1972), pp. 131–140 and *A general Hewitt-Yosida decomposition*, Canad. J. Math., 24 (1972), pp. 1164–1169.

By author's overlook in Remarks 3.13 the (necessary!) reference to [11] is omitted.

INSTYTUT OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY POZNAŃ

References

[1] J. K. Brooks, *Decomposition theorems for vector measures*, Proc. Amer. Math. Soc., 21 (1969), pp. 27–29.

[2] — R. S. Jewett, *On finitely additive vector measures*, Proc. Nat. Acad. Sci. (USA), 67 (1970), pp. 1294–1298.

[3] N. Bourbaki, *Topologie générale*, ch. 3, Paris 1960.

[4] R. B. Darst, *A decomposition for complete normed abelian groups with applications to spaces of additive set functions*, Trans. Amer. Math. Soc., 103 (1962), pp. 549–558.

[5] J. Diestel, *Applications of weak compactness and bases to vector measures and vectorial integration*, Rev. Roumaine Math. Pures Appl., 18 (1973), pp. 211–224.

[6] N. Dunford, J. T. Schwartz, *Linear Operators*, Part I, New York 1958.

[7] L. Drewnowski, *Topological rings of sets, continuous set functions, integration* I, II, III, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 20 (1972), pp. 269–276, 277–286, 439–445.

[8] E. Hewitt, K. Yosida, *Finitely additive measures*, Trans. Amer. Math. Soc., 72 (1952), pp. 46–66.

[9] I. Labuda, *Sur quelques généralisations des théorèmes de Nikodym et de Vitali–Hahn–Saks*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., 20 (1972), pp. 447–456.

[10] W. Orlicz, *Absolute continuity of vector-valued finitely additive set functions*. I, Studia Math. 30 (1968), pp. 121–133.

[11] C. E. Rickart, *Decomposition of additive set functions*, Duke Math. J., 10 (1943), pp. 653-665.
[12] Z. Semadeni, *Banach Spaces of Continuous Functions*, Warszawa 1971.
[13] J. J. Uhl, *Extensions and decompositions of vector measures*, Proc. London Math. Soc., (2), 3 (1971), pp. 672-676.
[14] D. A. Vladimirov, *Boolean Algebras* (in Russian), Moskva 1969.

Received May 12, 1972 (527)

STUDIA MATHEMATICA T. XLVIII. (1973)

Topological aspects of q-regular measures

by

RICHARD A. ALÒ (Pittsburgh, Penn.), ANDRÉ DE KORVIN (Terre Haute, Ind.) and LAURENCE E. KUNES (Terre Haute, Ind.)

Abstract. Let $L(E, F)$ be the set of bounded linear operators from the Banach space E to the Banach space F. If m is a measure defined on a ring $\mathscr{C}$ of subsets of T with values in $L(E, F)$, for each y^* in the dual F^*, one defines a mesaures m_{y^*} from $\mathscr{C}$ into E^*. Also for each A in $\mathscr{C}$ one may define a semi-norm $p_{m,A}$ on F^* in terms of the q-variation of m_{y^*}. Topologies are defined on the unit sphere δ^* of F^* utilizing these semi-norms. We then investigate the relationships of these topologies to the properties of the measures. We consider when the topologies are Hausdorff and when they are compact. We then consider operators on $\mathscr{L}^p_E(\mu)$ $(1 < p \leqslant \infty)$ using the above topologies. For example, if U is a continuous operator from $\mathscr{L}^p_E(\mu)$ into F and if U is absolutely continuous with respect to μ then U is compact if and only if the associated topology makes δ^* compact. Additional results for continuous and compact operators U which are absolutely continuous with respect to μ are obtained.

1. Introduction. The recent definitive work by W. Orlicz in [6] generates additional interest in the relationship of topologies placed on the unit sphere σ^* of a dual space F^* to the measure theoretic properties. In particular, in [4] and [5] a topology associated with a measure is defined as follows.

Let $L(E, F)$ be the set of bounded linear operators from the Banach space E into the Banach space F and let $\mathscr{C}$ be a ring of subsets of a non empty set T. If m is a measure defined on $\mathscr{C}$ with values in $L(E, F)$, then for each A in $\mathscr{C}$ a semi-norm $p_{m,A}$ is defined on the dual F^* of F by

$$p_{m,A}(y^*) = \overline{m_{y^*}}(A)$$

where $\overline{m_{y^*}}$ denotes the variation of the measure m_{y^*} that maps $\mathscr{C}$ into the dual E^* and is defined by

$$m_{y^*}(A) = \langle m(A), y^*\rangle.$$

The collection P of all such semi-norms for A in $\mathscr{C}$ generates a topology in the usual way. This topology when restricted to σ^*, the unit sphere of F^*, turns out to be of interest. Also of interest is the topology generated by $p_{m,A}$ for A in $\mathscr{C}$ where m is now an element in the set $r(E, F)$ of finitely additive set functions from $\mathscr{C}$ into $L(E, F)$. Among the numerous results contained in [4] and [5] one main property seems to be central.