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Observability for the one-dimensional heat equation

by

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Abstract. Let u be a solution of the heat equation on $S\times[0,T]$. On the closed linear span of $\{u(\vartheta,\cdot)\}$ in a linear topological space we define the linear mapping: $A_{\vartheta,T}\colon u(\vartheta,\cdot)\mapsto u(\cdot,T)$. In dependence on topologies in the domain and in the range we examine measure and metric properties of ϑs, for which $A_{\vartheta,T}$ is well-defined and bounded. Some related questions and an optimization problem are also concerned

1. Introduction. Consider a bounded open set S in R^m with piecewise smooth boundary ∂S. Let u be a solution of the heat equation

$$\frac{\partial u}{\partial t}=\Delta u,\qquad x\in S;\ t>0 \tag{1}$$

with the homogeneous boundary conditions:

$$u(x,t)+\alpha(x)u_n(x,t)=0,\qquad t>0 \tag{2}$$

at all the points of ∂S where the outward normal derivative u_n is defined. Depending on topologies in the domain and in the range, we ask whether the mapping

$$A_{\Xi,T}:\ u|_{\Xi\times(0,T)}\mapsto u(\cdot,T) \tag{3}$$

is well-defined and continuous ($\Xi\subset S$).

In [4] V. J. Mizel and T. I. Seidman posed the following question: "Does the temperature variation at an end of an insulated rod determine continuously the temperature distribution along the rod?"

Assuming that the domain and the range are L^2-spaces and that $S=(0,1)$, $u_x(0,t)=u_x(1,t)=0$, they showed that there exists a T_0 such that for all $T>T_0$ the operator $A_{0,T}$ is well-defined and bounded.

W. A. J. Luxemburg and J. Korevaar proved ([2]) that the same operator is well-defined and bounded for all $T>0$ and for any pair of L_p $(1\leqslant p<\infty)$ or C spaces.

The Mizel–Seidman paper [5] presents generalizations to the multidimensional case, where S is an insulated m-ball. They proved that $A_{\partial S,T}$ is compact for any $T>0$ provided that the domain is L_p $(2\leqslant p\leqslant\infty)$

and the range is L_r $(1 \leqslant r \leqslant \infty)$ and that $u_n|\partial_S = 0$. The present paper considers the continuity of the operators $A_{\vartheta,T}(A_{\Xi,T})$, where $\vartheta \in [0,1]$ $(\Xi \subset [0,1])$. It gives the formula for the extreme time of observation, i.e. for the number T_0 such that $A_{\vartheta,T}$ is well-defined and bounded for $T > T_0$ (for any pair of L_p $(1 \leqslant p < \infty)$ or C spaces) and is not bounded for $T < T_0$.

It shows that the Lebesgue measure of ϑs, for which $T_0 = 0$, is one; however, the other ϑs form a dense subset of $[0,1]$. Solutions of the very problem for other sets S and of several miscellaneous questions are also presented. The last section is devoted to an optimal observation problem. It is worth underlining that many proofs are partially based on the diophantine approximation theory.

2. Continuity criteria. To begin with, we examine the case of $S = (0,1)$. The condition (2) takes the form

$$u(0,t) = \alpha u_x(0,t), \qquad u(1,t) = \beta u_x(1,t) \tag{2'}$$

$0 \leqslant \alpha, -\beta \leqslant \infty$; $u = \infty u_x$ means that $u_x = 0$. A solution of (1)–(2′) can be expressed by an absolutely convergent series:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) \exp(-\lambda_n t) \tag{4}$$

where, respectively, λ_n are the eigenvalues and ψ_n the eigenfunctions such that $\|\psi_n\|_C = 1$ of $-\dfrac{d^2}{dx^2}$ with the conditions (2′). Let X denote $L_p(0,T)$ $(1 \leqslant p < \infty)$ or $C(0,T)$ and let Y denote $L_q[0,1]$ $(1 \leqslant q < \infty)$ or $C[0,1]$; thus $A_{\vartheta,T}$ (see (3)) is the mapping from a subset of X into Y.

THEOREM 1. (a) *If the series* $\sum_{n=1}^{\infty} \dfrac{\exp(-\lambda_n T)}{|\psi_n(\vartheta)|}$ *is convergent, the operator $A_{\vartheta,T'}$ is well-defined and bounded for $T' > T$ for any $\langle X, Y\rangle$.*

(b) *If this series is divergent, $A_{\vartheta,T'}$ is not bounded for $T' < T$ and for any $\langle X, Y\rangle$.*

If there is an n such that $\psi_n(\vartheta) = 0$, then $A_{\vartheta,T}$ is not well-defined and we put $T_0 = \infty$; if for all n $\psi_n(\vartheta) \neq 0$, then the Cauchy–Hadamard criterion for Dirichlet series (see S. Rolewicz [6], page 235) and Theorem 1 give:

COROLLARY 1. *The time of observation T_0 is given by*

$$T_0(\vartheta) = -\liminf \frac{\log|\psi_n(\vartheta)|}{\lambda_n}. \tag{5}$$

To prove Theorem 1 we shall need a lemma based on the considerations of [2].

Let $\{\lambda_n\}$ be a sequence of complex numbers satisfying the conditions

$$\sum \frac{1}{|\lambda_n|} < \infty. \tag{6}$$

There are ϱ and δ such that

$$|\lambda_n - \lambda_m| > \varrho|n-m|, \qquad m, n = 1, 2, \ldots, \tag{7}$$

$$\operatorname{Re}\lambda_n > \delta|\lambda_n| \qquad \text{as } n \to \infty. \tag{8}$$

Let $\Xi = \{\vartheta_1, \vartheta_2, \vartheta_3, \ldots\}$ and $X_\Xi = \{\langle u(\vartheta_1,\cdot), u(\vartheta_2,\cdot), \ldots\rangle\}$, where u is of the form (4) $(x \in S)$. X_Ξ is a subspace of $\mathop{\mathrm{P}}\limits_{k=1} X_k$ with the l_p or supremum product norm and with any sequence of norms in X_k.

LEMMA 1. *Under* (6), (7), (8) *and the following conditions:* a) *for each n there is a positive number B_n and there is a k such that ψ_n is continuous at ϑ_k and that*

$$B_n|\psi_n(\vartheta_k)| \geqslant |\psi_n(x)|, \tag{9}$$

b) *there is an ε so that*

$$\sum B_n \exp\big(-\operatorname{Re}\lambda_n(T-\varepsilon)\big) < \infty \tag{10}$$

$A_{\vartheta,T}$ is well-defined and bounded for any Y.

Proof. To see that $A_{\Xi,T}$ is well-defined, note that $|\psi_n(\vartheta_k)| > 0$ and that by [2] (page 36)

$$|c_n| \leqslant \big(d_n|\psi_n(\vartheta_k)|\big)^{-1}\|u(\vartheta_k,\cdot)\|_{X_k}$$

where d_n is the distance between x^{λ_n} and the closed linear hull of $\{x^{\lambda_m}\}_{m\neq n}$ in X_k-norm spaces. Since $d_n > 0$, $\|u\|_{X_\Xi} = 0$ implies that $c_n = 0$, $n = 1, 2, \ldots$ Now we shall prove the continuity:

$$\begin{aligned}\|u(\cdot,T)\|_Y &\leqslant \sum |c_n|\,\|\psi_n\|_Y \exp(-\operatorname{Re}\lambda_n T)\\ &\leqslant \max(P(S),1)\cdot\sum |c_n|\,|\psi_n(\vartheta_k)|\,B_n\exp(-\operatorname{Re}\lambda_n T)\\ &\leqslant P'\Big[\sum B_n d_n^{-1}\exp(-\operatorname{Re}\lambda_n T)\Big]\|u\|_X\end{aligned}$$

where P is the Lebesgue measure on R^m and P' is a constant. The series in square parentheses converges because of (10) and of the estimation for every ε and for each X:

$$d_n^{-1} \leqslant \exp(2\varepsilon\operatorname{Re}\lambda_n) \qquad \text{as } n \to \infty \tag{11}$$

(see [2], page 36).

LEMMA 2. *For each Y and for arbitrary $\varepsilon > 0$*

$$\|\psi_m\| > \frac{2}{\pi} - \varepsilon \quad as \quad m \to \infty. \tag{12}$$

We recall that ψ_m are defined under (4).

Proof. We shall observe only that for $y \in Y$. $\|y\|_{L^1} \leqslant \|y\|_{L^p} \leqslant \|y\|_C$, and we shall list six different forms of eigenfunctions according to various $\langle \alpha, \beta \rangle$.

1) $\langle \infty, -\infty \rangle$: $\lambda_n = \pi^2 n^2$, $\psi_n(x) = \cos \pi n x$.

2) $\langle 0, 0 \rangle$: $\lambda_n = \pi^2 n^2$, $\psi_n(x) = \sin \pi n x$.

3) $\langle 0, -\infty \rangle$: $\lambda_n = \pi^2 (n+1/2)^2$, $\psi_n(x) = \sin \pi (n+1/2) x$.

4) $\langle 0, \beta \rangle$, $0 < -\beta < \infty$, λ_n: $(\beta \sqrt{\lambda_n})^{-1} \sin \sqrt{\lambda_n} + \cos \sqrt{\lambda_n} = 0$, $\psi_n(x) = \sin \sqrt{\lambda_n} x$.

5) $\langle \infty, \beta \rangle$, $0 < -\beta < \infty$, λ_n: $(\beta \sqrt{\lambda_n})^{-1} \cos \sqrt{\lambda_n} + \sin \sqrt{\lambda_n} = 0$, $\psi_n(x) = \cos \sqrt{\lambda_n} x$.

6) $\langle \alpha, \beta \rangle$, $\alpha < \infty$, $-\beta < \infty$, λ_n: $(\beta - \alpha)\sqrt{\lambda_n} \cos \sqrt{\lambda_n} - (1 + \alpha\beta\lambda_n) \sin \sqrt{\lambda_n} = 0$, $\psi_n(x) = \cos\big(\sqrt{\lambda_n} x - \arccos(\sqrt{\lambda_n}\alpha : \sqrt{1+\lambda_n \alpha^2})\big)$.

Proof of Theorem 1. By the former proof, $\{\lambda_n\}$ fulfil (6), (7), (8). To prove part a) we put in Lemma 1, $B_n = |\psi_n(\vartheta)|^{-1}$.

b) If $\sum \frac{\exp(-\lambda_n T')}{|\psi_n(\vartheta)|} = \infty$, then it is not true that there is an N such that for all $n > N$ $\frac{\exp(-\lambda_n T')}{|\psi_n(\vartheta)|} < \frac{1}{n^2}$; thus there must be a subsequence $\{n_q\}$ for which the opposite inequality holds. We multiply both sides of this inequality by n_q^3 and we have

$$\frac{\exp\big(-\lambda_{n_q}(T' - \varepsilon)\big)}{|\psi_{n_q}(\vartheta)|} \geqslant n_q \quad \text{for } n_q \text{ large enough.}$$

Now, $A_{\vartheta,T}$ is not bounded if there is a subsequence $\{n_q\}$ such that

$$\varphi(n_q) \stackrel{\text{df}}{=} \frac{\|A_{\vartheta,T}(\exp(-\lambda_{n_q} t))\|_Y}{\|\exp(-\lambda_{n_q} t\|_X} \quad \text{tends to infinity.}$$

But $\varphi(n) = \frac{\exp(-\lambda_n T)\|\psi_n\|_Y}{|\psi_n(\vartheta)|\|\exp(-\lambda_n t)\|_X} \geqslant \frac{\exp(-\lambda_n T)\pi^{-1}}{|\psi_n(\vartheta)|}$ as $n \to \infty$. The last inequality is a consequence of Lemma 2 and of the remark that

$$\Big(\int_0^T \exp(-\lambda_n t p)\, dt\Big)^{1/p} \leqslant \sqrt[p]{\frac{1}{\lambda_n p}} \leqslant 1 \quad \text{as } n \to \infty.$$

3. Some properties of the function $\omega(T) = \|A_{\vartheta,T}\|$. We fix ϑ and $\langle X, Y \rangle$.

THEOREM 2. *Let E be an open interval in which $\omega(T)$ is bounded. Then ω is continuous in E.*

Proof. Let η be an arbitrary positive number and let T_1, T_2 belong to E. Then

$$|\omega(T_1) - \omega(T_2)| = \big|\|A_{\vartheta,T_1}\| - \|A_{\vartheta,T_2}\|\big| \leqslant \|A_{\vartheta,T_1} - A_{\vartheta,T_2}\|$$
$$= \sup_u \frac{\|(A_{\vartheta,T_1} - A_{\vartheta,T_2})u\|}{\|u\|}$$

and

$$\|(A_{\vartheta,T_1} - A_{\vartheta,T_2})u\|_Y \leqslant \sum |c_n|\,\|\psi_n\|_Y |\exp(-\lambda_n T_1) - \exp(-\lambda_n T_2)|$$
$$\leqslant \Big[\sum \frac{\exp(-\lambda_n T_1)}{d_n |\psi_n(\vartheta)|} \big|\big(1 - \exp\big(-\lambda_n(T_1 - T_2)\big)\big|\Big] \|u\|_X$$
$$= (I_N + I^N)\|u\|_X,$$

where I_N denotes the sum of the first N words of the above series and I^N denotes the remainder of the series.

Here without loss of generality we assume that $T_2 > T_1$, then

$$I^N \leqslant \sum_{n=N+1}^{\infty} \frac{\exp(-\lambda_n T_1)}{d_n |\psi_n(\vartheta)|}.$$

We choose an ε such that $T_1 - 2\varepsilon \in E$ and such that (11) holds for $n > N$. As there exists a T' such that $T' \in E$, $T_1 - 2\varepsilon > T'$ $(\omega(T') < \infty)$, $\sum \frac{\exp(-\lambda_n(T_1 - 2\varepsilon))}{|\psi_n(\vartheta)|}$ converges (by Theorem 1a, b). It is possible to enlarge N in order to obtain $I^N \leqslant \eta/2$. The above estimation does not depend on $|T_1 - T_2|$.

Now we demand that $I_N \leqslant \eta/2$.

Then the following inequality must hold:

$$|1 - \exp(-\lambda_N(T_2 - T_1))| \leqslant \Big[2 \sum_{n=1}^{N} \frac{\exp(-\lambda_n T_1)}{d_n \eta |\psi_n(\vartheta)|}\Big]^{-1}.$$

It is fulfilled if we take T_2 such that

$$|T_2 - T_1| < -\frac{1}{\lambda_N} \log\Big(1 - \Big(2 \sum_{n=1}^{N} \frac{\exp(-\lambda_n T_1)}{d_n \eta |\psi_n(\vartheta)|}\Big)^{-1}\Big) \quad \text{for } T_2 > T_1,$$

$$|T_2 - T_1| < \frac{1}{\lambda_N} \log\Big(1 + \Big(2 \sum_{n=1}^{N} \frac{\exp(-\lambda_n T_1)}{d_n \eta |\psi_n(\vartheta)|}\Big)^{-1}\Big) \quad \text{for } T_2 < T_1.$$

THEOREM 3. *When $Y = L^2[0,1]$, ω is decreasing in E, where E is an open interval in which ω is bounded.*

Proof. The operator $C\colon \psi_n \mapsto \frac{\exp(-\lambda_n T)}{\exp(-\lambda_n T')}\psi_n (T > T')$ is of norm smaller than one.

$$\omega(T) = \|A_{\vartheta,T}\| = \sup_u \frac{\|u(\cdot,T)\|_Y}{\|u(\vartheta,\cdot)\|_{X(0,T)}} \leqslant \|C\| \sup_u \frac{\|u(\cdot,T')\|_Y}{\|u(\vartheta,\cdot)\|_{X(0,T)}}$$

$$\leqslant \|C\| \sup \frac{\|u(\cdot,T')\|_Y}{\|u(\vartheta,\cdot)\|_{X(0,T')}} < \|A_{\vartheta,T'}\| = \omega(T').$$

4. Measure and metric theorems. We start with a lemma from the diophantine approximation theory. Let $\|x\| = \inf_{n\in Z}|x-n|$, P be the Lebesgue measure.

LEMMA 3 (S. Rolewicz). *Let*

(13) $$\alpha\in[0,1],\quad \nu>1,\quad 0<\mu<1/4,$$

Then

(14) $$P(I) = P\{\vartheta\in[0,1]\colon \|\nu\vartheta-\alpha\|<\mu\} < 8\mu.$$

THEOREM 4. *For P-almost all ϑs, $A_{\vartheta,T}$ is well-defined and bounded or all $T>0$ and for any $\langle X, Y\rangle$, i.e.*

(15) $$P\{\vartheta\in[0,1]\colon T_0(\vartheta) = 0\} = 1.$$

Proof. At first, note that $2\|x-\beta\| \leqslant |\sin\pi(x-\beta)| \leqslant \pi\|x-\beta\|$ and that ψ_n is of the form: $\psi_n(x) = \sin(\sqrt{\lambda_n}x-\beta_n)$ for some β_n. Let

$$\Gamma_n = \{\vartheta\colon |\psi_n(\vartheta)| < n^2\exp(-\lambda_n T')\}$$

$$\subset \left\{\vartheta\colon 2\left\|\frac{\sqrt{\lambda_n}\vartheta-\beta_n}{\pi}\right\| < n^2\exp(-\lambda_n T')\right\}$$

and let $\Omega = \{\vartheta\colon \bigwedge_n \psi_n(\vartheta) \neq 0\}$.

By Lemma 3, $P(\Gamma_n) < 4n^2\exp(-\lambda_n T')$ for large n, which implies the convergence of $\sum P(\Gamma_n)$. Assuming that $T' < T$, we have

$$\{\vartheta\colon \|A_{\vartheta,T}\|<\infty\} \supset \left\{\vartheta\colon \sum\frac{\exp(-\lambda_n T')}{|\psi_n(\vartheta)|} < \infty\right\}$$

$$\supset \{\vartheta\colon \bigvee_N \bigwedge_{n>N} |\psi_n(\vartheta)| \geqslant n^2\exp(-\lambda_n T')\}\cap\Omega$$

$$= \liminf_n \{\vartheta\colon |\psi_n(\vartheta)| \geqslant n^2\exp(-\lambda_n T')\}\cap\Omega$$

$$= \liminf(\Gamma_n)^c\cap\Omega = (\limsup\Gamma_n)^c\cap\Omega.$$

By the Borel–Cantelli lemma $P(\limsup\Gamma_n) = 0$. We must prove that Ω is of full measure. In fact, $\Omega^c = \bigcup_n\{\vartheta\colon \psi_n(\vartheta) = 0\}$ is a countable set thus of measure 0. Taking a sequence $\{T_n\}(T_n\to 0, T_n > T_{n+1})$, we have $\{\vartheta\in[0,1]\colon T_0(\vartheta) = 0\} = \bigcap_{n=1}\{\vartheta\colon \|A_{\vartheta,T_n}\|<\infty\}$, because $T<T'$ and $\|A_{\vartheta,T}\|<\infty$ imply that $\|A_{\vartheta,T'}\|<\infty$ (Corollary 1). The proof is complete: a countable intersection of full measure sets is of full measure.

THEOREM 5b. *The set of all ϑs for which $A_{\vartheta,T}$ is not bounded for any T is dense. The norm of the operator $A_{\vartheta,\infty}\colon u(\vartheta,\cdot)\mapsto\lim_{T\to\infty}(u,T)$ is 0.*

In cases 1) and 2) (Lemma 2) we can formulate

THEOREM 5a. *If $0<c<\infty$, then the set of irrational ϑs for which $\|A_{\vartheta,c}\| = \infty$ and $T_0(\vartheta) = c$ is dense;*

and

THEOREM 5. *Let c be a number $(0\leqslant c\leqslant\infty)$. The set $\{\vartheta\colon T_0(\vartheta) = c\}$ is dense in $[0,1]$.*

In July 1972 Professors V. T. Sos and E. Wirsing proved in two different ways that for every $\alpha\in[0,1]$ and for each sequence $\{\varphi_n\}$ $(\varphi_n>0)$ there exists a ϑ such that the following inequality holds for infinitely many n:

(16) $$\|n\vartheta-\alpha\|\leqslant\varphi_n.$$

Wirsing's idea constitutes the core of the following:

Proof of Theorem 5b.(1) By Theorem 1b it is sufficient to prove that $\sum\frac{\exp(-\exp(n))}{|\psi_n(\vartheta)|}$ is divergent for a dense set E of ϑs. We shall show that

(17) $$\exp(-\exp(n)) \geqslant \psi_n(\vartheta)|$$

has infinitely many solutions for $\vartheta\in E$.

As we regard the form of ψ_n (proof of Lemma 2), our task is to prove the proposition: For each closed interval $F\subset[0,1]$ and for each sequence $\{\langle\varkappa_n,\alpha_n,\varphi_n\rangle\}_n(\varkappa_n\to\infty, \alpha_n\in[0,1], \varphi_n>0)$ there is a $\vartheta\in F$ such that

(18) $$\|\varkappa_n\vartheta-\alpha_n\|\leqslant\varphi_n$$

has infinitely many solutions. Actually, we define $W_0 = F$ and for W_n we choose $\varkappa_{k_{n+1}}$ so that $(\varkappa_{k_{n+1}}W_n)\bmod 1\supset[0,1]$. Then we put

$$W_{n+1} = \{\vartheta\in W_n\colon \|\varkappa_{k_{n+1}}\vartheta-\alpha_{k_{n+1}}\|\leqslant\varphi_{k_{n+1}}\}.$$

Of course, each W_n is closed and non-empty and $W_n\supset W_{n+1}$; thus $\bigcap_n W_n \subset F$ is not empty.

(1) Professor A. Schinzel noticed that it is much easier to consider ϑ for which $A_{\vartheta,T}$ is not well defined.

Proof of Theorem 5a. (For preparatory remarks see Cassels [1].) For every $\vartheta \in R$ there is a unique sequence $1 = q_1 < q_2 < \ldots$ (finite iff ϑ is rational) and a sequence $p_1 < p_2 \ldots$ such that

(19) $$\|q_n\vartheta\| = |q_n\vartheta - p_n|,$$

(20) $$\|q_{n+1}\vartheta\| < \|q_n\vartheta\|,$$

(21) $$\|q\vartheta\| \geqslant \|q_n\vartheta\| \quad \text{for } 0 < q < q_{n+1}$$

($\frac{p_n}{q^n}$ is called a best approximation). There is the one-to-one correspondence between ϑ and its continued fraction $[a_0, a_1, a_2, \ldots]$, where a_0 is an integer, and a_n an integer greater than 0 for $n \geqslant 1$. For $n \geqslant 2$ we have:

(22) $$p_{n+1} = a_n p_n + p_{n-1},$$

(23) $$q_{n+1} = a_n q_n + q_{n-1}.$$

We shall quote three important statements:

(24) $$\|q_n\vartheta\| \leqslant \frac{1}{q_{n+1}},$$

(25) $$\|q_n\vartheta\| > \frac{1}{2q_{n+1}}.$$

(26) If $\vartheta = [a_0, a_1, a_2, \ldots, a_n, a_{n+1}, \ldots]$, $\vartheta' = [a_0, a_1, \ldots a_n, b_{n+1}, \ldots]$, then $|\vartheta - \vartheta'| < 2^{-(n-2)}$.

The proper proof will be carried out for case 1) (Lemma 2), because case 2) is analogous and much simpler.

Note that for any sequence $a_0, \ldots, a_{n-1}$ we can choose a_n, a_{n+1}, a_{n+2} in such a way that p_{n+2}, p_{n+3} are odd. Then, restricting a_{n+r} $(r > 2)$ to even numbers we obtain a sequence of odd numerators p_{n+r} $(r \geqslant 2)$. In fact,

if p_n odd, p_{n-1} even,	we put	a_n odd,	a_{n+1} even,	a_{n+2} even;
if p_n even, p_{n-1} odd	—	a_n odd,	a_{n+1} odd,	a_{n+2} even;
if p_n odd, p_{n-1} odd	—	a_n even,	a_{n+1} even,	a_{n+2} even.

The situation where p_n, p_{n-1} are both even cannot occur, for two successive best approximation numerators are relatively prime. This follows from the equality $|q_{n+1}p_n - q_n p_{n+1}| = 1$ (see [1]).

Given q_{n-1}, q_n for which p_{n-1}, p_n are odd, we take the smallest even integer a_n such that

(27) $$q_{n+1} = a_n q_n + q_{n-1} \geqslant q_n \exp(\pi^2 q_n^2 c).$$

Then

(28) $$q_{n+1} \leqslant q_n\big(\exp(\pi^2 q_n^2 c) + 2\big).$$

By (24) and (27)

$$\|q_n\vartheta\| \leqslant \frac{1}{q_{n+1}} < \frac{\exp(-\pi^2 q_n^2 c)}{q_n}$$

or, because p_n is odd

(29) $$q_n < \frac{\exp(-\pi^2 q_n^2 c)}{\|q_n\vartheta\|} \leqslant \frac{\exp(-\pi^2 q_n^2 c)}{\left\|q_n \frac{\vartheta}{2} - \frac{1}{2}\right\|}.$$

By (25) and (27) for large n

$$\|q_n\vartheta\| > \frac{1}{2q_{n+1}} \geqslant [2q_n(\exp(\pi^2 q_n^2 c) + 2)]^{-1} > q_n^2 \exp\big(-\pi^2 q_n^2 (c+\varepsilon)\big).$$

(21) entails for all large q

(30) $$\|q\vartheta\| > q^2 \exp\big(-\pi^2 q_n^2 (c+\varepsilon)\big).$$

Hence the following series is convergent for every $\varepsilon > 0$:

(31) $$\sum \frac{\exp\big(-\pi^2 q^2 (c+\varepsilon)\big)}{\left\|q\frac{\vartheta}{2} - \frac{1}{2}\right\|} < \infty.$$

It has already been mentioned that $2\|x - \frac{1}{2}\| \leqslant |\cos \pi x| \leqslant \pi \|x - \frac{1}{2}\|$. The final discussion of proof 1b and (29) imply

(32) $$\left\|A_{\frac{\vartheta}{2}, c}\right\| = \infty.$$

(31) and (32) imply $T_0(\vartheta/2) = c$; the application of (26) concludes the proof.

Let Ξ be an infinite sequence of $\vartheta_n \in [0, 1]$, $X = C(0, T)$, $Y = C[0, 1]$, $\alpha = 0 = \beta$. There is a Ξ such that $\|A_{\Xi, T}\| = \infty$ for all $T > 0$.

Example. $\vartheta_1 = 1$, $\vartheta_{k+1} = 3^{-3^{\frac{1}{\vartheta_k}}} \cdot \vartheta_k$,

(33) $$e_n = \langle \sin(\pi n \vartheta_1)\exp(-\pi^2 n^2 t),\ \sin(\pi n \vartheta_2)\exp(-\pi^2 n^2 t), \ldots\rangle,$$

(34) $$\frac{\|A_{\Xi,T}(e_n)\|}{\|e_n\|} = \frac{\exp(-\pi^2 n^2 T)}{\sup_k |\sin(\pi n \vartheta_k)|}.$$

Putting $q_n = \vartheta_n^{-1}$ we have $\|q_n\vartheta_k\| = 0$, if $k \leqslant n$, and $\|q_n\vartheta_k\| \leqslant 3^{-3^{\frac{1}{\vartheta_n}}}$, if $k > n$. Thus $\|A_{\Xi,T}\| \geqslant \dfrac{\exp(-\pi^2 q_n^2 T)}{3^{-3q_n}}$ is equal to infinity.

5. Other observability problems for a rod. The following theorem resolves the "mobile observer problem":

THEOREM 6. *For every* $a \in (0,1)$ *and for every* b $(0<b<\infty)$ *and for* $T>0$ *such that* $a+bT \leqslant 1$, $A\colon u(a+bt,t)\mapsto u(\cdot,T)$ *is well-defined and continuous.*

Proof. By (4) and the initial considerations of Lemma 2

$$u(a+bt,t) = \sum_{n=1}^{\infty} \tfrac{1}{2} c_n \cos\big(\sqrt{\lambda_n}(a+bt)-\alpha_n\big)\exp(-\lambda_n t) \tag{35}$$
$$= \sum_{n=1}^{\infty} c_n\big[\exp\big(i(\sqrt{\lambda_n}a-\alpha_n)\big)\exp\big((i\sqrt{\lambda_n}b-\lambda_n)t\big)+ \exp\big(-i(\sqrt{\lambda_n}a-\alpha_n)\big)\exp\big((-i\sqrt{\lambda_n}b-\lambda_n)t\big)\big].$$

The sequence $i\sqrt{\lambda_n}b-\lambda_n$, $-i\sqrt{\lambda_n}b-\lambda_n$ satisfies (6), (7), (8).

We have $\big\|\sum c_n\psi_n(x)\exp(-\lambda_n T)\big\| \leqslant \sum |c_n|\exp(-\lambda_n T)$ and

$$|c_n| = \big|c_n \exp\big(i(\sqrt{\lambda_n}a-\alpha_n)\big)\big|.$$

By (35) $|c_n| \leqslant \dfrac{1}{d_{n1}}\|u(a+bt,t)\|$. To complete the proof we should only recall (11).

It seems obvious that the temperature observed at a fixed point is, in fact, the temperature mean value over a certain segment of the rod. There arises the problem, whether, given $a, b \in [0,1]$, the variation of $\int_a^b u(x,\cdot)\,dx$ determines $u(\cdot,T)$ continuously.

THEOREM 7. a) *The set of* $\langle a,b\rangle$ *for which* $A_{a,b,T}\colon \int_a^b u(x,\cdot)\,dx \mapsto u(\cdot,T)$ *is well-defined and continuous for each* $T>0$ *and for every* $\langle X,Y\rangle$ *is of full measure.*

b) *The set of* $\langle a,b\rangle$ *for which* $A_{a,b,T}$ *is not well-defined is dense.*

Proof. We have

$$\int_a^b u(x,t)\,dx = \sum c_n \exp(-\lambda_n t)\,\frac{\sin(\sqrt{\lambda_n}b-\alpha_n)-\sin(\sqrt{\lambda_n}a-\alpha_n)}{\sqrt{\lambda_n}},$$

where u is of the form (4), $\psi_n(x) = \cos(\sqrt{\lambda_n}x-\alpha_n)$. By virtue of Lemma 1 slightly modified, $A_{a,b,T+\varepsilon}$ is well-defined and continuous if

$$\sum \frac{\sqrt{\lambda_n}\exp\big(-\lambda_n(T+\varepsilon/2)\big)}{|\sin(\sqrt{\lambda_n}b-\alpha_n)-\sin(\sqrt{\lambda_n}a-\alpha_n)|}$$

is convergent, which occurs if there is an N such that for all $n \geqslant N$

$$|\sin(\sqrt{\lambda_n}b-\alpha_n)-\sin(\sqrt{\lambda_n}a-\alpha_n)| \geqslant n^2\exp(-\lambda_n T) \tag{36}$$

and if for all n

$$\sin(\sqrt{\lambda_n}a-\alpha_n) \neq \sin(\sqrt{\lambda_n}b-\alpha_n). \tag{36'}$$

Finally (36) holds for all $\langle a,b\rangle \in \Omega = \liminf_n \Omega_n$, where

$$\Omega_n = \big\{\langle a,b\rangle\colon \big|2\sin\big(\tfrac{1}{2}\sqrt{\lambda_n}(b-a)\big)\sin\big(\tfrac{1}{2}\sqrt{\lambda_n}(b+a)-\beta_n\big)\big| \geqslant n^2\exp(-\lambda_n T)\big\}$$
$$\supset \Big\{\langle a,b\rangle\colon \big|\sin\big(\tfrac{1}{2}\sqrt{\lambda_n}(b-a)\big)\big| \geqslant \frac{n\exp(-\frac{1}{2}\lambda_n T)}{\sqrt{2}} \text{ and } \sin\big(\tfrac{1}{2}\lambda_n(a+b)-\beta_n\big) \geqslant \frac{n\exp(-\frac{1}{2}\lambda_n T)}{\sqrt{2}}\Big\}.$$

Now, it is sufficient to prove that $P(\limsup_n \Omega_n^c) = 0$ and that

$$P(\Delta) \overset{\mathrm{df}}{=} P\big\{\langle a,b\rangle\colon \bigvee_n \big(\sin(\sqrt{\lambda_n}b-\alpha_n) = \sin(\sqrt{\lambda_n}a-\alpha_n)\big)\big\} = 0, \tag{37}$$

$$\Omega_n^c \subset \Big\{\langle a,b\rangle\colon \Big\|\frac{\frac{1}{2}\sqrt{\lambda_n}(b+a)-\beta_n}{\pi}\Big\| < \frac{n\exp(-\lambda_n\frac{1}{2}T)}{\sqrt{2}}\Big\} \cup$$
$$\cup \Big\{\langle a,b\rangle\colon \Big\|\frac{\frac{1}{2}\sqrt{\lambda_n}(b+a)}{\pi}\Big\| < \frac{n\exp(-\lambda_n\frac{1}{2}T)}{\sqrt{2}}\Big\}; \qquad \beta_n = \alpha_n + \frac{\pi}{2}.$$

Since $\det\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix} = -2$, the Lebesgue measures of the components of the above sum are equal to the measures divided by 2 of

$$\Big\{\langle b-a, b+a\rangle\colon \Big\|\frac{\frac{1}{2}\sqrt{\lambda_n}(b+a)-\beta_n}{\pi}\Big\| < \frac{n\exp(-\lambda_n T/2)}{\sqrt{2}}\Big\}$$
$$= [0,1]\times\Big\{\tau\colon \Big\|\frac{\sqrt{\lambda_n}\tau/2-\beta_n}{\pi}\Big\| < \frac{n\exp(-\lambda_n T/2)}{\sqrt{2}}\Big\},$$

$$\Big\{\langle b-a, b+a\rangle\colon \Big\|\tfrac{1}{2}\sqrt{\lambda_n}\frac{(b-a)}{\pi}\Big\| < \frac{n\exp(-\lambda_n T/2)}{\sqrt{2}}\Big\}$$
$$= \Big\{\vartheta\colon \Big\|\sqrt{\lambda_n}\frac{\vartheta}{2\pi}\Big\| < \frac{n\exp(-\lambda_n T/2)}{\sqrt{2}}\Big\}\times[0,1],$$

respectively. By Lemma 3 $P(\Omega_n^c) \leqslant 8n\exp(-\lambda_n T/2)$, hence by the Borel–Cantelli lemma $P(\limsup \Omega_n^c) = 0$.

Now we are going to prove (37). We fix n and we choose a closed subinterval of $[0,1]$, where $\cos(\sqrt{\lambda_n}b-\beta_n)$ is monotonous, and a closed interval, where $\cos(\sqrt{\lambda_n}a-\beta_n)$ is monotonous. Taking the greatest intervals of this property, we have a finite number of them covering $[0,1]$. In each pair of the intervals we consider a function $a = f(b)$ such that $\cos(\sqrt{\lambda_n}b - \beta_n) = \cos(\sqrt{\lambda_n}a - \beta_n)$. Since f is continuous, its graph is closed in $[0, 1] \times [0, 1]$. Joining the graphs corresponding to all pairs of the intervals and to each n, we get a Borel set containing Δ. Since the b-section of this set has a countable number of points for all b, $P(\Delta) = 0$. The b) part of the theorem is trivial.

The observation of the temperature and the temperature derivative at any point gives complete information about the temperature distribution.

THEOREM 8. *For each ϑ and $T > 0$ the operator A: $\langle u(\vartheta, \cdot), u_x(\vartheta, \cdot)\rangle \mapsto u(\cdot, T)$ is well-defined and bounded for any $\langle X, Y\rangle$.*

6. Observability on a circle. In this section assertions similar to those of Theorems 1, 4, and 6 and of Corollary 1 will be presented. First of all, note that the circle observation operator is not well-defined unless Ξ has more points than one. Indeed, this becomes clear when we represent a solution of the heat equation for a circle in the expanded form:

$$u(x, t) = \sum (c_n \cos \pi n x + b_n \sin \pi n x) \exp(-\pi^2 n^2 t), \quad x \in [0, 2),\ t > 0. \tag{38}$$

THEOREM 9. a) *The convergence of $\sum \frac{\exp(-\pi^2 n^2 T)}{\|n(\vartheta - \tau)\|}$ entails the continuity of $A_{\vartheta,\tau,T'}$ $(T' > T)$ for any $\langle X_\vartheta, X_\tau, Y\rangle$ and any l_p or sup-product norm.*

b) *The divergence of the series implies the discontinuity of the operator $(T' < T)$.*

COROLLARY 2. *Let*

$$T_0(\vartheta, \tau]) = -\liminf \frac{\log(\|n(\vartheta - \tau)\|)}{\pi^2 n^2}; \tag{39}$$

$A_{\vartheta,\tau,T}$ is well-defined and bounded for $T > T_0$ and is not bounded for $T < T_0$.

Proof of Theorem 9. a).

$$\|u(\cdot, T)\|_Y \leqslant \sum (|c_n| + |b_n|) \exp(-\pi^2 n^2 T)$$

$$\leqslant \sum_n \left\{ \frac{[|c_n \cos \pi n \vartheta + b_n \sin \pi n \vartheta|^p + |c_n \cos \pi n \tau + b_n \sin \pi n \tau|^p]^{\frac{1}{p}}}{\max\limits_{\chi = \vartheta, \tau} |c_n \cos \pi n \chi + b_n \sin \pi n \chi|} \times \right.$$

$$\left. \times (|c_n| + |b_n|) \exp(-\pi^2 n^2 T) \right\}$$

$$\leqslant \left(\sup_{\langle c_n\rangle} \sum \frac{\exp(-\pi^2 n^2 T)}{d_n \frac{1 + |c_n|^2}{1 + |c_n|} \max(|\cos(\pi n - a_n)|, |\cos(\pi n \tau - a_n)|)} \right) \times$$

$$\times \|\langle u(\vartheta, \cdot), u(\tau, \cdot)\rangle\|_X$$

where $a_n = \arccos \frac{c_n}{\sqrt{1 + |c_n|^2}}$. By (11) the operator $A_{\vartheta,\tau,T}$ is well-defined and bounded if there is an ε such that

$$\sum \frac{\exp(-\pi^2 n^2 (T - \varepsilon))}{\inf\limits_{\langle a_n\rangle} \max(|\cos(\pi n \vartheta - a_n)|, |\cos(\pi n \tau - a_n)|)} < \infty. \tag{40}$$

The denominator can be estimated from above and below by

$$\inf_{\langle a_n\rangle} \max\left(\left\| n\vartheta - \frac{a_n}{\pi} - \frac{1}{2} \right\|, \left\| n\tau - \frac{a_n}{\pi} - \frac{1}{2} \right\| \right),$$

which is equal to

$$\min\left(\left\| \frac{n(\vartheta - \tau)}{2} \right\|, \left\| \frac{n(\vartheta - \tau)}{2} - \frac{1}{2} \right\| \right) = \frac{1}{2} \|n(\vartheta - \tau)\|.$$

THEOREM 10. *$P\{\langle \vartheta, \tau\rangle : T_0(\vartheta, \tau) = 0\} = 1$ where P is the normalized Lebesgue measure on $[0, 2) \times [0, 2)$.*

THEOREM 11. *The "mobile observer problem" for a circle has the affirmative answer.*

7. The case of a nonhomogeneous rod and its applications. Quite similar results can be obtained for more complex problems. For instance, the practical temperature prediction all over a helicopter landing platform on a ship can be reduced to the following form:

$$\frac{\partial u}{\partial t} - Au = \sigma \tau \chi, \quad x \in [0, 1],\ t > 0 \tag{41}$$

where σ, τ are numbers, χ is the characteristic function of $(0, 1)$, $A = \frac{\partial^2}{\partial x^2}$ for $x < 1$ and $A = a^2 \frac{\partial^2}{\partial x^2}$ for $x > 1$; $a > 0$, $k > 0$.

Boundary and consistency conditions are given:

$$u(0, t) = x_x(0, t) = 0, \tag{42}$$

$$u(1-, t) = u(1+, t), \tag{43}$$

$$u_x(1-, t) = k u_x(1+, t). \tag{44}$$

We have proved that the eigenvalues of $-A$ satisfy (6), (7), (8).

8. Optimal observation. Let Ξ be a finite sequence $(\langle \vartheta_1, \vartheta_2, \ldots, \vartheta_K\rangle$, $\vartheta_k \in (0, 1))$. We endow Y and each $X_k = \{u(\vartheta_k, \cdot)\}$ with the supremum norm, the product norm of $X = \mathop{\mathsf{P}}\limits_{k=1}^{K} (X_k \cap \operatorname{lin}\{\exp(-\lambda_1 t), \exp(-\lambda_2 t)\})$

being also of the supremum type. Assume $\alpha = \infty = \beta$. We define the mapping

(45) $$A_{\Xi,T}\colon \langle \hat{u}(\vartheta_1, \cdot), \hat{u}(\vartheta_2, \cdot) \ldots \rangle \mapsto \hat{u}(\cdot, T),$$

where $\hat{u}$ is the projection of u on $\operatorname{lin}\{\psi_1 \exp(-\lambda_1 t), \psi_2 \exp(-\lambda_2 t)\}$. The general optimal observation seems to be very difficult, and thus we deal with the restricted operator.

The optimal observation principle claims:

THEOREM 12.

a) *For any Ξ we have $\langle \vartheta_1, \vartheta_2 \rangle$ such that* $\|A_{\vartheta_1,\vartheta_2,T}\| < \|A_{\Xi,T}\|$.

b)

(46) $$\inf_{\Xi} \|A_{\Xi,T}\| = \exp(-\pi^2 T).$$

c) ϑ_1^m *increasing*, ϑ_2^m *decreasing*

$$\|A_{\vartheta_1^m,\vartheta_2^m,T}\| \to \exp(-\pi^2 T) \quad \textit{iff} \ \lim \vartheta_1^m = 1 \ \textit{and} \ \lim \vartheta_2^m = 0.$$

d) $$\inf_{\vartheta} \|A_{\vartheta,T}\| > \exp(-\pi^2 T).$$

Proof. Here we write $\Xi = \langle x_1, x_2, \ldots, x_N \rangle$.

$$\inf_{\langle x_1, x_2, \ldots \rangle} \|A_{\Xi,T}\| =$$

$$\inf_{\Xi} \sup_{\langle c_1, c_2 \rangle} \frac{\sup\limits_{0 \leqslant x \leqslant 1} |c_1 \cos \pi x + c_2 \cos 2\pi x|}{\max\limits_{n} \sup\limits_{0<t'<T'} |c_1 \cos \pi x_n \exp\big(\pi^2 (T'-t')\big) + c_2 \cos 2\pi x_n \exp\big(4\pi^2 (T'-t')\big)|}.$$

Substituting $T = \pi^2 T'$, $t = \pi^2 t'$, we convert this expression into:

(47) $$\sup_{\langle x_1, x_2, \ldots \rangle} \inf_{c} \max_{n} \sup_{t} \left| \frac{c}{1+|c|} \cos \pi x_n e^T e^{-t} + \frac{1}{1+|c|} \cos 2\pi x_n e^{4T} e^{-4t} \right| \stackrel{\mathrm{df}}{=} \sup_{\langle x_1, x_2, \rangle} \inf_{c} \max_{n} H(x_n, c).$$

Firstly, we examine the function $\psi(t) = ae^{-t} + be^{-4t}$. If $ab \geqslant 0$, then $|\psi|$ is decreasing. If $ab < 0$, then ψ has the only extreme point t_0 such that $e^{-t_0} = \sqrt[3]{\frac{-a}{4b}}$. Besides, $\lim\limits_{t \to -\infty} |\psi(t)| = \infty$, $\lim\limits_{t \to \infty} |\psi(t)| = 0$. In our case $a = \frac{c}{1+|c|} \cos \pi x e^T$, $b = \frac{1}{1+|c|} \cos 2\pi x e^{4T}$ and $e^{-t_0} = -\sqrt[3]{\frac{c \cos \pi x}{4 \cos 2\pi x}}\, e^{-T}$ $(\cos 2\pi x \neq 0)$. If $c \cos \pi x \cos 2\pi x \geqslant 0$, then $H(x, c) = \left| \frac{c}{1+|c|} \cos \pi x e^T + \frac{1}{1+|c|} \cos 2\pi x e^{4T} \right|$. H takes the same form if $c \cos \pi x \cos 2\pi x < 0$ and

a) $c \leqslant \dfrac{-4 \cos 2\pi x}{\cos \pi x} e^{3T}$ and $\cos \pi x \cos 2\pi x > 0$,

b) $c \geqslant \dfrac{-4 \cos 2\pi x}{\cos \pi x} e^{3T}$ and $\cos \pi x \cos 2\pi x < 0$.

If $\cos 2\pi x = 0$, $H(x, c) = \left| \frac{c}{1+|c|} \cos \pi x e^T \right|$. Thus, tending to plus infinity with fixed x, we enter the area where H is of the above mentioned form. Since $\lim\limits_{c \to \infty} H(x, c) = |\cos \pi x| e^T$, (47) is not greater than e^T. To prove d) note that for each x there is a c such that $H(x, c) < e^{T-\varepsilon}$. This is obvious when $x \neq 0$ or 1. But $H(0, -1-4e^{3T}) = \left| \frac{-1-4e^{3T}}{2+4e^{3T}} e^T + \frac{1}{2+4e^{3T}} e^{4T} \right| < e^T$ and $H(1, -c) = H(0, c)$.

To prove a) and b) note that $\inf\limits_{c} \max\big(H(0, c), H(1, c)\big) = e^T$.

c) For all large $c > 0$ $H(0, c) < e^T$ and $H(1, c) < e^T$.

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