maximal ideal space $\mathfrak{M}$. *If there exist two real functions* f_1, f_2 *in* B *and a positive non-zero Borel measure* ν *on* $\mathfrak{M}$, *with compact support, such that*

$$\int_{R^2} (u^2+v^2)\,\|\nu\,\mathrm{Exp}\,[i(uf_1+vf_2)]\|_{B^*}\,du\,dv < \infty \tag{4.1}$$

then B *contains a closed ideal which is not generated by a single function.*

COROLLARY. *Let* $A_p(G)$ *be the Banach algebras defined as in* [6]. *If* $1 < p < \infty$ *and* G *is not discrete,* $A_p(G)$ *contains a closed ideal which is not singly generated.*

Proof. It follows from Lemma 2.2 that condition (4.1) is satisfied for $A_p(G)$.

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A remark on the d-characteristic and the d_Ξ-characteristic of linear operators in a Banach space

by

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Abstract. Let X be a Banach space, and Ξ a total space of continuous linear functionals on X which is also a Banach space. It is proved that $I+T$ is a Φ_Ξ-operator provided $T: X \to X$ is compact and Ξ is preserved by the conjugate operator T'. The paper is closely related to the work of D. Przeworska-Rolewicz and S. Rolewicz.

Let X be a linear space (over the field of real or complex numbers) and let A be a linear operator mapping X into itself and such that Ax is defined for all $x \in X$ ($D_A = X$). Let the set of all such operators be denoted by $L_0(X)$.

We denote by

$$N(A) = \{x \in X;\ Ax = 0\}$$

the kernel of the operator A, and by

$$R(A) = \{y \in X;\ y = Ax,\ x \in X\}$$

the range of the operator A, and define

$$\alpha_A = \dim N(A), \qquad \beta_A = \dim X/R(A)$$

(dim denotes the dimension of a linear set and $X/R(A)$ means the quotient space). The ordered pair (α_A, β_A) is called the *d-characteristic of the operator* A. The index of the operator A is the number

$$\operatorname{ind} A = \beta_A - \alpha_A.$$

By X' the space of all linear functionals on X is denoted. Let $\Xi \subset X'$ be a total space of linear functionals on X, i.e. if $\xi(x) = 0$ for all $\xi \in \Xi$ then $x = 0$. We write

$$N_\Xi(A') = \{\xi \in \Xi;\ \xi(Ax) = 0 \quad \text{for all } x \in X\}$$

and define

$$\beta_A^\Xi = \dim N_\Xi(A').$$

The ordered pair (α_A, β_A^Ξ) is called the *d_Ξ-characteristic of* A.

For a given total space $\Xi \subset X'$ and $A \in L_0(X)$ we define the conjugate operator A': $\Xi \to X'$ by the relation

$$A'\xi(x) = \xi(Ax) \quad \text{for all } \xi \in \Xi,\ x \in X.$$

We have evidently

$$N(A) = \{\xi \in \Xi;\ A'\xi = 0\} = \{\xi \in \Xi;\ A'\xi(x) = 0 \text{ for all } x \in X\} = N_\Xi(A')$$

and therefore also

(1) $$\alpha_{A'} = \beta_A^\Xi.$$

For a given total space $\Xi \subset X'$ and $A \in L_0(X)$ the inclusion $R(A') \subset \Xi$ does not always hold. We say that the space Ξ is preserved by the conjugate operator A' if $R(A') \subset \Xi$.

LEMMA 1 (cf. Theorem 1.2 from A III, § 1 in [3]). *If $A \in L_0(X)$ and the total space $\Xi \subset X'$ is preserved by A' then*

(2) $$\alpha_{A'} \leqslant \beta_A$$

and also

(3) $$\beta_A^\Xi \leqslant \beta_A.$$

It is evident that the d-characteristic and d_Ξ-characteristic of an operator $A \in L_0(X)$ are not equal in general (cf. the examples in [1]). For a comparison of the given concepts of the dimensional characteristic of A it is sufficient to study the numbers β_A and β_A^Ξ. Operators for which the d-characteristic equals the d_Ξ-characteristic are called Φ_Ξ*-operators* (cf. [1], [3]).

THEOREM 1. *Let X be a linear space, and $\Xi \subset X'$ a total space of linear functionals on X, $A \in L_0(X)$ such that Ξ is preserved by the conjugate operator A'.*

If $\operatorname{ind} A = 0$ *and* $\alpha_A \leqslant \alpha_{A'}$ *then* $\beta_A = \beta_A^\Xi$, *i.e. A is a* Φ_Ξ*-operator.*

Proof. The assumption $\operatorname{ind} A = 0$ implies $\alpha_A = \beta_A$. Hence the inequality $\alpha_A \leqslant \alpha_{A'}$ can be written in the form $\beta_A \leqslant \alpha_{A'}$. Using (1) we obtain $\beta_A \leqslant \beta_A^\Xi$. This inequality together with (3) yields our assertion.

LEMMA 2. *Let X be a linear space, $\Xi \subset X'$ a total space, Θ a linear set in X' such that $\Xi \subset \Theta$. Let $A \in L_0(X)$ and A', $\tilde{A}$ are the conjugate operators with respect to Ξ, Θ respectively (Θ is evidently also a total space of linear functionals on X). Then*

(4) $$\alpha_{A'} \leqslant \alpha_{\tilde{A}}.$$

Proof. If $\xi \in \Xi \subset \Theta$, then we have by definition $A'\xi = \tilde{A}\xi$. Hence $N(A') = N(\tilde{A}) \cap \Xi$ and therefore $\dim N(A') \leqslant \dim N(\tilde{A})$.

If X is a linear space and $\Xi \subset X'$ a total space, then for a given $x \in X$ the relation $F_x(\xi) = \xi(x)$ evidently defines a linear functional on Ξ, i.e. for any $x \in X$ we have an element $\varkappa x = F_x \in \Xi'$. The map $\varkappa$: $X \to \Xi'$

is called the *natural embedding of X into Ξ'* and is a monomorphism. The image $\varkappa X$ of X in Ξ' is a total space of linear functionals on Ξ. (For these facts see [3], A III, § 1.)

Since $\varkappa X \subset \Xi'$ is a total space, we can define for a given operator $A' \in L_0(\Xi)$ the conjugate operator A'': $\varkappa X \to \Xi'$ by the relation

$$A''\varkappa x(\xi) = \varkappa x(A'\xi) = (A'\xi)(x) \quad \text{for } x \in X,\ \xi \in \Xi.$$

If, moreover, for $A \in L_0(X)$ the total space $\Xi \subset X'$ is preserved by A', then also $\varkappa X$ is preserved by A'' because we have $A''\varkappa x = \varkappa A x \in \varkappa X$ for any $x \in X$. Hence it follows that in this case the operator A'' conjugate to A' is (up to the natural embedding $\varkappa$) identical with the operator A.

In the sequel let X be a Banach space, and X^+ the space of all continuous linear functionals on X. X^+ is also a Banach space. Further, let $\Xi \subset X^+$ be a total space of continuous linear functionals on X. Then we have

$$|\xi(x)| \leqslant \|\xi\|_{X^+} \|x\|_X$$

for $\xi \in \Xi$, $x \in X$ (norms in X, X^+ respectively). For $\xi \in \Xi$ we write $\|\xi\|_\Xi = \|\xi\|_{X^+}$. Ξ with the norm $\|\ \|_\Xi$ forms a normed space and we have

(5) $$|\xi(x)| \leqslant \|\xi\|_\Xi \|x\|_X$$

for all $\xi \in \Xi$, $x \in X$.

For any $x \in X$ the natural embedding $\varkappa$ described above defines a linear functional on Ξ ($\varkappa x(\xi) = \xi(x)$). By (5), $\varkappa x \in \Xi'$ is for any $x \in X$ a continuous linear functional, i.e., we have $\varkappa x \in \Xi^+$ for any $x \in X$, $\varkappa$: $X \to \Xi^+$. The image $\varkappa X$ of X in Ξ^+ is a total space and the map $\varkappa$: $X \to \Xi^+$ is a monomorphism.

For a given continuous $A \in L_0(X)$ we define $A^+ \in L_0(X^+)$ by the relation

$$A^+ f(x) = f(Ax) \quad \text{for } f \in X^+,\ x \in X,$$

and similarly $A'^+ \in L_0(\Xi^+)$ can be defined for a continuous $A' \in L_0(\Xi)$ as follows:

$$A'^+\varphi(\xi) = \varphi(A'\xi) \quad \text{for } \varphi \in \Xi^+,\ \xi \in \Xi.$$

If $A' \in L_0(\Xi)$ is continuous, then it is possible to define the conjugate operator A'': $\varkappa X \to \Xi'$ as in the above case of a general linear space X. In the case under consideration $A''\varkappa x$ is a continuous linear functional for every $x \in X$, i.e. A'': $\varkappa X \to \Xi^+$.

LEMMA 3. *Let X be a Banach space, and let $\Xi \subset X^+$ be a total space of continuous linear functionals on X. If $A \in L_0(X)$ is a continuous operator such that Ξ is preserved by the conjugate operator A', then*

(6) $$\alpha_{A''} = \dim N(A'') \leqslant \dim N(A'^+) = \alpha_{A'^+}$$

holds.

Proof. The natural embedding $\varkappa X$ of X into Ξ^+ is a total space of continuous linear functionals on Ξ ($\varkappa X \subset \Xi^+$). The operators A'', A'^+

are the conjugates to $A' \in L_0(\Xi)$ with respect to the total spaces $\varkappa X$, Ξ^+ respectively. The assertion of our Lemma follows directly from Lemma 2.

Using the fact that the operator A'' is (up to the monomorphism $\varkappa$) identical with the operator A, we obtain $\alpha_{A''} = \alpha_A$, and Lemma 3 can be reformulated as follows:

LEMMA 4. *If the assumptions of Lemma 3 are fulfilled, then*

$$\alpha_A \leqslant \alpha_{A'^+}. \tag{7}$$

THEOREM 2. *Let X be a Banach space, and $\Xi \subset X^+$ a total space of continuous linear functionals on X. Let $A \in L_0(X)$ be a continuous operator and let Ξ be preserved by the conjugate operator A'.*

If $\operatorname{ind} A = 0$ *and* $\alpha_{A'^+} \leqslant \alpha_{A'}$ *then the d-characteristic of A is equal to the d_Ξ -characteristic of A.*

Proof. By the assumption $\alpha_{A'^+} \leqslant \alpha_{A'}$ and by Lemma 4 we obtain $\alpha_A \leqslant \alpha_{A'}$. This inequality together with $\operatorname{ind} A = 0$ gives by Theorem 1 the equality $\beta_A = \beta_A^\Xi$ and our Theorem is proved.

THEOREM 3. *Let X be a Banach space, and let $\Xi \subset X^+$ be a total space of continuous linear functionals on X. Let $A \in L_0(X)$ be continuous and let Ξ be preserved by A'.*

If $\operatorname{ind} A = \operatorname{ind} A' = 0$, *then the d-characteristic of A is equal to the d_Ξ-characteristic of A.*

Proof. By (1) we have $\alpha_{A'^+} = \beta_{A'}^{\Xi^+}$. Then inequality (3) from Lemma 1 assumes in our case the form $\beta_{A'}^{\Xi^+} \leqslant \beta_{A'}$ and hence we obtain

$$\alpha_{A'^+} \leqslant \beta_{A'}.$$

Since $\operatorname{ind} A' = 0$, we have $\alpha_{A'} = \beta_{A'}$ and the last inequality has the form $\alpha_{A'^+} \leqslant \alpha_{A'}$. Hence all the assumptions of the previous Theorem 2 are fulfilled and our assertion is a consequence of this theorem.

THEOREM 4. *Let X be a Banach space, and $\Xi \subset X^+$ a total space of continuous linear functionals on X which is also a Banach space. Let $T \in L_0(X)$ be a compact (completely continuous) operator such that Ξ is preserved by the conjugate operator T'. Then the d-characteristic of the operator $A = I+T$ is equal to its d_Ξ-characteristic (I is the identical operator in X).*

Proof. The operator $A = I+T \in L_0(X)$ is continuous. It is well known that if T is compact then $\operatorname{ind} A = 0$. We recall that the operator $T' \in L_0(\Xi)$ is also compact (see Theorem 7.4 in C III from [3]). Hence we have $\operatorname{ind} A' = 0$ and the theorem follows from Theorem 3.

Remark 1. In paper [1] a theorem which is similar to our Theorem 4 is established under the additional assumption that the compact operator $T \in L_0(X)$ is approximable by finite-dimensional operators. This theorem is followed in [1] by the authors' remark (cf. page 121, line 3 from below) which gives the impression that this result without the assumption of approximability of T by a finite-dimensional operator would also be of interest.

P. Enflo has recently shown that the answer to the approximation problem (approximation of compact operators by finite-dimensional ones) is negative. Hence our Theorem 4 cannot be simply derived from the above-quoted Theorem of D. Przeworska-Rolewicz and S. Rolewicz.

Remark 2. If we use the terminology of § 7, C III in [3] then, provided the assumptions of Theorem 4 are fulfilled, we can state that if T is an element of the ideal of compact operators in the algebra $B_0(X, \Xi)$ of continuous linear operators on X with conjugates preserving Ξ then $A = I+T$ is a Φ_Ξ -operator. This gives also the answer to the question formulated in Remark on p. 247 in [2] and in Remark on p. 297 in [3].

Finally let us mention that the assumption that Ξ is also a Banach space in Theorem 4 seems to be necessary since without this assumption the equation $\operatorname{ind} A' = 0$ is not satisfied in general.

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