On commutative approximate identities
and cyclic vectors of induced representations

by

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Abstract. It is shown that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity. This is applied to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

The aim of this paper is twofold. To show that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity and apply this to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

A construction of commutative approximate identity for a $C^*$-algebra was given by J. F. Aarnes and R. V. Kadison [1]. Their method uses $C^*$-algebras technique and does not apply to the group algebras. It would be interesting to know whether there exists an approximate identity for $L_1(G)$ consisting of commuting continuous functions with compact support.

The fact that for a first countable group representations induced by cyclic representations are cyclic was first proved by F. Greenleaf and M. Moskowitz [5] and [6] and a construction of a cyclic vector for such representations was claimed by the authors [7]. Unfortunately [7] makes use of a statement in [2], p. 49, which is false, as it has been recently discovered by R. Goodman. The construction presented here avoids this difficulty and (for induced representations) improves the construction given in [7].

Very briefly the idea is the following. For a Lie group $G$ the fundamental solution $u(x,t) = p^t(x)$ of the heat equation is a one-parameter semi-group of non-negative functions $p^t$, that is $p^s \ast p^t = p^{s+t}$ for all positive real $s, t$. Moreover $p^t f$ tends to $f$ as $t$ tends to zero, and for a fixed $t$ the function $p^t f$ decreases faster than exponentially at infinity. In short, functions $p^t, t \in \mathbb{R}^+$, form an approximate identity for $L_1(G)$ consisting of commuting rapidly decreasing functions.
On the other hand, the positive-definite measures μ which define representations induced by cyclic representations are of at most exponential growth. Thus functions φ′ can be used in the same way as functions "p in [7] to construct a cyclic vector for the representation P′.

Another method of saving the construction in [7] for Lio groups has been recently developed by H. Goodman [4].

1. The space of very rapidly decreasing functions. Let G be a locally compact group. We say that a non-negative function ϕ on G is submultiplicative if

\[ \varphi(gh) \leq \varphi(g)\varphi(h) \quad \text{for all} \quad g, h \in G. \]

Let \( \Phi \) denote the set of all submultiplicative continuous functions on G. It is clear that \( \Phi \) is closed under multiplication and that the modular function \( \Delta_\varphi \) is in \( \Phi(G) \).

Let \( \{ G_i \}_{i \in I} \) be the family of all compactly generated open subgroups of G directed by inclusion. For an \( i \) in \( I \) we define \( E(G_i) \) to be the space of continuous functions \( f \) such that \( \text{supp} f \) is contained in \( G_i \) and for each \( \varphi \) in \( \Phi(G) \)

\[ \sup \{|f| \varphi(g)|: g \in G_i \} = |f|_{i} \leq \infty. \]

The set of pseudo-norms \( |f|_{i} \) define a locally convex topology in \( E(G_i) \). The space \( E(G) = \bigcup_{i \in I} E(G_i) \) is topologized as the inductive limit of the spaces \( E(G_i) \). It is easy to see that \( E(G) \) is complete and that the functions with compact support form a dense subset of \( E(G) \). \( E(G) \) is invariant under left and right translations and both left and right regular representation of \( G \) on \( E(G) \) are jointly continuous.

**Proposition 1.1.** If \( G \) is compactly generated, there is a \( \varphi_0 \) in \( \Phi(G) \) such that \( \varphi_0 \varphi(g)\varphi^{-1}(g) \leq \infty \) and for every \( \varphi \) in \( \Phi(G) \) there is a constant \( M \) and a positive integer \( k \) such that

\[ \varphi(g) \leq M\varphi_0(g)^k \quad \text{for all} \quad g \in G. \]

**Proof.** Let \( U \) be a precompact neighbourhood of the unit element in \( G \) such that \( \bigcup_{n=1}^{\infty} U^n = G \) and \( F \) a finite set such that \( U^2 \subset FU \). Then, of course, \( U^{\infty} \subset F^{n-1}U \). For a \( g \in G \) we put

\[ \tau_\varphi(g) = \inf \{ n: g \in U^n \} \]

and

\[ \tau_\varphi(g) = |U|^{-1} \int |\tau(hg)|dh. \]

Since \( \tau_\varphi \) is left uniformly continuous on \( G \), the function

\[ \sigma(g) = \sup \{ |\tau_\varphi(gh) - \tau_\varphi(h)|: h \in G \} \]

is a non-negative function on \( G \) continuous at the unity and

\[ \sigma(gh) \leq \sigma(g) + \sigma(h), \]

\[ \sigma(g) \geq \tau_\varphi(g) \geq \tau(g) - 2. \]

The continuity of \( \sigma \) at a point \( h \) in \( G \) follows from the inequalities

\[ - \sigma(g^{-1}) \leq \sigma(g) - \sigma(h) \leq \sigma(g). \]

The function \( \varphi_0 \) is defined as follows

\[ \varphi_0(g) = G^{\infty}, \quad \text{where} \quad \varphi = \text{card} F. \]

We have

\[ \int g^{-1}(g)dg = \sum_{n=1}^{\infty} \int \tau(g)dg \leq \sum_{n=1}^{\infty} \sum_{m=1}^{n} G^{-2(n+1)} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{n} G^{-2(n+1)} \leq G\sum_{n=1}^{\infty} C^{-n} < \infty. \]

Now let \( \varphi \) be an arbitrary function in \( \Phi(G) \) and let \( M_\varphi = \sup \{|\varphi(g)|: g \in U\} \). For an element \( g \) in \( G \) we have \( g = g_1 \cdots g_n \) with \( g_j \in U \) and \( \tau(g) = n \). Hence

\[ \varphi(g) \leq \varphi(g_1) \cdots \varphi(g_n) \leq M_\varphi^n = M_\varphi^{|U|} \leq C^{n(g+1)} = M_\varphi^n(g)^k, \]

where \( k = \frac{1}{\log_2 M_\varphi} \) and \( M = M_\varphi^k \).

**Corollary 1.2.** \( E(G) \) is a *-subalgebra of \( L_1(G) \).

**Corollary 1.3.** If \( \varphi \in E(G) \), then for every \( \varphi \) in \( \Phi(G) \) the function \( \varphi \) belongs to \( L_1(G) \).

**Corollary 1.4.** Suppose \( G \) is generated by a relatively compact neighbourhood \( U \) of the unit element of \( G \). If a set \( F \) in \( E(G) \) is bounded, i.e. \( \sup_{f \in F} \|f\|_E \leq C \) for every \( f \) in \( F \) and \( \varphi \) in \( \Phi(G) \), then for every \( \varphi \) in \( \Phi(G) \) the function \( \varphi \) tends to zero uniformly on \( F \) as \( g \) tends to infinity.

**Proof.** Take \( \varphi \) in \( \Phi(G) \) and \( \varepsilon > 0 \). Let \( \varphi_0 \) be the function defined in (1.2) by means of \( U \). Since \( U \) is bounded, we have

\[ \|f\varphi\varphi_0(g)\varphi_0^{-1}(g)\| \leq C_{\varphi_0} \|M_\varphi^n\| \leq C_{\varphi_0}M_\varphi^n \leq \varepsilon, \]

for all \( g \in U^n \) and \( g \) large enough.

2. Approximate identity for \( E(G) \). In this section we prove the existence of a commutative approximate identity in \( E(G) \). For Lio groups this is a consequence of several well-known properties of the fundamental
solution of the heat equation. For other locally compact groups we obtain the result by Yamabe approximation theorem.

Let $G$ be a Lie group. We denote by $BC^0(G)$ the space of bounded infinitely differentiable functions whose all the derivatives are bounded. Let $L(G)$ be the algebra of linear operators on $BC^0(G)$ which commute with right translations on $G$. The Lie algebra $LG$ of $G$ viewed as right-invariant differential operators of the first order on $G$ is contained in $L(G)$ and so is the enveloping algebra of $LG$.

Let $G'$ be another Lie group and let

$$\pi: G \to G'$$

be a homomorphism of $G$ onto $G'$. For a function $f$ in $BC^0(G')$ we correspond a function $\pi*f$ in $BC^0(G)$ defined by $\pi*f(g) = f(\pi(g))$. Clearly, since $\pi$ is onto, $\pi*$ is one-to-one and the subspace $\pi^*BC^0(G')$ of $BC^0(G)$ is stable under $L(G)$.

For a function $f$ on $G$ which is constant on the cosets modulo $\ker \pi$ let $\tilde{f}$ be the unique function on $G'$ such that $\pi*\tilde{f} = f$. We have the algebra homomorphism

$$\pi_*: L(G') \to L(G)$$

defined by

$$(\pi_*T)f = (T\pi*\tilde{f})^*.$$ Clearly

$$\pi_*LG = L(G')$$

and $\pi_*$ coincides with $\pi_*$ if $LG$ and $L(G')$ are viewed as tangent spaces at the unit elements of $G$ and $G'$, respectively.

Let $p \in L_0(G)$ and let $T_p f = p*p$, where $f \in BC^0(G)$. Then $T_p \in L(G)$ and

(2.1) \[ \pi_*T_p = T_{p\pi}. \]

where $p(g) = \int p(\lambda g) d\lambda$.

We say that a metric locally compact group $G$ is a Yamabe group, if there exists a sequence $U_1 \supset U_2 \supset \ldots$ of relatively compact neighbourhoods of the unit element in $G$ with $\bigcap_{n=1}^{\infty} U_n = 1$ and a descending sequence $N_1, N_2, \ldots$ of normal subgroups of $G$ such that $X_G = U_1$ and $G_j = G/N_j$ is a Lie group. (A discrete group is regarded as a 0-dimensional Lie group.)

Let $G$ be a Yamabe group and let $\pi$ and $\pi_j$ be the natural homomorphisms

$$\pi: G \to G_j, \quad \pi_j: G_{j+1} \to G_j.$$ Of course, $\pi^{j+1} = \pi_{j+1} \pi_j$.

By an easy inductive procedure, for each $j$ we select a basis

$$X_{j1}, \ldots, X_{j\nu(j)}$$

in the Lie algebra $L_{G_j}$ of $G_j$ in such a way that

$$\pi_{j-1} X_{j \lambda} = \begin{cases} X_{j-\lambda} & \text{for } k < \dim G_{j-1}, \\ 0 & \text{for } k > \dim G_{j-1}. \end{cases}$$

Then, if

$$A_j = \sum_{T_{X_j}} X_{j1},$$

we have $A_j \in L(G_j)$ and, moreover,

(2.2) \[ \pi_{j-1} A_j = A_{j-1}. \]

The following properties of the operator $A_j$ were established by L. Garding [3] and E. Nelson [9], cf. also [10].

(i) $A_j$ is a self-adjoint negative operator on $L_2(G_j)$,

(ii) for real positive $t$ the operator $\exp(tA_j)$ is of the form $\exp(tA_j) = p_j(t) f_j$, where $p_j(t) \geq 0$ and $\int_0^t p_j(t) dt = 1$, consequently

(iii) $\exp(tA_j) \in L(G_j)$,

(iv) for each $\varphi \in \Phi(G_j)$, for every positive real $s$ there is a constant $C$ such that $\int \frac{|p_j(t)\varphi(t)|^s dt}{\varphi(t)} \leq C$ for all $t < s$, whence also

$$\int \frac{|p_j(t)\varphi(t)|^s dt}{\varphi(t)} \leq C$$

for all positive $t$,

(v) for every $f$ in $L_0(G_j)$, $1 < p < \infty$, $\lim_{t \to 0^+} \int |p_j(t)f(t)|^p dt = 0$ and

also, if $f$ is uniformly continuous, $\lim_{t \to 0^+} |p_j(t)f(t)|^p = 0$.

**Lemma 2.1.** For all $t \in \mathbb{R}^+$ we have $p_j(t) \in B(G_j)$.

The proof follows from the following evolution and (2.3) iv.

$$\int p_j(t)\varphi(t) \delta_{G_j}(g) = \int p_j(\lambda) p_j(\lambda^{-1}) \varphi(t) \delta_{G_j}(\lambda^{-1}) g) d\lambda$$

$$\leq \int p_j(\lambda) p_j(\lambda^{-1}) \varphi(t) \delta_{G_j}(\lambda^{-1}) d\lambda$$

$$\leq \left( \int |p_j(\lambda)\varphi(t)|^2 d\lambda \right)^{1/2} \left( \int |p_j(\lambda^{-1})\varphi(t)|^2 d\lambda \right)^{1/2}$$

$$= \left( \int |p_j(\lambda)\varphi(t)|^2 d\lambda \right)^{1/2} \left( \int |p_j(\lambda)\varphi(t)|^2 d\lambda \right)^{1/2}.$$
Proof. We take an $f$ in $E(G_j)$ and a $\psi$ in $\Phi(G_j)$ and we note that for an $s > 0$ there is a constant $C$ such that

$$\|\widehat{p_j^* f} \psi(g)\| \leq C$$

for all $0 < t < s$.

In fact,

$$\|\widehat{p_j^* f} \psi(g)\| \leq \int p_j^*(h) |f(h^{-1})g(h)\psi(h)\psi(h^{-1})g| dh$$

$$\leq \int p_j^*(h) \psi(h) dh \sup_{\lambda} \int |f(h^{-1})g(h)\psi(h^{-1})g| dh$$

$$= \int p_j^*(h) \psi(h) dh \cdot \|f\|_\psi,$$

whence (2.4) follows by (2.3) iv.

Now for an $s > 0$, in virtue of (2.4), we apply Corollary 1.4 to select a compact set $A$ in $G_j$ such that

$$\|\widehat{p_j^* f} \psi(g)\| \leq s/2$$

and $|f(g)\psi(g)| \leq s/2$

for all $g$ outside $A$ and $t < s$. Then

$$\|\widehat{p_j^* f} - f\| \leq \sup_{\psi, g, \lambda} \|\widehat{p_j^* f} - f\|_\psi$$

for $g$ in $A$ and all $t \in \mathbb{R}^+$, and

$$\|\widehat{p_j^* f} - f\| \leq s$$

for $g \in A$ and $t < s$,

which, in virtue of (2.3) v, completes the proof of the lemma.

Now let $m_j$ be the normalized Haar measure on $X_j$ viewed as a Borel measure on $G$. Since $X_j$ is a normal subgroup, then for all continuous functions $f$ on $G$ we have

$$m_j f = m_j,$$

$$j = 1, 2, \ldots$$

Moreover, for $k \leq j$, we have

$$m_k m_j = m_j.$$

For a positive $t$ and $j = 1, 2, \ldots$ we write

$$\overline{p_j^* (g)} = p_j^* (e^t g), \quad g \in G.$$
Since every locally compact group contains an open Yamabe subgroup, cf. [3], chapter IV, we may assume that $G$ is Yamabe.

Now let $p_j, j = 1, 2, \ldots$ be defined as in (2.7). We prove that they satisfy (i)-(v) of the theorem.

(i) and (ii) follow immediately from the definition. (iii) is an easy consequence of (2.5) and (2.6). Thus only (iv) and (v) require any proving.

To this end we prove first

**Lemma 2.3.** For every $\varphi \in \Phi(G)$ there is a constant $C_\varphi$ such that

\[
\int p_j(g)\varphi(g)\,dg \leq C_\varphi.
\]

**Proof.** Let $\varphi \in \Phi(G)$ and $\varphi(g) > 0$ for all $g \in G$ and let $\varphi_j$ be defined as in (2.8) for $j = 1$. Then, since all functions $\tilde{\varphi}_j$ and the measures $m_j$ are non-negative, by (2.6), we have

\[
\int p_j^*\varphi_j(g)\,dg \geq \int \tilde{\varphi}_j^*m_j(g)\varphi(g)\,dg.
\]

Hence

\[
\int \tilde{\varphi}_j^*(g)\varphi(g)\,dg \leq \int \tilde{\varphi}_j^*(h^{-1}g)\,dm_j(h)\varphi(h^{-1}g)\,dg.
\]

Thus, by (2.9), the lemma follows.

From Lemma 2.3 property (iv) follows almost immediately. Let $\varphi \in \Phi(G)$ and let $C_\varphi$ be the constant established in Lemma 2.3. Then, for an $f \in E(G)$, we have

\[
|p_j^*f(g)|\varphi(g) \leq \int p_j(h)(f(h^{-1}g))\varphi(g)\,dh
\]

and $j = 1, 2, \ldots$, where $K = \int \varphi(g)^{-1}\,dm_1(g)^{-1}$ and $C$ is the bound for $\int \tilde{\varphi}_j^*(g)\varphi(g)\,dg$ stated in (2.3) iv. Thus, by (2.9), the lemma follows.

Now, by Lemma 2.3, from (2.11) we infer

\[
|p_j^*m_j^*f-f^*f|_p < \varepsilon/3
\]

for all $i > j_1$, which together with (2.12) and (2.11) again proves (v) and at the same time completes the proof of Theorem 2.1.

**3. Cyclic vectors of induced representations.** Let $G$ be a locally compact group. Let $K(G)$ denote the space of continuous functions with compact support on $G$. As we have noticed before, $K(G)$ is dense in $E(G)$. The classical Blattner construction shows that a unitary representation of $G$ induced by a cyclic representation is defined by a positive-definite measure $\mu \in K(G)$, e.g. [2]. We shall prove later that $\mu$ is bounded on $E(G)$, that is $\mu$ belongs to $E(G)'$. Let us first recall Blattner construction.

A Radon measure $\mu$ on $G$ is bounded on $E(G)$ if $\int |\varphi|d|\mu| < \infty$, we then write $\mu \in E(G)'$.

Let $\mu \in E(G)'$. If $\mu$ is positive-definite, i.e. $\langle f^*f, \mu \rangle > 0$ for all $f \in E(G)$, it defines a unitary representation of $G$ as follows. Let $L^2_\mu = E(G)/I$ be a pre-Hilbert space with a strictly positive-definite inner product $\langle \cdot, \cdot \rangle$.

Then $L^2_\mu$ is natural mapping of $E(G)$ onto $E(G)/I$. Moreover, if $L^2_\mu = L^2_\mu$, then $L^2_\mu$ is stable under $L^2_\mu$, and so $L^2_\mu$ acts on $L^2_\mu$ and is unitary respect to $\cdot, \cdot$. As such it extends to the completion $L^2_\mu$ and is denoted by $L^2_\mu$. The mapping $\varphi \rightarrow L^2_\mu$ is the required representation. For a function $f \in E(G)$ we write

\[
L^2_\mu = \langle f^*f, \mu \rangle.
\]

Let $H$ be an open compactly generated subgroup of $G$ and let $p_j, j = 1, 2, \ldots$ be the approximate identity in $E(G)$ which satisfies (i)-(v) of Theorem 2.1, and supp $p_j \subset H$. Let $a_j$ be a sequence of positive numbers such that the series

\[
\sum_{j=1}^\infty a_j|p_j^*p_j|
\]

is convergent to a function $\xi$ in $E(G)$. To see that such $a_j, j = 1, 2, \ldots$ exist we take the function $\varphi_\xi$ in $\Phi(H)$ as defined in Proposition 1.2 in $\Phi(H)$ and select $a_j$ such that $\sum_{j=1}^\infty a_j|p_j^*p_j| < \infty$.

**Theorem 3.1.** If $\mu$ is a positive-definite measure bounded on $E(G)$, then the vector $\xi$ where $\xi$ is defined by (3.3) is cyclic for the representation $G\mu \rightarrow L^2_\mu$.

**Proof.** Let $P$ be an orthogonal projection in $H^\ast$ which commutes with every $L^2_\mu, \mu \in G$, and $P^2 = 0$. Then $P$ commutes with every $L^2_\mu$,
\[ f \in E(G). \] For an arbitrary \( i \) we have
\[
(3.3) \quad 0 = \left( P_{i} (p_{*,p_{i}})^{-} \right) = \sum_{j=1}^{a_{j}} \left( \left. P_{i} (p_{*,p_{i}})^{-} \right| (p_{*,p_{i}})^{-} \right).
\]
But for all \( i, j = 1, 2, \ldots \), by (3.1), we have
\[
\left( P_{i} (p_{*,p_{i}})^{-}, (p_{*,p_{i}})^{-} \right) = \left( (L_{0} f_{i}, (p_{*,p_{i}})^{-}) \right) = \left( P_{i} (p_{*,p_{i}})^{-}, (p_{*,p_{i}})^{-} \right) = \left( (p_{*,p_{i}})^{-}, (p_{*,p_{i}})^{-} \right).
\]
Since \( a_{j} > 0 \), (3.3) implies \( P_{i} (p_{*,p_{i}})^{-} = 0 \) for all \( i, j = 1, 2, \ldots \). Hence, since \( p_{j} \) is an approximate identity in \( E(G) \), \( P_{i} \) is equal to zero for all \( i, j = 1, 2, \ldots \).

Finally, for arbitrary \( f, h \) in \( E(G) \)
\[
(P_{i}, h) = \lim_{i \to \infty} \left( P_{i} (p_{*,p_{i}})^{-}, h \right) = \lim_{i \to \infty} \left( P_{i} (f^{*} h^{*})^{-}, 0 \right) = 0.
\]
Which shows that \( P \) is zero and so \( \mathcal{F} \) is cyclic.

**Theorem 3.2.** If \( \mu \) is a non-negative positive-definite Radon measure on \( G \), then \( \mu \) is bounded on \( E(G) \).

**Proof.** It suffices to show that if \( H \) is an open compactly generated subgroup of \( G \) then \( E(H) \subset L_{1}(\mu) \). Since \( \mu \) is positive definite, for every non-negative continuous function \( f \) with compact support in \( H \), \( f^{*} \mu^{*} f^{*} \) is a non-negative positive-definite continuous function and as such is bounded. If \( \varphi_{x} \) is a function in \( F(H) \) as defined in Proposition 1.1, \( \varphi_{x}^{-1} \) is integrable.

\[ \int f^{*} \mu^{*} f^{*}(g) \varphi_{x}(g)^{-1} dg = \int f^{*} \varphi_{x}^{-1}(g) \mu(g). \]

But, since \( \varphi_{x} \) is submultiplicative, there exists a positive constant \( M \) such that
\[
M \varphi_{x}^{-1}(g) \leq \int f^{*} \varphi_{x}^{-1}(f(g)) dg.
\]

In fact,
\[
\int f^{*} \varphi_{x}^{-1}(f(g)) dg = \int f^{*}(s) \varphi_{x}^{-1}(g^{*} s^{-1} g) \varphi_{h}(h^{-1}) d \mu(h) \geq \int f^{*}(s) \varphi_{x}^{-1}(s^{-1} g) d s \int \varphi_{h}(h^{-1}) \varphi_{x}^{-1}(h) d h \cdot \varphi_{x}^{-1}(g).
\]
Consequently,
\[
\int \varphi_{x}^{-1}(g) d s < \int f^{*} \mu^{*} f^{*}(s) d \mu(s) \leq \int \varphi_{x}^{-1}(g) d s < \infty.
\]

But, for every \( f \in E(H) \), \( f |_{\varphi_{x}} < C \), hence \( |f| < C \varphi_{x}^{-1} \), which shows that \( f \in L_{1}(\mu) \), what we had to prove.

**Theorem 3.3.** A unitary representation of a first countable, locally compact group \( G \) induced by a cyclic representation of a subgroup \( H \) is cyclic.

**Proof.** The induced representation is of the form \( g \mapsto \mathcal{E}_{g} \), where \( \mu \) is the Radon measure defined as follows
\[
\int f^{*}(g) d \mu(g) = \int f^{*}(h) \delta_{x}^{(0)}(h) \delta_{t}^{-1}(h) \delta_{x}^{(1)}(h) d \mu(h) ,
\]
where \( g(h) \) is a normalized positive-definite continuous function on \( H \) (cf. [2]).

We show that \( \mu \in E(G)^{\prime} \). In fact,
\[
\int f^{*}(g) d \mu(g) \leq \int f^{*}(h) \delta_{x}^{(0)}(h) \delta_{t}^{-1}(h) d \mu(h) ,
\]
whence, since \( \delta_{x}^{(0)}(h) \delta_{t}^{-1}(h) d \mu(h) \) is a positive, positive-definite measure, by Theorem 3.2, the right hand side of (3.4) is finite for all \( f \in E(G) \), so \( \mu \in E(G)^{\prime} \) and the theorem follows from Theorem 3.2.

**References**


