(b) In the case of a real space the proof is simpler. The set $\Omega = \{ x \in \mathbb{R} : |x + |y| > |z| \} (R$ is the real field) contains at least one of the rays $[0, +\infty), (-\infty, 0]$ and by assumption $\omega$ is negative. Then, $\omega^k \Omega$ for some positive integer $k$.

Remark. It can be easily seen, by taking the supremum norm on $C^1$, for example, that there may be no positive integer $k$ such that $|x + \omega^k y| > |x|$. 

PROPOSITION 2. Let $X$ be a real or complex normed vector space and let $A$ be a linear operator from $X$ into itself such that $\|Ax\| \leq \|x\|$ for all $x \in X$ (i.e., $A$ is a contraction). Let $\lambda$, $\mu$ be eigenvalues of $A$ such that $|\lambda| = 1$ and $\lambda \neq \mu$. If $u$ and $v$ are eigenvectors of $A$ corresponding to $\lambda$ and $\mu$, respectively, then $u$ is orthogonal to $v$.

Proof. Let $\alpha$ be an arbitrary scalar. We have for all positive integers $k$,

$$(u + \alpha v) \geq \|A^k(u + \alpha v)\| = \|A^k u + \alpha \mu^k v\|.$$

whence, denoting $\omega = \mu/\lambda$,

$$(*) \quad \|u + \omega \alpha v\| \leq \|u + \omega \alpha v\|.$$

If $|\alpha| < 1$, then $|\alpha| < 1$ and letting in $(*)$ $k \to +\infty$, we obtain $|u| \leq |u + \omega \alpha v|$. If $|\alpha| = 1$, then we have $0 < \arg \omega < 2\pi$ (since $\omega \neq 1$) and making use of Proposition 1 we obtain from $(*)$ that $|u| \leq |u + \omega \alpha v|.$

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(522)
If \( f \in \mathcal{X}(G) \) then

\[
|f|_{S} = \sup_{h \in \mathcal{H}} \sum_{\gamma} |f(\gamma, h)|
\]

and, because \( G \) is unimodular, if \( \gamma = (e, h) \) then

\[
||R_{x}f||_{S} = \sup_{x \in \mathcal{H}} \sum_{\gamma} |f(x \gamma, h) |
\]

\[
\geq \sum_{\gamma} |f(\gamma)| = \sum_{\gamma} |f(\gamma, e h)|.
\]

Let \( E_{1}, E_{2}, \ldots \) be disjoint open subsets of \( H \) and \( \gamma_{1}, \gamma_{2}, \ldots \) elements of \( \Gamma \) with \( \gamma_{i} h_{i} \in E_{i} \) for all \( i \). Clearly if \( i \neq j \) then \( \gamma_{i} \neq \gamma_{j} \). Let \( \nu_{i} \) be a continuous function \( G \to [0, 1] \) with support contained in \( (\gamma_{i}) \times E_{i} \) and \( \nu_{i}(\gamma_{i}, \gamma_{i} h_{i}) = 1 \). Then \( w = \sum_{i} \nu_{i} e_{i} \) converges in \( S \) because

\[
\sum_{i} \sum_{j} j^{-1} \nu_{i}(\gamma_{j}, h) = k^{-1} \nu_{i}(\gamma_{i}, h) = 1 \quad \text{if} \quad h \in E_{i} \quad \text{for some} \quad h > 1,
\]

\[
= 0 \quad \text{otherwise}.
\]

However

\[
||R_{x}u||_{S} \geq \sum_{i} |\nu_{i}(\gamma_{i}, \gamma_{i} h)| = \sum_{i} i^{-1} = \infty
\]

so that \( S \) is not symmetric.

2. Assymetry of a class of Wiener-like Segal algebras. Let \( G \) be a locally compact group and \( \Gamma \) a discrete subgroup such that the left coset space \( G/\Gamma \) is compact. On \( \mathcal{X}(G) \), the set of continuous complex valued functions on \( G \) with compact support, we define the norm

\[
||f||_{S} = \sup_{x \in \mathcal{H}} \sum_{\gamma} |f(\gamma, h)|.
\]

The completion of \( \mathcal{X}(G) \) in this norm is the Segal algebra \( S \) with which we are concerned ([51], p. 23).

Let \( H \) be a compact group, \( \Gamma \) a group of automorphisms of \( H \) and let \( \mathcal{H} = \{ \gamma h; \gamma \in \Gamma \} \) infinite. Let \( G \) be the semi-direct product of \( I \) and \( H \), that is the product space \( I \times H \) (\( I \) has the discrete topology) with multiplication \( (\gamma, h)(\gamma', h') = (\gamma \gamma', (h \gamma')h) \). Identifying \( I \) with \( (\{\gamma\}, \gamma \in \Gamma) \) and \( H \) with \( \{ (e, h); h \in H \} \), we see that the pair \( G, \Gamma \) satisfy the conditions of the preceding paragraph and we shall show that in this case the Segal algebra is not symmetric.

3. Approximate units in the augmentation ideal of a group algebra. The augmentation ideal \( I_{0}(\mathcal{G}) \) of \( L^{1}(\mathcal{G}) \) is \( \{ f \in L^{1}(\mathcal{G}); \int_{\mathcal{G}} f = 0 \} \) where \( \mathcal{I} \) is a left Haar measure on the locally compact group \( G \). We shall say that a Banach algebra \( A \) has a bounded right approximate unit if there is \( C > 0 \) such that \( \gamma \in A, \gamma > 0 \) then there is \( \gamma \in A \) with \( \|\gamma\| \leq C \) and \( \|\gamma - e_{0}\| < \epsilon \). A has a right approximate unit if for all \( \gamma \in A, \gamma > 0 \) there is \( \gamma \in A \) with \( \|\gamma - e_{0}\| < \epsilon \). If a group \( A \) is the product of two others. In the convolution algebra \( L^{1}(\mathcal{G}) \) a product is a continuous function so this algebra cannot have a bounded approximate unit but the usual construction for a bounded approximate unit in \( L^{1}(\mathcal{T}) \) gives an unbounded approximate unit in \( L^{1}(\mathcal{T}) \). Thus some Banach algebras
have unbounded but no bounded right approximate units. In [3] it was shown that if 
G is amenable if and only if \( L_0(G) \) has a bounded approximate right unit. In this example we will show that if \( G = F_2 \) the free group

on two generators \( a, b \) then \( L_0(G) \) does not even have an unbounded approximate right unit.

It will be convenient to represent elements of \( L\left(F_n\right) \) as linear combinations of group elements. Let \( r = a + b - b = a \in L_0(F_n) \). We shall define a function \( \sigma : F_2 \to \{0, 1\} \) such that for all \( g \in G \)

\[
\sigma(g) + \sigma(ag) - \sigma(bg) - \sigma(bg) = 1.
\]

If \( g \in L_0(G) \) is defined by

\[
\sigma(\xi a_{\xi} g) = \sum a_{\xi} \sigma(g) \quad \text{where} \quad a_{\xi} \in C
\]

and \( \Sigma \) means \( G \times G \), then \( \sigma(r) = 1 \) and

\[
\sigma(r) = \sigma(\xi a_{\xi} g) = \sum a_{\xi} \sigma(\xi a_{\xi} g) = \sum a_{\xi} \sigma(\xi a_{\xi} g) + \sigma(\xi a_{\xi} g) = \sum a_{\xi} \sigma(\xi a_{\xi} g) = \sum a_{\xi} = 0.
\]

Thus \( r \) does not lie in the closed right ideal it generates.

To define \( \sigma \), for \( g \in F_n \) let \( [g] \) denote the sum of the absolute values of the exponents of \( a \) and \( b \) when \( g \) is written in reduced from, so that \( |g| = 0, |a^n b^m a^{-n} b^{-m}| = |g| \) and so on. We define \( \sigma \) by induction on \( |g| \) taking \( \sigma(g) = 1 \) if \( |g| \leq 1 \) and when \( |g| = n + 1, n \geq 1 \) we take \( \sigma(g) = 1 \) if

(i) \( g = a h, a^{-1} h \), or \( b^{-1} h, h = g, \)

(ii) \( g = a b h, a^{-1} b h, h = n - 1, \sigma(h) = \sigma(a h) = \sigma(b h) = 0 \) or \( g = b a^{-1} h, h = n - 1, \sigma(h) = \sigma(b h) = 1, \sigma(b h) = 0 \) otherwise.

Note that for all \( g \in F_n \) we have \( \sigma(bg) \leq \sigma(g) \); this follows by (i) if \( |bg| < |g| \), is obvious if (ii) or by (i) in the remaining case. Put

\[
\tau(g) = \sigma(g) + \sigma(bg) - \sigma(bg) - \sigma(bg).
\]

Suppose \( |g| < |ag| \). Then \( \sigma(bg) = 1 \) by (i) (or by specific definition if \( g = a \)) and by the above remark \( \sigma(g), \sigma(bg) = (1, 0), (0, 0) \) or \((1, 1) \). In the first of these cases \( \sigma(bg) = 1 \) by (ii) and so \( \tau(g) = 1 \) whereas in the second and third \( \sigma(bg) = 0 \) and again \( \tau(g) = 1 \). A similar argument applies if \( |g| > |ag| \).

4. Translation invariant subspaces which are not mapped onto closed subspaces by the canonical quotient map. If \( G \) is a locally compact group and \( \mathcal{G} \) a closed normal subgroup with a left Haar measure \( \lambda \) then \( T : L(G) \to L(G) \) is defined by \( (T\xi)(a\xi) = \int f(\xi(a\xi)) d\lambda(g) \).

If \( G \) is amenable then the image under \( T \) of a closed \( G \) invariant subspace of \( L(G) \) is a closed subspace of \( L(G) \) ([4], p. 177).

**Theorem.** If \( L_0(G) \) does not contain an approximate right unit then \( L(G) \), where \( G = \mathbb{Z} \times G \), contains a closed \( G \) invariant subspace \( E \) such that \( T E \) is not closed.

**Proof.** We shall consider \( G = \{0, g \}; g \in G \) = \( T \) and \( L(G) \) as embedded in \( M(T) \) in this way. Choose \( \phi \in L_0(G) \) with \( |\phi| = 1 \), \( |\phi' - \phi| \geq \delta > 0 \) for all \( \phi' \in L_0(G) \) and \( \phi \in L(G) \) with \( |\phi| = \|\phi\| = 1 \). For \( n \in \mathbb{Z} \) let \( f_n : L(G) \to \mathbb{C} \) be defined by

\[
\begin{align*}
\phi_n(n, g) &= \phi(g), \\
\phi_n(-n, g) &= n^{-1} \phi(g), \\
\phi_n(m, g) &= 0 \quad \text{for} \quad |m| \neq n
\end{align*}
\]

and let \( E \) be the closed right \( G \) invariant subspace generated by \( \{f_n \in \mathbb{Z} \} \).

We shall show that

\[
(f_n \in L_0(G)) = f_n \in L(G) \cap \ker T
\]

Clearly the first set is contained in the second. If \( E \) lies in the second set then \( T E = 0 \) and there is a sequence \( \{a_n\} \) from \( L(G) \) with \( f_n a_n \to E \).

Thus \( T \{f_n \in L(G)\} \to 0 \). As

\[
\begin{align*}
T(f_n a_n) &= \int \int n^{-1} \phi(h) a_n n^{-1} \delta(h) d\lambda(h) d\lambda(k) \\
&= n^{-1} \int \phi(h) a_n d\lambda(k) \\
&= n^{-1} \int a_n d\lambda(k)
\end{align*}
\]

This implies \( f_n a_n \to 0 \) as \( m \to \infty \) so that \( \lim m \to \infty f_n (a_m - \psi f_n a_n d\lambda) = f_n \)

and \( f_n (a_m - \psi f_n a_n d\lambda) \) is in \( f_n \in L_0(G) \).

As a Banach space \( L(G) \) is the direct sum of a sequence of copies of \( L(G) \), \( \ker T \) is a direct sum of its intersections with these copies and \( E \) is a direct sum of the \( f_n \in L(G) \) (by a direct sum of Banach spaces \( X_1, X_2, \ldots \) we mean \( \{x : x \in X_1 \times X_2 \times \cdots \} \) with norm \( \|x\| = \sum_{i=0}^{\infty} \|x_i\| \) we see

\[
\delta(f_n, E \cap \ker T) = \delta(f_n, f_n L(G) = \inf_{\|f_n \psi - \phi\| = \delta} \|f_n \psi - \phi\|)
\]

where \( \delta \) denotes the distance from the point to the subspace.

\( T \) gives a one to one map \( r \) of the Banach space \( E/E \cap \ker T \) onto
5. Failure of the multiplier result in $L^\infty \otimes L^\infty$. The theory of multiplier spaces introduced by Figiel–Talansky is now well developed. The basis of the theory as given by Rieffel [6] is that, with the usual identification of the operator space $\mathcal{L}(E, E')$ with the space $(E \otimes E')'$, the operators from $L^p(E)$ into $L^p(G)$ which commute with right translations correspond to those elements $(L^p(G) \otimes L^p(G))'$ which are zero on tensors of the form $g \otimes y - y \otimes g y$ where $g \in L^1(G)$, $g y \in L^1(G)$, $y \in G$ and $y y$ denotes the right and left translates of $y$. Thus these operators can be identified with elements of $L^\infty(G) \otimes L^\infty(G)*$ (for convenience we are assuming $G$ is unimodular and $L^1(G)$ and $L^\infty(G)$ are paired by the bilinear form $\int_g f(y) g(y^{-1}) dy$. The most difficult part of this theory is to show that if $\pi$ is defined on $L^p(G) \otimes L^p(G)$ by $\pi(x \otimes y) = x y$ then $\pi$ is the closed linear span of the tensors of the form $g y - y \otimes g y$ so that $L^\infty(G) \otimes L^\infty(G)$ can be identified with $L^\infty(G \otimes G$. In a similar way operators from $L^p(E)$ into $L^p(G)$ commuting with convolution by $L^1(G)$ functions can be identified with the dual of the quotient of $L^\infty(G) \otimes L^\infty(G)$ by the closed linear span of the tensors $x \otimes y - y \otimes x y$ (as $L^1(G)$). However in most cases an operator commutes with translation if and only if it commutes with convolution by $L^1(G)$ functions.

Certain cases in which $p$ and $q$ are infinite are not covered by [6] and some of the results are false in these cases.

**Lemma.** There is a non zero functional in $L^\infty(T)$ which is translation invariant and zero on $C(T)$.

**Proof.** Let $\tau_1, \tau_2, \ldots$ be an enumeration of the rationals and $E_n = \bigcup \{ x \in E : | x - \tau_i | < 2^{-n} \}$. If $\tau$ is any non void open arc in $T$ and $w_1, \ldots, w_n \in T$ then $I$ contains a point of $[w_n, w_1) \cap E_\tau$ and hence some non void open subarc $I_n$ of $w_n E_n$. Thus $I$ contains a non void open subarc $I_n$ of $w_n E_n$. Eventually we find a non void open arc $I_n$ contained in $I \cap w_n E_n$ and $w_n E_n$. Let $\chi$ be the characteristic function of $E_\tau$ and $f$ a convex combination of the translates of $\chi$ by the $w_n$. Then $f = 1$ on $I_n$ so if $\lambda \in C$ with $| f - \lambda I_{I_n} | < \delta$ then $| \lambda | < \delta$. However if $\lambda = \int \chi d\mu$ then $\mu \leq \pi$ so that

$$\int \chi f(x) \mu = \int x f(x) \mu = k \leq \pi$$

and $| f - \lambda I_{I_n} | < \delta$ implies $2 \pi | \lambda | < k + \frac{1}{8} \pi < \frac{3}{8} \pi$ and hence $| \lambda | < \frac{1}{8} \pi$, which contradicts $| \lambda | < \frac{1}{8} \pi$. Thus in $L^\infty(T)$ the closed convex hull $K$ of the translates of $\chi$ is distance at least $1/4$ from the constant functions. Hence, by the Hahn–Banach theorem there is $\beta$ in $L^\infty(T)$ with $\beta(\lambda) = 0$, $\| \beta \| \geq 1$ for all $\lambda \in K$. Let $M$ be a translation invariant mean on $L^p(T)$, the space of all bounded functions on $T$ ([2]), Theorem 17.5, and define

$$a(f) = M(\beta(f))$$

where $(\beta (f))(w) = \beta(f)(w)$, $f \in L^\infty(T)$, $w \in T$. Clearly $a L^\infty(T)$, $a$ is translation invariant, $a(T) = 0$ and $Re a \geq 1$ on $K$ so that $a \neq 0$, $a | G(T)$ is thus, by uniqueness, a multiple of Lebesgue measure and so is zero because $a(l) = 0$.

When the multiplier result $\pi = \text{Span}(\sigma \otimes y, \sigma \otimes \varphi \otimes y)$, $y \in L^1(G)$, holds, $T$ is an operator from $L^p(E)$ into $L^p(G)$ which commutes with convolution by $L^1(G)$ functions and $\varphi \otimes \varphi(y) \neq 0$ then $T(\varphi(y)) = 0$. A similar remark applies to operators commuting with translation. When $p = q = \infty$ we define the convolution $\sigma \otimes \varphi = L^1(G)$, $\sigma \otimes \varphi(y) \neq 0$ by $(\sigma \otimes \varphi)(F) = (\sigma \otimes \varphi)(F)$ and the translation $gg(y) = (g, g y)$. In particular if $\sigma$ is translation invariant then $\sigma(y) = 0$ and if $y$ is zero on $G(T)$ then $a(y) = 0$.

Let $\sigma, \tau \in L^\infty(T) \setminus C(T)$ such that $\tau \sigma = 0$ (we could choose $\tau, \sigma$ such that $\tau = -\tau = -\tau = -\tau$ for all $\tau \in T$) and let $\sigma$ be a $\tau \in L^\infty(T)$ with $\sigma = 0$ on $C(T)$, $\sigma \neq 0$, $\sigma \neq 0$ defines $\tau$ by $T(\sigma) = -\tau$ so that $T$ is an operator from $L^\infty(T)$ into $L^\infty(T)^\ast$. Because $a$ is zero on $C(T)$, we see that $T(\sigma a) = 0$ and because $a \neq 0$ we see that $T(\sigma a F) = 0$ for all $a \in L^1(T)$, $F$ in $L^1(T)$, and so $T$ commutes with convolution by $L^1(T)$ functions. However $\tau \sigma = 0$ and $T(\tau) = \sigma \tau(\tau) \sigma - 0$ so $\sigma \neq 0$ is not the closed span of the tensors $x \otimes y - y \otimes x y$ (as $L^1(T)$, $x \otimes L^1(T)$, $a L^1(T)$).

In a similar way defining $T : L^\infty(T) \rightarrow L^\infty(T)^\ast$ by $T(F) = a(F) \lambda$ where $a$ is the functional in the Lemma and $\lambda$ is the Lebesgue integral $\lambda(F)$,$$
abla = \int F(x) d\mu,$$ we get $g(\lambda) = a(F)g(\lambda) - a(F) \lambda$ and $T(g(\lambda)) = a(F) \lambda$, so that $T$ commutes with translations. However if $\varphi \in L^\infty(T)$ with $\varphi \neq 0$ then $\varphi \lambda(\tau) - 0 = 0$ and then $T(\varphi(y)) = 0$ so that $\varphi \neq 0$ and $T(\varphi(y)) = 0$. However if $\varphi \neq 0$ then $\varphi \sigma \neq 0$ and hence $T(\varphi(y)) = 0$, and $\varphi \sigma \neq 0$ so that $T(\varphi(y)) = 0$ so that $T$ does not commute with convolution by $a$. $\Box$

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On commutative approximate identities and cyclic vectors of induced representations

by

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Abstract. It is shown that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity. This is applied to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

The aim of this paper is twofold. To show that every locally compact group has a commutative approximate identity for $L_1(G)$ which consists of continuous positive functions which decrease very rapidly at infinity and apply this to a construction of a cyclic vector for a representation of a locally compact first countable group induced by a cyclic representation.

A construction of commutative approximate identity for a $C^*$-algebra was given by J. F. Aarnes and R. V. Kadison [1]. Their method uses $C^*$-algebras technique and does not apply to the group algebras. It would be interesting to know whether there exists an approximate identity for $L_1(G)$ consisting of commuting continuous functions with compact support.

The fact that for a first countable group representations induced by cyclic representations are cyclic was first proved by F. Greenleaf and M. Moskowitz [5] and [6] and a construction of a cyclic vector for such representations was claimed by the authors [7]. Unfortunately [7] makes use of a statement in [2], p. 49, which is false, as it has been recently discovered by R. Goodman. The construction presented here avoids this difficulty and (for induced representations) improves the construction given in [7].

Very briefly the idea is the following. For a Lie group $G$ the fundamental solution $u(g, t) = e^{t}g)$, of the heat equation is a one-parameter semi-group of non-negative functions $e^{t}$, that is $e^{s}e^{t} = e^{s+t}$ for all positive real $s$, $t$. Moreover $e^{t}f$ tends to $f$ as $t$ tends to zero, and for a fixed $t$ the function $e^{t}$ decreases faster than exponentially at infinity. In short, functions $e^{t}$, $t \in R^+$, form an approximate identity for $L_1(G)$ consisting of commuting rapidly decreasing functions.