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On contractions of normed vector spaces

by

EMERIC DEUTSCH* (New York)

Abstract. Let X be a real or complex normed vector space, let $x, y \in X$, and let ω be a scalar such that $0 < \arg\omega < 2\pi$. It is shown that there exists a positive integer k such that $\|x+\omega^k y\| \geqslant \|x\|$. As an immediate consequence one obtains that the eigenvector corresponding to a unimodular eigenvalue of a contraction A is orthogonal, in the sense of G. Birkhoff, to the eigenvector corresponding to any other eigenvalue of A.

Let X be a normed vector space. Koehler and Rosenthal [5] have proved that if A is a linear isometry on X, then the eigenvectors corresponding to distinct eigenvalues of A are orthogonal (we say that x is orthogonal to y ($x, y \in X$) if $\|x\| \leqslant \|x+\alpha y\|$ for all scalars α [2]). The same result has been obtained by I. Istrăţescu [4] for a linear contraction A acting in a Banach space X and whose spectrum lies on the unit circumference. In both [4] and [5], semi-inner-product [6] and Banach limit [1] (generalized limit in [7]) techniques are used.

The purpose of this note is to prove, by a simpler method, a generalization of the above mentioned theorems. In our approach, the brunt of the proof is taken by a proposition concerning the geometry of a normed vector space and which may present interest in its own right.

PROPOSITION 1. *Let X be a real or complex normed vector space, let $x, y \in X$, and let ω be a scalar such that $0 < \arg\omega < 2\pi$. Then, there exists a positive integer k such that $\|x+\omega^k y\| \geqslant \|x\|$.*

Proof. (a) First we consider the case of a complex space. We denote $\Omega = \{\zeta \in C: \|x+\zeta y\| \geqslant \|x\|\}$, where C is the complex field. It is easy to prove that the complement Φ of Ω is a convex bounded set and $0 \notin \Phi$, $0 \in \bar{\Phi}$. Then Ω contains a closed semiplane $\Sigma = \{\zeta \in C: \beta - \pi/2 \leqslant \arg\zeta \leqslant \beta + \pi/2\}$, say. Let $\omega = \varrho e^{i\theta}$, where $0 < \theta < 2\pi$. We may assume $\varrho \neq 0$. If $\theta/2\pi$ is rational, then clearly $\omega^k \in \Sigma \subset \Omega$ for some positive integer k. If $\theta/2\pi$ is irrational, then by a generalization of Dirichlet's theorem (see, for example, [3]), there exist positive integers k and p such that $|k\theta - 2p\pi - \beta| < \pi/2$, whence $\omega^k \in \Sigma \subset \Omega$.

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(b) In the case of a real space the proof is simpler. The set $\Omega_r = \{\zeta \in R: \|x+\zeta y\| \geqslant \|x\|\}$ (R is the real field) contains at least one of the rays $[0, +\infty)$, $(-\infty, 0]$ and by assumption ω is negative. Then, $\omega^k \in \Omega$ for some positive integer k.

Remark. It can be easily seen, by taking the supremum norm on C^2, for example, that there may be no positive integer k such that $\|x+\omega^k y\| > \|x\|$.

PROPOSITION 2. *Let X be a real or complex normed vector space and let A be a linear operator from X into itself such that $\|Ax\| \leqslant \|x\|$ for all $x \in X$ (i. e. A is a contraction). Let λ, μ be eigenvalues of A such that $|\lambda| = 1$ and $\lambda \neq \mu$. If u and v are eigenvectors of A corresponding to λ and μ, respectively, then u is orthogonal to v.*

Proof. Let a be an arbitrary scalar. We have for all positive integers k,

$$\|u+av\| \geqslant \|A^k(u+av)\| = \|\lambda^k u + a\mu^k v\|.$$

whence, denoting $\omega = \mu/\lambda$,

$$(*) \qquad \|u+\omega^k(av)\| \leqslant \|u+av\|.$$

If $|\mu| < 1$, then $|\omega| < 1$ and letting in $(*)$ $k \to +\infty$, we obtain $\|u\| \leqslant \|u+av\|$. If $|\mu| = 1$, then we have $0 < \arg\omega < 2\pi$ (since $\omega \neq 1$) and making use of Proposition 1 we obtain from $(*)$ that $\|u\| \leqslant \|u+av\|$.

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POLYTECHNIC INSTITUTE OF BROOKLYN
BROOKLYN, NEW YORK, U.S.A.

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Some examples in harmonic analysis

by
B. E. JOHNSON (Newcastle upon Tyne)

Abstract. The paper consists essentially of five examples as follows.
(1) A Segal algebra on a commutative group which is not *closed.
(2) A Wiener-like Segal algebra which is not *closed.
(3) A group algebra such that the ideal of functions with Haar integral zero does not have an unbounded approximate unit.
(4) A group Γ with a closed normal subgroup G and a G-invariant subspace E of $L^1(\Gamma)$ such that TE is not closed where T is the canonical map of $L^1(\Gamma)$ onto $L^1(\Gamma/G)$.
(5) A compact group G such that the kernel of the convolution product map from $L^\infty(G)\hat{\otimes}L^\infty(G)$ is not the closed linear span of the tensors $\varphi * a \otimes \psi - \varphi \otimes a * \psi$, $a \in L^1(G)$, $\varphi, \psi \in L^\infty(G)$.

In this paper we give a number of examples arising in various parts of harmonic analysis. The first four are connected with the work of H. Reiter.

1. Symmetry and *symmetry in Segal algebras. A Segal algebra S ([5], p. 16) is a dense left translation invariant subset of $L^1(G)$, G a locally compact group, which is a Banach space under some left translation invariant norm $\|\ \|_S$ dominating the L^1 norm and such that the left regular representation of G on S is strongly continuous. S is symmetric if in addition $\|\ \|_S$ is right invariant and the right regular representation is strongly continuous. If G is abelian every Segal algebra is symmetric.

The Segal algebra S is *symmetric if it is stable under the hermitian involution* on $L^1(G)$. We shall construct an example with $G = \boldsymbol{R}$ of a (necessarily symmetric) Segal algebra which is not *symmetric.

Let $f \in L^1(\boldsymbol{R})$. Define

$$S_f = \{g;\ g \in L^1(\boldsymbol{R}), f*g \in C_0(\boldsymbol{R})\},$$

$$\|g\|_S = \|g\|_1 + \|f*g\|_\infty$$

where $f*g \in C_0(G)$ means $f*g$ differs from a C_0 function on a set of measure zero and $\|\ \|_p$ is the L^p norm. As S_f contains all continuous functions with compact support, S_f is dense in $L^1(\boldsymbol{R})$ and it is easy to check that S_f is a Segal algebra. Consider the case

$$f(x) = (x|\log x|)^{\frac{1}{2}} \quad 0 < x < \tfrac{1}{2},$$
$$= 0 \quad x \leqslant 0 \text{ or } x \geqslant \tfrac{1}{2}.$$