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Topological algebras of continuous functions over valued fields

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Abstract. In this note we consider the algebra of continuous functions $C(T, F)$ mapping the 0-dimensional Hausdorff space T to the complete rank one nonarchimedean nontrivially valued field F. The first two sections of the paper are concerned with the development of analogs $\hat{\beta}(T)$ and $\hat{\nu}(T)$ of the Stone–Čech and realcompactifications of T. An analog of the Gelfand–Kolmogoroff theorem is presented. The kernels of homomorphisms of $C(T, F)$ into F are characterized in a fashion analogous to Hewitt's characterization for real algebras, where F is a complete discretely valued field whose residue class field has nonmeasurable cardinal.

In the final section it is shown that in the case where F is complete and discretely valued with residue class field having nonmeasurable cardinal, the algebra $C(T, F)$ endowed with compact-open topology is F-bornological if and only if $\hat{\nu}(T) = T$.

In the present note we study algebras $C(T, F)$ of continuous functions with pointwise operations from a 0-dimensional Hausdorff space T to a complete rank one nonarchimedean nontrivially valued field F. $C(T, F)$ carries the compact-open topology and is therefore a locally F-convex ([12]) topological vector space. Nachbin ([9]) and Shirota ([11]) obtained a necessary and sufficient condition for real algebras $C(X, R)$ of continuous functions mapping the Tychonoff space X into the real numbers R to be bornological: $C(X, R)$ is bornological if and only if X is a Q-space (as defined in [5] or [6]). Here, in Theorem 7, we obtain a necessary and sufficient condition for $C(T, F)$ to be F-bornological (in the sense of [12]) when F is a complete discretely valued field whose residue class field ([1]) has nonmeasurable cardinal ([5]). To accomplish this, we bypass the real-number-dependent machinery used in analyzing the algebras $C(X, R)$.

Throughout this paper T denotes a 0-dimensional Hausdorff topological space, F a complete rank one nonarchimedean nontrivially valued field. Our use of "0-dimensionality" is that there is a base for the topology consisting of closed and open (clopen) sets. The F-valued characteristic function of a subset E of T is denoted by k_E.

1. Maximal ideals of $C(T, F)$. In this section we develop an analog $\hat{\beta}(T)$ of the classical Stone–Čech compactification $\beta(T)$ of T. We also develop an analog $\hat{\nu}(T)$ of the classical realcompactification $\nu(T)$ of T. We list (Theorems 1–3) relationships between the points $s \in \hat{\beta}(T)$ and maximal ideals M of the algebra $C(T, F)$ analogous to those of [5]. The validity of Props. 1–3 enables us to omit the proofs of Theorems 1–3 (cf. [5], pp. 10–108).

Definition 1. A subset E of T is a C_δ set if there exists a countable collection of clopen sets (S_n) such that $E = \bigcap S_n$. A subset E of T is an F-zero set if there exists $f \in C(T, F)$ such that $E = f^{-1}(0)$. We then also denote E by $z(f)$.

Proposition 1. *E is an F-zero set if and only if E is a C_δ set.*

Proof. If E is an F-zero set, then $E = z(f)$ for some $f \in C(T, F)$ and $E = \bigcap \{t \in T \mid |f(t)| < 1/n\}$.

Conversely, if $E = \bigcap S_n$ where each S_n is a clopen subset of T, then choosing $a \in F$ such that $|a| < 1$ and setting $f = \sum a^n k_{CS_n}$, we see that $z(f) = E$.

From Prop. 1 we see that the F-zero sets are the same for any field F and may be referred to simply as *zero sets*. We now show that disjoint zero (C_δ) sets can be separated by clopen sets.

Proposition 2. *If $E \cap L = \emptyset$ where E and L are C_δ sets, then there exists a clopen set $S \subset T$ such that $E \subset S$ while $L \subset CS$.*

Proof. For the sake of the proof, we choose F to be a field such that $\sqrt{-1} \notin F$ (e. g. a p-adic number field Q_p where $p = 4n+3$ for some positive integer n). Since $E = z(f)$ and $L = z(g)$ for some $f, g \in C(T, F)$, we observe that as $\sqrt{-1} \notin F$, then $z(f^2+g^2) = \emptyset$. Consequently $h = f^2/(f^2+g^2) \in C(T, F)$ and the set $S = \{t \in T \mid |h(t)| < 1/2\}$ will satisfy the conditions of the proposition.

Definition 2. Let $\mathfrak{T}$ be a collection of nonempty C_δ-sets such that

(a) If $E_1, E_2 \in \mathfrak{T}$ then $E_1 \cap E_2 \in \mathfrak{T}$,

and

(b) if $E_1 \in \mathfrak{T}$ and $E_1 \subset E_2$ where E_2 is a C_δ set, then $E_2 \in \mathfrak{T}$. Then $\mathfrak{T}$ is called a *z-filter*. If $\mathfrak{T}$ is maximal under set inclusion, then $\mathfrak{T}$ is called a *z-ultrafilter*.

As an intersection of finitely many (even denumerably many) C_δ sets is a C_δ set, it can be shown that every z-filter can be extended to a z-ultrafilter.

Proposition 3. *The mapping $M \to Z(M) = \{z(f) \mid f \in M\}$ establishes a 1-1 correspondence between the maximal ideals M of $C(T, F)$ and the z-ultrafilters on T.*

Proof. We note that the proof of the analogous statement as presented in [5] would be applicable to this situation if it could be shown that when $f, g \in M$, then $z(f) \cap z(g) \in Z(M)$. Thus we show that when $f, g \in M$ it follows that $z(f) \cap z(g) \in Z(M)$.

We begin by showing that if $f, g \in M$ then $z(f) \cap z(g) \neq \emptyset$. If $z(f) \cap z(g) = \emptyset$, then by Prop. 2 it follows that there is a clopen set $S \subset T$ such that $z(f) \subset S$ while $z(g) \subset CS$. Let

$$f'(t) = \begin{cases} 0 & \text{if } t \in S, \\ f(t)^{-1} & \text{if } t \in CS, \end{cases}$$

$$g'(t) = \begin{cases} 0 & \text{if } t \in CS, \\ g(t)^{-1} & \text{if } t \in S. \end{cases}$$

Then $ff' + gg' = k_T \in M$ which is contradictory.

We now show that if $f \in M$ and $z(g) = z(f)$ for some $g \in C(T, F)$, then it follows that $g \in M$. To do this we simply observe that $z(g) \cap z(h) \neq \emptyset$ for all $h \in M$ and therefore the ideal generated by M and the function g is a proper ideal. Thus $g \in M$.

Now we can show that if $f, g \in M$ then $z(f) \cap z(g) \in Z(M)$. To begin we note that $z(f) = \bigcap S_n$ and $z(g) = \bigcap W_n$ where the sets S_n and W_n are clopen subsets of T. Choosing $a \in F$ such that $0 < |a| < 1$ and setting $f' = \sum a^{2n} k_{CS_n}$ and $g' = \sum a^{2n+1} k_{CW_n}$, we observe that $z(f) = z(f')$, $z(g) = z(g')$, and $z(f'+g') = z(f') \cap z(g') = z(f) \cap z(g)$. Since $f'+g' \in M$, the proof is seen to be complete.

Definition 3. Let F be a local field and $V = \{a \in F \mid |a| \leqslant 1\}$ be the valuation ring of F. Let $\mathfrak{H} = \{f \in C(T, F) \mid f(T) \subset V\}$ and consider the map

$$e\colon T \to V^{\mathfrak{H}}, \quad t \to (f(t))_{f \in \mathfrak{H}}.$$

As in [7] the mapping e of T into the product space $V^{\mathfrak{H}}$ is a topological embedding. We define the closure $\overline{e(T)}$ of $e(T)$ in $V^{\mathfrak{H}}$ to be the F-Stone–Čech compactification of T and denote it by $\hat{\beta}_F(T)$.

By standard arguments we may show that the compactifications $\hat{\beta}_F(T)$ are equivalent compactifications of T for all F, so we may refer to this compactification of T as $\hat{\beta}(T)$ — the nonarchimedean Stone–Čech compactification of T.

As in [7], if T and T^* are both 0-dimensional Hausdorff spaces, a continuous function $f\colon T \to T^*$ can be extended to a continuous function $\hat{f}\colon \hat{\beta}(T) \to \hat{\beta}(T^*)$.

We now list a group of results whose proofs are similar to proofs of analogous results in [5]. We emphasize that unless otherwise stated, F is any complete nonarchimedean nontrivially valued field.

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THEOREM 1 ("Gelfand–Kolmogoroff"). *There exists a* 1-1 *correspondence between the points* $s \in \hat{\beta}(T)$ *and maximal ideals* M *of* $C(T, F)$ *where*

$$s \to \{f \in C(T, F) \mid s \in cl_{\hat{\beta}(T)} z(f)\} = M(s)$$

establishes the correspondence.

DEFINITION 4. The *nonarchimedean realcompactification* $\hat{\nu}(T)$ of T is defined to be the collection of points $s \in \hat{\beta}(T)$ such that if (W_n) is any denumerable collection of clopen neighborhoods of s in $\hat{\beta}(T)$, then $\bigcap (W_n) \cap T \neq \emptyset$. If $\hat{\nu}(T) = T$, then T is called a $\hat{Q}$-space.

THEOREM 2. *The following statements are all equivalent.*

(a) $s \in \hat{\nu}(T)$,

(b) *If* $z(f_n) \in M(s)$ $(n = 1, 2, \ldots)$, *then* $\bigcap z(f_n) \in Z(M(s))$,

(c) *If* $z(f_n) \in M(s)$ $(n = 1, 2, \ldots)$, *then* $\bigcap z(f_n) \neq \emptyset$.

THEOREM 3. *If* F *is a local field, then* $M(s)$ *is the kernel of a homomorphism of* $C(T, F)$ *into* F *if and only if* $s \in \hat{\nu}(T)$.

THEOREM 4. *If* f *is a continuous function taking* T *into* T^* *and* $\hat{f}$ *is the continuous extension of* f *taking* $\hat{\beta}(T)$ *into* $\hat{\beta}(T^*)$, *then* $\hat{f}(\hat{\nu}(T)) \subset \hat{\nu}(T^*)$.

Proof. Let $s \in \hat{\nu}(T)$. To show that $\hat{f}(s) \in \hat{\nu}(T^*)$, it is shown that if (W_n) is a denumerable collection of clopen neighborhoods of $\hat{f}(s)$ in $\hat{\beta}(T^*)$, then $\bigcap W_n \cap T^* \neq \emptyset$. To demonstrate this we observe that as $s \in \hat{\nu}(T)$, it follows that $f^{-1}(\bigcap W_n \cap T^*) = \bigcap f^{-1}(W_n) \cap T \neq \emptyset$.

2. The homomorphisms of $C(T, F)$ into F. A set S is said to have *nonmeasurable cardinal* [5] if every ultrafilter $\mathfrak{T}$ of subsets of S, closed with respect to the formation of denumerable intersections, is *fixed* $(\bigcap \mathfrak{T} \neq \emptyset)$. This is equivalent to the requirement that every countably additive 0–1 measure on the σ-algebra of all subsets of S be concentrated at a point of S. A "measurable" cardinal has never been exhibited. Moreover, the collection of nonmeasurable cardinals is a subclass of the class of all cardinals which is closed with respect to the standard operations on the class of cardinals [5].

In this section we show that Theorem 3 can be generalized to include all fields F such that is F complete, discretely valued, and the residue class k ([1] or [10]) has nonmeasurable cardinal. As a complete discretely valued field F is a local field if and only if k is a finite field [10], we see from the above remarks that this constitutes a considerable broadening of Theorem 3.

PROPOSITION 4. *If* $M(s) \subset C(T, F)$ *and* $M(s)$ *is the kernel of a homomorphism* h *taking* $C(T, F)$ *into* F, *then* $s \in \hat{\nu}(T)$.

Proof. It is sufficient to show that if (S_n) is a pairwise disjoint clopen cover of T, then $S_j \in Z(M(s))$ for some integer j. To prove this let $a \in F$ be such that $0 < |a| < 1$ and let $f = \sum a^n k_{CS_n}$. Since $f - h(f) k_T \in M(s)$, then $z(f - h(f) k_T) \in Z(M(s))$. But $z(f - h(f) k_T) = S_j$ for some j.

THEOREM 5. *Let* F *be a complete and discretely valued field whose residue class field* k *has nonmeasurable cardinal. Then* $M(s) \subset C(T, F)$ *is the kernel of a homomorphism of* $C(T, F)$ *into* F *if and only if* $s \in \hat{\nu}(T)$.

Proof. To prove this we must show that if $g \notin M(s)$, then for some $a \in F$, $g - a k_T \in M(s)$. Let $(a_\mu)_{\mu \in U}$ be a collection of representatives of k and choose a scalar $b \in F$ such that $|b| < 1$ is a generator of the value group $|F^*|$ of F.

Since $F = \bigcup_{-\infty}^{0} b^j V$, then $T = \bigcup_{-\infty}^{0} g^{-1}(b^j V)$ and, as $Z(M(s))$ is closed under the formation of denumerable intersections, it follows that for some integer j, $g^{-1}(b^j V) \in Z(M(s))$. We observe that $b^j V = \bigcup_{\mu \in U} (b^j a_\mu + b^{j+1} V)$. Since for any subset H of U, $S_H = \bigcup_{\mu \in H} (b^j a_\mu + b^{j+1} V)$ is a clopen subset of F, we see that the sets $H \subset U$ such that $g^{-1}(S_H) \in Z(M(s))$ are an ultrafilter of subsets of U. Since U has nonmeasurable cardinal, there exists $u_0 \in U$ such that $g^{-1}(b^j a_{u_0} + b^{j+1} V) \in Z(M(s))$. Similarly, there exists u_1 such that $g^{-1}(b^j a_{u_0} + b^{j+1} a_{u_1} + b^{j+2} V) \in Z(M(s))$. In this way we construct a nest $S_n = b^j a_{u_0} + \ldots + b^{j+n} a_{u_n} + b^{j+n+1} V$ of subsets of T such that $\operatorname{diam} S_n \to 0$ and $g^{-1}(S_n) \in Z(M(s))$. As F is a complete field, $\bigcap S_n = \{a\}$ for some $a \in F$ and it follows that $g^{-1}(a) = \bigcap g^{-1}(S_n) \in Z(M(s))$.

EXAMPLE 1. In [8] Michael showed that if an algebra $A \subset C(X, R)$ is "closed under inverses" (if $f \in A$ and $f^{-1} \in C(X, R)$, then $f^{-1} \in A$) and A satisfies conditions (a) and (b) below, then the nontrivial homomorphisms of A into R are generated by the points of X as evaluation map homomorphisms.

(a) Given $f_1, \ldots, f_n \in A$ such that $\bigcap_{i=1} z(f_i) = \emptyset$, then there exist $g_1, \ldots, g_n \in A$ such that $\sum f_i g_i = k_T$.

(b) There exist $h_1, \ldots, h_m \in A$ such that for any $a_1, \ldots, a_m \in R$, $\bigcap_{i=1}^{m} z(h_i - a_i k_r)$ is compact.

The proof of ([8], p. 51) may be applied to the setting of this paper and it may be noted therefore that if $A \subset C(T, F)$ is closed under inverses and A satisfies (a) and (b), then the nontrivial homomorphisms of A into F are generated by the points of T.

By Prop. 3, if $A = C(T, F)$, then A satisfies (a). If we take $T = F$, then $C(F, F)$ satisfies (b). Thus F is an F-Q (in the sense of [2]) space.

THEOREM 6. *If* F *is a complete discretely valued field whose residue class field has nonmeasurable cardinal, then* F *is a* $\hat{Q}$-*space.*

Proof. Apply Theorem 5 and Example 1.

It is known that a complete discretely valued field may be constructed whose residue class field k has arbitrary cardinality. From the results of this section, it can be shown that if the statement of Theorem 5 is true for all complete discretely valued fields, then all cardinals are nonmeasurable.

3. The algebra $C(T, F)$ with compact-open topology. In this section we examine the topological algebra $C(T, F)$ (endowed with compact-open topology). Principal among the results obtained (Theorem 7) is a necessary and sufficient condition for $C(T, F)$ to be F-bornological when F is a complete discretely valued whose residue class field k has nonmeasurable cardinal.

DEFINITION 5. Let F be complete and discretely valued and let $(a_u)_{u\in U}$ be a complete set of representatives for the cosets of the residue class field of F. We may assume that U is totally ordered and that the representative determined by the first element of U is the scalar 0. Choose an element $\pi \in F$ such that $|\pi| < 1$ is a generator of the value group $|F^*|$ of F. For any two elements, a and b, of F there is ([1]) an integer N and sequences $(a_{u_i})_{i\geqslant N}$ and $(a_{\lambda_i})_{i\geqslant N}$ of representatives such that

$$a = \sum_{i\geqslant N} a_{u_i}\pi^i \quad \text{and} \quad b = \sum_{i\geqslant N} a_{\lambda_i}\pi^i.$$

We define $\sup(a, b)$ to be a if $u_N > \lambda_N$ or if $a_{u_i} = a_{\lambda_i}$ $(i = N, \ldots, j)$ while $u_{j+1} > \lambda_{j+1}$. Under these circumstances we also say that $\inf(a, b) = b$. If $a = b$ we take $\sup(a, b) = \inf(a, b) = a$.

We note that if $|a| > |b|$, then $\sup(a, b) = a$ and $\inf(a, b) = b$. Propositions 5 and 6 concerning this notion of sup and inf follow easily.

PROPOSITION 6. *If F is a complete discretely valued field and $f, g \in C(T, F)$ then the functions defined by $\sup(f(t), g(t))$ and $\inf(f(t), g(t))$ are continuous.*

PROPOSITION 7. *Let F be a complete discretely valued field and let V be an absolutely F-convex subset of T with the following property: there is a positive number a and a compact subset K of T such that $\sup|f(T)| \leqslant a$ or f vanishes in some neighborhood of K implies that $f \in V$. Then there is a positive number b such that $\sup|f(K)| \leqslant b$ implies that $f \in V$.*

DEFINITION 6. For $f, g \in C(T, F)$ we say that $f \leqslant g$ if $\inf(f, g) = f$. The *interval* $[f, g]$ is the set $\{h \in C(T, F) \mid f \leqslant h \leqslant g\}$.

Note that $\inf(f, g) = f$ if and only if $\sup(f, g) = g$. In addition, $h \in [f, g]$ only when $|f(t)| \leqslant |h(t)| \leqslant |g(t)|$ for each $t \in T$.

Specializing van Tiel's notions of „bornivorous" and "bornological space" to function algebras yields:

DEFINITION 7. An absolutely F-convex subset V of $C(T, F)$ is an *F-bornivore* if it absorbs every bounded subset of $C(T, F)$. If all F-bornivores are neighborhoods of 0, then $C(T, F)$ is *F-bornological*.

Note that intervals $[f, g]$ are bounded in $C(T, F)$ and are therefore absorbed by bornivores. We may now present our principal theorem.

THEOREM 7. *Let F be a complete discretely valued field whose residue class field k has nonmeasurable cardinal. Then $C(T, F)$ is F-bornological if and only if T is a $\hat{Q}$-space.*

Proof. First assume that T is a $\hat{Q}$-space. To prove that $C(T, F)$ is F-bornological, it suffices to show that if V is an absolutely F-convex set which absorbs all intervals $[f, g]$, then V is a neighborhood of 0. Consider then such an interval absorbing absolutely F-convex set. A closed-hence compact-subset K of $\hat{\beta}(T)$ is a *support set* for V if when f vanishes on an open superset of $K \cap T$, it follows that $f \in V$. Clearly $\hat{\beta}(T)$ itself is a support set for V. Since for any $a \in F$ there is a scalar $b \in F$ such that $[0, ak_T] \subset bV$, it follows that for some $r > 0$, $\sup|f(T)| \leqslant r$ implies that $f \in V$. If we show that there is a support set for V which lies in T then, by Prop. 7, V is a neighborhood of 0.

Let $\mathfrak{H}$ be the collection of all support sets for V. It is readily shown that if $L, K \in \mathfrak{H}$ and $L \cap K = \emptyset$, then $C(T, F) = V$. If $L \cap K \neq \emptyset$ and $L \cap K \subset S$ where S is a clopen subset of $\hat{\beta}(T)$, then L and $K \cap CS$ are disjoint. Thus there is a clopen set $U \subset \hat{\beta}(T)$ such that $L \subset U$ while $K \cap CS \subset CU$. Let $f \in C(T, F)$ be a function which vanishes on $S \cap T$. Since $fk_{U\cap T}$ vanishes on $(S \cup CU) \cap T$ and $K \cap T \subset (S \cup CU) \cap T$, it follows that $fk_{U\cap T} \in V$. Similarly $fk_{CU\cap T} \in V$ and therefore $f \in V$. Thus it follows that $L \cap K \in \mathfrak{H}$.

It is clear that $L = \bigcap \mathfrak{H}$ is a support set for V. It will now be shown that $L \subset T$. Let $s \in \hat{\beta}(T) - T$. Since T is a $\hat{Q}$-space, it follows that there is a sequence (W_n) of clopen neighborhoods of s in $\hat{\beta}(T)$ such that $\bigcap W_n \subset \hat{\beta}(T) - T$. We may assume that $W_{n+1} \subset W_n$ for all n. For each n suppose that f_n vanishes on $(\hat{\beta}(T) - W_n) \cap T$ and $f_n \notin V$. Let $\mu \in F^*$ be such that $|\mu| < 1$ and consider $g = \sup(\mu^n f)$. Since $\bigcup T - W_n = T$ and f_k vanishes on $T - W_n$ for all $k \geqslant n$, it follows that $g \in C(T, F)$. Since $[0, g] \subset aV$ for some $a \in F$ and $\mu^n f_n \in [0, g]$ for every n, then $f_n \in V$ for every n such that $|a|/|\mu|^n \leqslant 1$. This, however, contradicts the way in which the f_n were chosen and it follows that $\hat{\beta}(T) - W_n \in \mathfrak{H}$ for some n. Thus $s \notin L$ and $L \subset T$.

To prove the converse suppose that T is not a $\hat{Q}$-space. Then there is some $s \in \hat{\nu}(T) - T$. By the results of Theorems 4 and 6, every function $f \in C(T, F)$ can be extended to $\hat{f} \in C(\hat{\nu}(T), F)$. By the Tietze–Ellis extension theorem ([4]) or the results of [3], the mapping $h(f) = \hat{f}(s)$ is a discontinuous homomorphism of $C(T, F)$ into F. If it can be shown that h is bounded, it follows that $C(T, F)$ is not bornological.

If $X \subset C(T, F)$ is bounded and $h(X)$ is unbounded, there must be a sequence (f_n) from X such that $|\hat{f}_n(s)| \to \infty$. Letting $W_n = \{s' \in \hat{\nu}(T) \mid |\hat{f}(s')| \geqslant |\hat{f}_n(s)| - 1\}$, $s \in \bigcap W_n$ and since $s \in \hat{\nu}(T)$, it follows that $\bigcap W_n \cap \cap T \neq \emptyset$. Thus there exists $t \in \bigcap W_n \cap T$ and since $|f_n(t)| \to \infty$, X is not bounded and the proof is done.

Definition 8. A closed set $X \subset T$ is *relatively F-precompact* if all functions in $C(T, F)$ are bounded on X.

Proposition 8. *A closed subset X of T is relatively F-precompact if and only if* $\mathrm{cl}_{\hat{\beta}(T)} X \subset \hat{\nu}(T)$.

Proof. Suppose first that there is a point $s \in C\hat{\nu}(T)$ such that s is in the closure of X. Let (W_n) be a descending denumerable sequence of neighborhoods of s such that $\bigcap W_n \cap T = \emptyset$. Choose $a \in F$ such that $|a| > 1$. Let $S_n = (W_n - W_{n+1}) \cap T$ and $f = \sum a^n k_{S_n}$. It is clear that f is unbounded on X and therefore X is not relatively F-precompact.

Suppose conversely that X is not relatively F-precompact. Let $f \in C(T, F)$ be unbounded on X and $S_n = \{t \in T \mid |f(t)| > n\}$. Consider a set $Y = \{t_n \in T \mid t_n \in S_n \cap X\}$ where with no loss of generality we may assume the relationship $t_j \neq t_i$ if $i < j$ holds. As $\hat{\beta}(T)$ is compact, there exists $s \in \hat{\beta}(T)$ such that $s \in \mathrm{cl}_{\hat{\beta}(T)} Y$. Thus, of course, $s \in \mathrm{cl}_{\hat{\beta}(T)} X$. However, $s \in \mathrm{cl}_{\hat{\beta}(T)} S_n$ for all integers n and therefore S_n belongs to the z-ultrafilter $Z(M(s))$ for each integer n. Since $\bigcap S_n = \emptyset$, it follows that $s \in C\hat{\nu}(T)$.

From the preceeding result it can be seen that relative F-precompactness of X is independent of the field F and we may therefore refer to X as *relatively precompact*.

Proposition 9. *If T is a $\hat{Q}$-space, then every relatively precompact set is compact.*

Proof. To prove this it must be shown that X is closed in $\hat{\beta}(T)$. However, since $T = \hat{\nu}(T)$ and $\mathrm{cl}_{\hat{\beta}(T)} X \subset \hat{\nu}(T)$, the proof is seen to be complete.

In [2], [3] it is shown that if F is complete and discretely valued, then $C(T, F)$ is F-barreled if and only if every relatively precompact subset of T is compact. R. L. Ellis proved this result for spherically complete fields and never published it. By the result of Proposition 8 we see that the property of T which is necessary and sufficient for $C(T, F)$ to be F-barreled (F a spherically complete field) is entirely dependent on T and its relationship to $\hat{\beta}(T)$. Thus we have the following result.

Theorem 8. *Let F and K be spherically complete fields. Then $C(T, F)$ is F-barreled if and only if $C(T, K)$ is K-barreled.*

Theorem 9. *Let F be a complete discretely valued field whose residue class field has nonmeasurable cardinal. Then if $C(T, F)$ is F-bornological, it follows that $C(T, F)$ is F-barreled.*

Proof. We apply Theorem 7, Proposition 9, and the comments following Proposition 9.

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