EXAMPLE 2. Let E be a Banach space with a basis $\{x_n\}$ such that the closed linear subspace $[f_n]$ of E^* spanned by the coefficient functionals $\{f_n\}$ is of characteristic 1, i.e. [3],

$$\|x\| = \sup_{\substack{f\in[f_n]\\ \|f\|=1}} |f(x)| \quad (x\in E); \tag{12}$$

such a basis is e.g. the unit vector basis in $E = c_0$ or $E = l^p$, $1 \leqslant p < \infty$. Furthermore, let $F = E$ and define u by (9). Then, as above, $u^*(F^*)$ is a norm-dense subspace of $[f_n]$ and hence, by (12),

$$\|x\| = \sup_{\substack{f\in u^*(F^*)\\ \|f\|=1}} |f(x)| \quad (x\in E). \tag{13}$$

On the other hand, since u is one-to-one, whenever the equation $u(x) = y$ has a solution x_0, we have $\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \|x_0\|$. Consequently (2), and hence (1), is satisfied, although $u(E)$ is not closed.

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A trace inequality for generalized potentials*

by

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Abstract. In the spirit of the Sobolev–Il'in inequality, the trace or restriction of generalized potentials of L_p functions to arbitrary measurable sets in Euclidean space are studied. When the potential kernel is homogenous, then necessary and sufficient conditions are given for the trace inequality to hold.

Introduction. In [1] the author showed that the necessary and sufficient condition for the continuous imbedding of $L_p(\boldsymbol{R}^n, l_n)$ into $L^q(\boldsymbol{R}^n, \nu)$, $1 < p < q < \infty$ via the Riesz potential operator $T\colon f \to h_a * f(x) = \int |x-y|^{a-n} f(y)\,dy$, $0 < a < n$, is that the maximal function of ν of dimension d, $0 < d \leqslant n$, be bounded $(d/q = n/p - a)$, i.e.,

$$M_d(\nu)(x) \equiv \sup_{r>0} r^{-d}\nu\big(B(x, r)\big)$$

is a bounded function of x. Here $B(x, r) = \{y\in \boldsymbol{R}^n\colon |x-y| < r\}$ and $L_p(\boldsymbol{R}^n, \nu)$ denotes the usual Lebesgue pth power summable functions on $\boldsymbol{R}^n$ with respect to the Borel measure ν. l_n = Lebesgue n-dimensional measure on $\boldsymbol{R}^n$. When d is a positive integer, this result has, as a corollary, the well known Sobolev-Il'in theorem concerning the restrictions of Riesz potentials of L_p functions to smooth manifolds in $\boldsymbol{R}^n$ of dimension d. That is, the *trace* of the potential belongs to L_q on the manifold with respect to some d-dimensional measure with a bounded d-dimensional maximal function, e.g., surface measure. See [5].

The purpose of this note is not only to extend this result to a more general class of potentials, but to give a much more direct and simplified approach than presented in [1], even in the case of Riesz potentials. Furthermore, the method of proof allows for a much more accurate estimate of the norm of the operator T than known before. In particular, we get $\|T\|_{pq} \approx \sup_x M_d(\nu)(x)^{1/q}$, see corollary to Theorem B.

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We shall consider potentials of the form $k(f\mu, y) = \int k(x, y) f(x)\, d\mu(x)$, where k is a non-negative function on the product space $(X \times Y, \mu \times \nu)$, ν σ-finite, and find conditions on the measures μ and ν such that the linear map $T\colon f \to k(f\mu, \cdot)$ is continuous from $X_p(\mu) = L_p(X, \mu)$ into $Y_q(\nu) = L_q(Y, \nu)$, $p < q$, Theorem A. These conditions are expressed in terms of a k dependent maximal function of the measures and the methods used also allow for cases when the maximal functions are not necessarily bounded. When k is a homogenous kernel, necessary conditions on ν are obtained for $T\colon X_p(l_n) \to Y_q(\nu)$ which agree with the sufficient conditions of Theorem A, Theorem B. In particular, these results extend the theorem of [1] to the parabolic Riesz potentials, i.e.,

$$\Gamma_\alpha * f(y, s) = \int_{-\infty}^{s} \int_{\boldsymbol{R}^n} (s-t)^{(\alpha-n-2)/2} \exp\left(-\frac{|y-x|^2}{4(s-t)}\right) f(x, t)\, dx\, dt,$$

$0 < \alpha < n+2$. See for example [6] for various properties of parabolic potentials.

Finally, in Section 4 we remark about some special cases of potentials on the half space $\boldsymbol{R}_+^{n+1}$ where a positive result in the limiting case $p = q$ is known.

1. Preliminaries. Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ denote two measure spaces with ν σ-finite. Functions on X or Y will always be assumed extended real valued and measurable. Throughout, $k(x, y)$ will be a non-negative extended real valued $\mathfrak{A} \times \mathfrak{B}$ measurable function. $X_p(\mu)$ will denote $L_p(X, \mathfrak{A}, \mu)$ with norm $X_p(f; \mu) = \{\int |f(x)|^p\, d\mu(x)\}^{1/p}$, $1 \leqslant p < \infty$. $X_p^*(\mu)$ is weak-$X_p(\mu)$, $p > 0$, i.e., f such that $\sup_{t>0} t\mu(\{X\colon |f(x)| > t\})^{1/p} < \infty$. $F\mu$ is the signed measure $F(x)\, d\mu(x)$ on X. $k(f\mu, y) = \int k(x, y) f(x)\, d\mu(x)$. Similar notation is used for $Y_p(\nu)$, $Y_p^*(\nu)$ and integration over Y. $X_p(F\mu) \cap X_p(\mu)$ is the Banach space with norm $X_p(\cdot; F\mu) + X_p(\cdot; \mu)$, $F \geqslant 0$.

If $T\colon f \to k(f\mu, \cdot)$ we denote the continuous imbedding of $X_p(F\mu) \cap X_p(\mu)$ into $Y_q(v)$ by

$$T\colon X_p(F\mu) \cap X_p(\mu) \to Y_q(\nu)$$

and similarly for $X_p(F\mu) \cap X_p(\mu)$ into $Y_q^*(\nu)$. In each case, the norm of the operator T is indicated by $\|T\|_{pq}$ or $\|T\|_{pq}^*$.

If $e_\lambda = \{(x, y)\colon k(x, y) > \lambda\}$, then $e_\lambda(x)$ and $e_\lambda(y)$ are the x and y-sections of e_λ. The generalized maximal function of μ (of order $r > 0$) with respect to k is

$$M_r(\mu)(y) \equiv \sup_{\lambda>0} \lambda^r \mu(e_\lambda(y)).$$

Similarly, $M_r(\nu)(x)$ is defined using the x-section of e_λ. Note that when $k = |x-y|^{\alpha-n}$, $0 < \alpha < n$, on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ and $r = n/(n-\alpha)$, then $M_r(\mu)$ is just the usual Hardy–Littlewood–Wiener maximal function of μ. Also when X is a locally compact group and $k = k(x-y)$ on $X \times X$, then the condition "$M_r(\mu)$ is bounded" means that $k(x)$ and all its translates belong to $X_r^*(\mu)$ uniformly. This is always the case if, for example, μ is a Haar measure on X and $k \in X_r^*(\mu)$.

2. Trace inequality. With the above notation, we are in a position to prove

THEOREM A. (i) *$T\colon X_1(M_s(\nu)\cdot\mu) \cap X_1(\mu) \to Y_s^*(\nu)$, provided $s > 1$. $\|T\|_{1s}^* \leqslant \max\left(\frac{1}{s-1}, 1\right)$.* (ii) *If $M_r(\mu)(y) \leqslant A < \infty$ for all y, then*

$$T\colon X_p(M_s(\nu)\cdot\mu) \cap X_p(\mu) \to Y_q(\nu),$$

provided $1 < p < q < \infty$, $s/q = r/p + 1 - r$. $\|T\|_{pq}^* \leqslant \frac{q}{s} A^{1/p'} \max\left(\frac{p}{q-p}, 1\right)$. $p' = p/(p-1)$.

COROLLARY. *Under the assumptions of Theorem* A (i) *or* (ii):

(a) *If $M_s(\nu)(x) \leqslant B < \infty$ for all x, then $T\colon X_p(\mu) \to Y_q(\nu)$, $p > 1$; $T\colon X_1(\mu) \to Y_s^*(\nu)$, $p = 1$.*

(b) *If $M_s(\nu)(x) \geqslant b > 0$ for all x, then $T\colon X_p(M_s(\nu)\cdot\mu) \to Y_q(\nu)$ $p > 1$; $T\colon X_1(M_s(\nu)\cdot\mu) \to Y_s^*(\nu)$, $p = 1$.*

The corollary is an obvious consequence of the theorem. In proving Theorem A, we give only the argument for (ii) which is a consequence of

LEMMA 1: *If $M_r(\mu)(y) \leqslant A < \infty$ for all y, then*

$$t\nu(E_t)^{1/q} \leqslant A^{1/p'} \frac{q^2}{s(q-p)} X_p(f; M_s(\nu)\cdot\mu)^{p/q} X_p(f; \mu)^{1-p/q}, \tag{1}$$

where $E_t = \{y\colon k(|f|\mu, y) > t\}$, $t > 0$.

Proof. The fact that $\nu(E_t) < \infty$, $t > 0$, when the right side of (1) is finite is part of the conclusion of the lemma. But since ν is σ-finite, there are sets $Y_i \in \mathfrak{B}$, $i = 1, 2, \ldots$ with $\nu(Y_i) < \infty$, and $Y_i \uparrow Y$. Hence establishing the lemma for E_t replaced by $E_t \cap Y_i$ and letting $i \to \infty$ implies that at the outset we can assume $\nu(E_t) < \infty$ for $t > 0$. (As simple examples will show, the σ-finiteness of ν is a necessary condition for (1) to hold.) With this done, we estimate as follows:

$$t\nu(E_t) \leqslant \int |f(x)| \left\{\int k(x, y)\, d\nu_t(y)\right\} d\mu(x)$$

$$= \int_0^\infty \left\{\int |f(x)| \nu_t(e_\lambda(x))\, d\mu(x)\right\} d\lambda$$

$$= \int_0^\sigma (\ldots)\, d\lambda + \int_\sigma^\infty (\ldots)\, d\lambda = I_1 + I_2,$$

where ν_t denotes ν restricted to E_t and $\sigma > 0$ is to be specified later. Writing

$$\nu_t(e_\lambda(x)) \leqslant \nu_t(e_\lambda(x))^{1/p'} M_s(\nu)(x)^{1/p} \lambda^{-s/p},$$

we have

$$I_2 \leqslant X_p(f; M_s(\nu)\cdot\mu) \int_\sigma^\infty \Big(\int \nu_t(e_\lambda(x))\, d\mu(x)\Big)^{1/p'} \lambda^{-s/p}\, d\lambda.$$

But $\int \nu_t(e_\lambda(x))\, d\mu(x) \leqslant \nu(E_t)\lambda^{-r}\sup_x M_r(\mu)(x)$. Hence

$$I_2 \leqslant X_p(f; M_s(\nu)\cdot\mu) A^{1/p'} \nu(E_t)^{1/p'} \frac{pq}{s(q-p)} \sigma^{s\left(\frac{1}{q}-\frac{1}{p}\right)}$$

provided $p < q$. Also

$$I_1 \leqslant X_p(f;\mu)\nu(E_t)^{1/p} \int_0^\sigma \Big(\int \nu_t(e_\lambda(x))\, d\mu(x)\Big)^{1/p'} d\lambda$$

$$\leqslant X_p(f;\mu)\nu(E_t)^{1/p} A^{1/p'} \frac{q}{s} \sigma^{s/q}$$

provided $q < \infty$.

For $t > 0$ such that $\nu(E_t) > 0$ choose $\sigma = \sigma(t)$ so as to minimize the sum of the estimates for I_1 and I_2 in σ. An elementary calculation shows that the proper choice is a constant (in t) multiple of $\nu(E_t)^{-1/s}$ which when substituted for σ gives (1). Finally, (1) trivially holds when $\nu(E_t) = 0$.

(ii) of Theorem A now follows easily from Lemma 1 by first applying Young's inequality $\left(a^\theta b^{1-\theta} \leqslant \theta a + (1-\theta)b,\ a, b \geqslant 0,\ 0 \leqslant \theta \leqslant 1\right)$ and interpolating in p and q in the sense of Marcinkiewicz. (See for example [4] and [8] — the arguments given there can easily be adapted to the present setting of the sum of two L_p norms.)

3. Homogenous kernels. We now consider some necessary conditions on ν by restricting to the case of homogenous kernels k on $\boldsymbol{R}^n$. Let π: $\boldsymbol{R}_+ \to \mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$, (i.e., a continuous linear map of the positive real numbers into the linear transformations of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$) satisfying

(1) $\pi(\lambda\mu) = \pi(\mu)\pi(\lambda)$, $\pi(1) = I$ = identity,

(2) $0 < \lambda \leqslant 1$, $\|\pi(\lambda)\| \leqslant \lambda$, $\|\cdot\|$ = norm in $\mathscr{L}(\boldsymbol{R}^n, \boldsymbol{R}^n)$.

It is well known that $\pi_\lambda = \pi(\lambda) = \exp(U\log\lambda)$, U a real $n\times n$ matrix (the infinitesimal generator for the group).

Definition. A function $k(x) \not\equiv 0$ and locally summable on $\boldsymbol{R}^n$ satisfying

(1) $k(\pi_\lambda x) = \lambda^{-\beta \operatorname{tr} U} k(x)$, $x \neq 0$, $0 < \beta < 1$, and

(2) $k(x) \to 0$ as $x \to \infty$,

is called a *homogenous kernel* for the group $\{\pi_\lambda\}$. $\operatorname{tr} U$ = trace of U.

When $\beta = 1$, the reader might refer to [7] where the corresponding singular integrals for homogenous kernels are studied. The case when π_λ is a diagonal matrix, $0 < \beta < 1$, corresponds to the semi-elliptic potential kernels — e.g., the Riesz potential kernels, elliptic and parabolic.

Let $Q_a(x^0) = \{x \in \boldsymbol{R}^n\colon |x_i - x_i^0| \leqslant a,\ i = 1, \ldots, n\}$, $a > 0$ and $x^0 \in \boldsymbol{R}^n$. Both l_n and dx refer to Lebesgue n-measure.

Lemma 2. *If* $T\colon X_p(l_n) \to Y_q(\nu)$, *then*

$$\nu(\pi_\varrho Q_1(x^0))^{1/q} \int_{\pi_\varrho Q_1(0)} k(x)\, dx \leqslant 2^{n/p} \|T\|_{pq} \varrho^{\operatorname{tr} U/p}$$

for all $\varrho > 0$ *and* $x^0 \in \boldsymbol{R}^n$.

Proof. Let $f(x)$ = indicator function of $\pi_\varrho Q_2(x^0)$, then with a little matrix theory

$$X_p(f; l_n) = l_n(\pi_\varrho Q_2(x^0))^{1/p} = 2^{n/p} \varrho^{\operatorname{tr} U/p}.$$

Also

$$Y_q(k*f;\nu)^q \geqslant \int_{\pi_\varrho Q_1(x^0)} \Big(\int_{\pi_\varrho Q_2(x^0)} k(y-x)\, dx\Big)^q d\nu(y). \tag{2}$$

But for $y \in Q_1(x^0)$, $\pi_\varrho Q_1(y) \subset \pi_\varrho Q_2(x^0)$, hence the right side of (2) always exceeds

$$\nu(\pi_\varrho Q_1(x^0)) \Big(\int_{\pi_\varrho Q_1(0)} k(x)\, dx\Big)^q$$

and the conclusion follows from the imbedding.

Theorem B. *If k is a homogenous kernel for the group $\{\pi_\lambda\}$, then $T\colon X_p(l_n) \to Y_q(\nu)$ implies $M_s(\nu)(x)$ is bounded, $s = q(r/p+1-r)$, $r = 1/\beta$.*

Proof. From Lemma 2, $\nu(\pi_\varrho Q_1(x^0)) \leqslant C\varrho^{q\operatorname{tr} U\left(\frac{1}{p}+\beta-1\right)}$ for all $\varrho > 0$ and $x^0 \in \boldsymbol{R}^n$ since

$$\int_{\pi_\varrho Q_1(0)} k(x)\, dx = \int_{Q_1(0)} k(\pi_\varrho y) \varrho^{\operatorname{tr} U} dy$$

$$= \varrho^{\operatorname{tr} U(1-\beta)} \int_{Q_1(0)} k(y)\, dy.$$

Here $C = 2^{nq/p}\|T\|_{pq}^q \Big(\int_{Q_1(0)} k\, dy\Big)^{-q}$.

To conclude, we show $\{y\colon k(y-x) > \lambda\} \subset \pi_\varrho Q_a(x)$, $\lambda = \varrho^{-\beta \operatorname{tr} U}$, for all $\lambda > 0$ and some $a > 0$ independent of x and λ. To see this, first note that we need only consider the case $x = 0$. Hence if $y = \pi_\varrho z$, then we need $\{\pi_\varrho z\colon k(z) > 1\} \subset \pi_\varrho Q_a(0)$ or $\{z\colon k(z) > 1\} \subset Q_a(0)$ for some $a > 0$. But since $k(z) \to 0$ as $z \to \infty$, the set where $k > 1$ is bounded. Thus we have

$$M_s(\nu)(x) \leqslant C\|T\|_{pq}^q, \quad \text{for all } x.$$

COROLLARY. *If k is a homogenous kernel* ($\beta = 1/r$), *then*

$$T\colon X_p(l_n) \to Y_q(\nu),\ 1 < p < q < \infty,\ s/q = r/p + 1 - r$$

if and only if

$$M_s(\nu) \text{ is bounded.}$$

Furthermore, there is a constant C independent of ν such that

$$C^{-1} \sup_x M_s(\nu)(x)^{1/q} \leqslant \|T\|_{pq} \leqslant C \sup_x M_s(\nu)(x)^{1/q}.$$

Proof. For the upper estimate on $\|T\|_{pq}$, refer to (1) which now gives

$$t\nu(E_t)^{1/q} \leqslant C \sup_x M_s(\nu)(x)^{1/q} X_p(f; l_n).$$

Applying the Marcinkiewicz interpolation theorem of [8] (where the interpolation constant is carefully estimated), we get the desired result.

4. Remarks. We now consider two special kernels on the half space $\boldsymbol{R}_+^{n+1} = \{(x, t)\colon x \in \boldsymbol{R}^n, t > 0\}$, namely

$$\Gamma_a(x, t) = t^{(a-n-2)/2} \exp(-|x|^2/4t),\ t > 0$$

and $\Gamma_a = 0$, $t \leqslant 0$; $0 < a < n+2$. And

$$P_a(x, t) = t(|x|^2 + t^2)^{(a-n-3)/2},\ t > 0$$

and $P_a = 0$, $t \leqslant 0$; $0 < a < n+1$. P_2 is the Poisson kernel for the Laplace equation in $\boldsymbol{R}_+^{n+1}$ and Γ_2 is the fundamental solution of the heat equation in $\boldsymbol{R}_+^{n+1}$.

Let $P_a(t)*f(y) = \int P_a(y-x, t) f(x)\, dx$ and similarly for $\Gamma_a(t)*f(y)$.

THEOREM C. *Let $Y = \boldsymbol{R}_+^{n+1}$ and ν a Borel measure on Y. Suppose that p and q satisfy $1 < p < q < \infty$, $d/q = n/p + 2 - a$, $0 < d \leqslant n+1$, then the necessary and sufficient condition for*

$$Y_q(P_a(\cdot)*f; \nu) \leqslant C X_p(f; l_n) \tag{3}$$

or all $f \in X_p$, C independent of f is that

$$\sup_{\substack{r>0 \\ x^0 \in \boldsymbol{R}^n}} r^{q(2-a)-d} \int_0^r \int_{|y-x^0|<r} t^{(a-2)q}\, d\nu(y, t) < \infty. \tag{4}$$

Similarly, if P_a is replaced by Γ_a, (4) is replaced by

$$\sup_{\substack{r>0 \\ x^0 \in \boldsymbol{R}^n}} r^{q(2-a)-d} \int_0^{r^2} \int_{|y-x^0|<r} t^{(a/2-1)q}\, d\nu(y, t) < \infty, \tag{5}$$

$0 < d \leqslant n+2$.

We outline the proof. The necessity of (4) or (5) follows as in Lemma 2, while the sufficiency is a consequence of Theorem A taking $d\mu = dx\, d\delta_0(t)$, δ_0 = Dirac measure at zero. Writing $j(x, t) = t^{a-2} P_a(x, t)$ and assuming (4) holds for a fixed p_0, q_0 such that $1 < p_0 < q_0 < \infty$, $d/q_0 = n/p_0 + 2 - a$, the homogeneity of P_a together with Theorem A then implies that (3) holds for all p and q such that $1 < p < q < \infty$, $q_0/q = p_0/p$. The result for Γ_a is even easier.

The purpose in pointing out Theorem C is that when $d = n$, we have $n\left(\frac{1}{p} - \frac{1}{q}\right) = a-2$ and $2 < a < 2 + n/p$, and the limiting case $p = q$ (which is *not* a consequence of Theorem A) is an important result of Carleson–Hormander involving the theory of maximal functions. See [3].

In the same vein, Theorem A also gives:

$$Y_q(P_a(\cdot)*f; \nu) \leqslant C\left[\frac{p}{q} X_p(f; M_s(\nu) l_n) + \left(1 - \frac{p}{q}\right) X_p(f; l_n)\right]$$

$$1 < p < q < \infty,\ s = n/(n+2-a),\ n\left(\frac{1}{p} - \frac{1}{q}\right) = a - 2.$$

Again the limiting case $p = q$ is known, [2]. Unfortunately, $C = 0\left(\frac{1}{a-2}\right)$ here.

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