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Some generalizations of the hypersingular integral operators

by

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Abstract. Estimates are given for hypersingular integral operators, G^α, with variable kernel. More general Schwartz distributions inducing operators of the form G^α are considered and estimates are given for the resulting operators. G^α is shown to be continuous on certain spaces with weighted norm. Commutators of certain G^α with multiplication by certain bounded functions are studied.

1. Introduction. In this paper we shall study several topics related to the hypersingular integral operators which were studied in [3], [4]. These operators were first studied in [11] using the Marcinkiewicz interpolation theorem and the method of rotation for values of the parameter in the range $0 < \alpha < 2$. In [3], [4] complex powers of operators, as in Komatsu's theory [6], were used to evaluate and estimate the hypersingular singular integral operators G^α for $\mathrm{Re}(\alpha) \geqslant 0$, $\alpha \neq 0$, and it was shown in [4] that the family G^α is closely related to the singular integral operators of Calderón and Zygmund.

We shall establish some notation and quote the main theorem from [3], [4] before discussing the topics of this paper. Let E denote N-dimensional Euclidean space and let dx be Lebesgue measure on E. Let Σ denote the unit sphere in E and let $d\sigma$ be Lebesgue measure on Σ. If $\beta = (\beta_1, \beta_2, \ldots, \beta_N)$ is a multi-index of non-negative integers, $|\beta| = \sum_{i=1}^{N} \beta_i$, $x^\beta = x_1^{\beta_1} \cdot x_2^{\beta_2} \cdot \ldots \cdot x_N^{\beta_N}$ for $x = (x_1, x_2, \ldots, x_N) \in E$, $\beta! = \prod_{k=1}^{N} (\beta_k!)$ and $D^\beta = D_1^{\beta_1} D_2^{\beta_2} \ldots D_N^{\beta_N}$ when $D_k = \dfrac{\partial}{\partial x_k}$. If f is a smooth function with compact support in E, set

$$R_k(f, y)(x) = f(x-y) - \sum_{|\beta| \leqslant k} \frac{D^\beta f(x)}{\beta!} (-y)^\beta.$$

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Let Ω be a complex valued function on E which is homogeneous of degree zero and integrable over Σ. Set

$$G^{a}f(x) = \lim_{\varepsilon\to 0}\int_{\|y\|>\varepsilon} R_k(f,y)(x)\frac{\Omega(y)}{\|y\|^{N+a}}dy$$

for $k<\mathrm{Re}(a)<k+1$; G^a is a hypersingular integral operator of order a. If $\int_\Sigma \omega^\beta \Omega(\omega)d\sigma(\omega) = 0$ when $|\beta| = k$, then $G^a g(x)$ exists when $\mathrm{Re}(a) = k>0$.

Let $P_tf(x) = C_N\int_E f(x-y)t(\|y\|^2+t^2)^{-(N+1)/2}\,dy$ denote the Poisson integral over E and set

$$J^a f(x) = \Gamma(a)^{-1}\lim_{\varepsilon\to 0^+}\left[\int_\varepsilon^\infty P_tf(x)t^{a-1}e^{-t}dt+\frac{\varepsilon^a f}{a}\right]$$

for $\mathrm{Re}(a)\geqslant 0$; J^a is the ath order Bessel potential over E. J^a is an analytic semi-group on $L_p(E)$, $1<p<\infty$, in $|\arg(a)|<\pi/2$ and a strongly continuous semi-group on $L_p(E)$, $1<p<\infty$, in $\mathrm{Re}(a)\geqslant 0$. $L_p(E)$ denotes $L_p(E, dx)$; fix p in $1<p<\infty$. Let $L_p^a(E)$ denote the range of J^a acting on $L_p(E)$. If $\mathrm{Re}(a)\geqslant 0$, J^a is one-to-one and the range of J^a is dense in $L_p(E)$; the norm in $L_p^a(E)$ is defined by $\|g\|_{p,a} = \|f\|_p$ when $J^a(f) = g$.

The main theorem from [3], [4] concerning the hypersingular integral operators is:

THEOREM 1.1. *If* $0<k<\mathrm{Re}(a)<k+1$,

$$G^aJ^af(x) = \Gamma(-a)\int_\Sigma (D_\omega^a J^a)f(x)\,\Omega(\omega)\,d\sigma(\omega)$$

where D_ω^a is the ath power of the derivative in the direction ω. $S_a = D_\omega^a J^a$ is an analytic semi-group of bounded operators on $L_p(E)$ for $1<p<\infty$ in $|\arg(a)|<\pi/2$ and a strongly continuous semi-group of bounded operators on $L_p(E)$ in $\mathrm{Re}(a)\geqslant 0$. If $\mathrm{Re}(a) = k$, if $a\neq k$, and if $\int_\Sigma\omega^\beta\Omega(\omega)d\sigma(\omega) = 0$ for each $|\beta| = k$, then

$$G^aJ^af(x) = \Gamma(-a)\int_\Sigma (D_\omega^a J^a)f(x)\,\Omega(\omega)\,d\sigma(\omega).$$

If $a = k>0$ and if $\int_\Sigma\omega^\beta\Omega(\omega)d\sigma(\omega) = 0$ for each $|\beta| = k$, then

$$G^aJ^af(x) = \frac{(-1)^{k+1}}{k!}\int_\Sigma \frac{\partial}{\partial a}(D_\omega^a J^a)f(x)|_{a=k}\,\Omega(\omega)\,d\sigma(\omega).$$

This representation for G^aJ^a shows that G^a is a bounded operator from $L_p^a(E)$ to $L_p(E)$ for $\mathrm{Re}(a)\geqslant 0$, $a\neq 0$. See [4] for the case $a = 0$, Calderón–Zygmund operators.

In this paper we shall continue to rely on Komatsu's series of papers [6] on complex powers of operators to study the following topics related to the operators G_a: Since the boundedness of G^aJ^a on $L_p(E)$ follows from the properties of $D_\omega^aJ^a$, Theorem 1.1 admits an immediate generalization for certain kernels $\Omega(\omega, x)$ and for certain vector valued kernels. In the third section of the paper we notice that G^a is determined by the application of a certain distribution to the translation semi-group $T_{r\omega}f(x) = f(x-r\omega)$ and we study other distributions which define similar operators. In the next section we consider modified operators G_v^a which depend analytically on a parameter v; these operators are used later to study G^a on spaces with weighted norms. In Section 6 of the paper the commutators of certain G^a with multiplication by certain bounded functions are studied. Throughout the paper, we emphasize the operators $D_\omega^{i\gamma}$, $D_\omega^{k+i\gamma}$, $G^{i\gamma}$, and $G^{k+i\gamma}$, k an integer, since corresponding results for other indices a can be made to follow by interpolation.

For analysis on Euclidean space, we use the notation introduced above. If T is a linear operator with domain and range in $L_p(E)$, $D(T)$ denotes the domain of T and $R(T)$ denotes the range of T. M, $M(p)$, $M(p,a)$, etc. (K, $K(p)$, $K(p,a)$, etc.) denote positive (complex) constants which depend only on the parameters shown and whose values vary with the occasion of their use. We frequently let C_N denote a normalizing constant which depends on the dimension of E. $\langle f,g\rangle = \int_\Sigma f(t)g(t)\,dt$ denotes the dual pairing between $L_p(E)$ and $L_q(E)$ when $1/p+1/q = 1$. We shall frequently let p' denote the index conjugate to p, $1/p+1/p' = 1$.

2. Hypersingular integrals with variable kernel. Let $\Omega(y,x)$ be a homogeneous function of degree zero in y, regard Ω as a function from the unit sphere in E to $L_r(E)$, $1\leqslant r\leqslant\infty$, and suppose that $\int_\Sigma\|\Omega(\omega,\cdot)\|_r d\sigma(\omega) < \infty$. If f is a smooth function with compact support on E, set

$$R_k(f,y)(x) = f(x-y)-\sum_{|\beta|\leqslant k}\frac{D^\beta f(x)}{\beta!}(-y)^\beta = \int_0^1\frac{(1-t)^k}{k!}\left(\frac{\partial}{\partial t}\right)^{k+1}f(x-ty)\,dt$$

and define

$$G^af(x) = \lim_{\varepsilon\to 0}\int_{\|y\|>\varepsilon}R_k(f,y)(x)\frac{\Omega(y,x)}{\|y\|^{N+a}}dy$$

when $k \leqslant \operatorname{Re}(\alpha) < k+1$. Since we may use polar coordinates to write

$$G^\alpha f(x) = \int_\Sigma \int_{0^+}^\infty R_k(f, r\omega) r^{-\alpha-1} dr\, \Omega(\omega, x)\, d\sigma(\omega),$$

we have that for $k < \operatorname{Re}(\alpha) < k+1$,

$$G^\alpha f(x) = \Gamma(-\alpha) \int_\Sigma D_\omega^\alpha f(x)\, \Omega(\omega, x)\, d\sigma(\omega);$$

see [4] for a short summary of the theory of complex powers of operators which was given in [6]. If J^α denotes the αth order Bessel potential, $D_\omega^\alpha J^\alpha$ is an analytic semi-group of bounded operators on $L_p(E)$ in $|\arg(\alpha)| < \pi/2$ and $D_\omega^\alpha J^\alpha$ is a strongly continuous semi-group of bounded operators on $L_p(E)$ for $\operatorname{Re}(\alpha) \geqslant 0$. Thus

THEOREM 2.1. *If* $\int_\Sigma \|\Omega(\omega, \cdot)\|_r d\sigma(\omega) < \infty$, *if* $1/p + 1/r = 1/q$, *and if* $0 \leqslant k < \operatorname{Re}(\alpha) < k+1$, *then*

$$G^\alpha J^\alpha f(x) = \Gamma(-\alpha) \int_\Sigma D_\omega^\alpha J^\alpha f(x)\, \Omega(\omega, x)\, d\sigma(\omega).$$

If $\operatorname{Re}(\alpha) = k \geqslant 0$, *if* $\alpha \neq k$, *and if* $\int_\Sigma \omega^\beta \Omega(\omega, x)\, d\sigma(\omega) = 0$ *(a.e.) when* $|\beta| = k$, *then*

$$G^\alpha J^\alpha f(x) = \frac{(-1)^{k+1}}{k!} \int_\Sigma \frac{\partial}{\partial \alpha} (D_\omega^\alpha J^\alpha) f(x)|_{\alpha=k}\, \Omega(\omega, x)\, d\sigma(\omega).$$

In all cases,

$$\|G^\alpha J^\alpha f\|_q \leqslant M(p, \alpha) \int_\Sigma \|\Omega(\omega, \cdot)\|_r d\sigma(\omega)\, \|f\|_p.$$

Proof. Since the proof is essentially the same as that given for the constant valued kernel in [3] for $\operatorname{Re}(\alpha) \neq k$ and in [4] for $\operatorname{Re}(\alpha) = k$, we shall only sketch the development. If $0 < \operatorname{Re}(\delta) < 1$, then $\Gamma(-\delta)^{-1} \int_{0+}^\infty \big(f(x - r\omega) - f(x)\big) t^{-\delta-1} dt = D_\omega^\delta f(x)$ for smooth functions f and elementary computations show that if $\alpha = k + \delta$, $0 < \operatorname{Re}(\delta) < 1$, then

$$\int_{0+}^\infty R_k(f, r\omega) r^{-\alpha-1} dr = \Gamma(-\alpha) D_\omega^\delta D_\omega^k f = \Gamma(-\alpha) D_\omega^\alpha f.$$

It was shown in [4] that $S_\alpha = D_\omega^\alpha J^\alpha$ is an analytic semi-group on $\operatorname{Re}(\alpha) > 0$ and that the boundary value group is strongly continuous and consists of bounded operators. In $k < \operatorname{Re}(\alpha) < k+1$ $G^\alpha J^\alpha$ has the desired form

and estimate. If $\operatorname{Re}(\alpha) = k$, if $\alpha \neq k$, and if $\int_\Sigma \omega^\beta \Omega(\omega, x)\, d\sigma(\omega) = 0$ (a.e.) for all $|\beta| = k$, then $\int_\Sigma R_k(f, r\omega)\, d\sigma(\omega) = \int_\Sigma R_{k-1}(f, r\omega)\, d\sigma(\omega)$ and one can use the fact [4] that

$$D_\omega^{i\gamma} f(x) = \Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0} \left[\int_\varepsilon^\infty f(x - r\omega) r^{-i\gamma-1} dr - \frac{\varepsilon^{-i\gamma} f(x)}{i\gamma} \right]$$

and some identities of the Γ-function to conclude that $G^\alpha J^\alpha$ has the desired form. When $\alpha = k > 0$ and

$$\int_\Sigma \omega^\beta \Omega(\omega, x)\, d\sigma(\omega) = 0 \text{(a.e.) for all } |\beta| = k,$$

$$\int_\Sigma D_\omega^k J^k f(x)\, \Omega(\omega, x)\, d\sigma(\omega) = 0 \text{(a.e.)},$$

and one can use the fact that

$$\begin{aligned}\Gamma(-\alpha) &= \Gamma(1+\alpha)\Gamma(-\alpha)\Gamma(1+\alpha)^{-1} = -\pi/\big(\Gamma(1+\alpha)\sin\pi\alpha\big)^{-1} \\ &= -\pi\Gamma(1+\alpha)^{-1}\big((\sin\pi\alpha - \sin\pi k)/(\alpha - k)\big)^{-1}(\alpha - k)^{-1}\end{aligned}$$

and the fact that $D_\omega^\alpha J^\alpha$ is an analytic semi-group to show that

$$\lim_{\alpha\to k} G^\alpha J^\alpha f(x) = \frac{(-1)^{k+1}}{k!} \int_\Sigma \frac{\partial}{\partial\alpha} (D_\omega^\alpha J^\alpha) f(x)|_{\alpha=k}\, \Omega(\omega, x)\, d\sigma(\omega).$$

Since $\frac{\partial}{\partial\alpha} S_\alpha|_{\alpha=k} = -i \frac{\partial}{\partial\gamma} S_{k+i\gamma}|_{\gamma=0}$, one uses the formula for the infinite smal generator of $D_\omega^{i\gamma}$, Theorem 9 of [4], and the fact that $\int_\Sigma D_\omega^k \frac{\partial}{\partial\alpha} J^\alpha f(x)|_{\alpha=k} \Omega(\omega, x)\, d\sigma(\omega) = 0$ (a.e.), to conclude that $G^k J^k f = \lim_{\alpha\to k} G^\alpha J^\alpha f$.

The estimates for the $G^\alpha J^\alpha$ now follow from the estimates for $D_\omega^\alpha J^\alpha$ [3] and for $\frac{\partial}{\partial\alpha}(D_\omega^\alpha J^\alpha)$ [4], Hölder's inequality, and Minkowski's integral inequality. This completes the proof.

Several of the generalized forms of the hypersingular integral operators which are to be discussed in the sequel admit variable kernel generalizations. We will not make these generalizations explicit. It is not hard to see that the theorem of this section extends to the situation where f is a Banach space valued function and Ω has values in the continuous linear operators from the range space of f to another Banach space. One needs to show first that $D_\omega^\alpha J^\alpha$ is an analytic semi-group of bounded operators with a strongly continuous boundary value group on vector valued functions; this can be proved by using the vector valued version of M. Riesz's Theorem on the Hilbert transform and by following the argument in [3] for the constant valued case since the kernel Ω_α for $D^\alpha_\omega J^\alpha$ depends only on the one dimensional subspace of E generated by ω. See the argument in the proof of Theorem 4 of [3] for details.

3. Schwartz distributions and hypersingular integrals. The hypersingular integrals discussed in the introduction and in Section 2 of this paper arise from the application of the Schwartz distribution $t_+^{-\alpha-1}$, $\mathrm{Re}(\alpha)\geqslant 0$, $\alpha\neq 0$, to the semi-group of operators $T_tf(x)=f(x-t\omega)$ when $\omega\in E$ and $t\geqslant 0$. From the Theorem 1.1 mentioned in the introduction, one surmises that there are other distributions over the positive half-line, R^+, which define similar families of operators. It is the purpose of this section to mention several classes of these distributions and to investigate the operators arising from the application of some of these distributions to the time variable of a semi-group. The distributions which we shall use are discussed in detail in [5].

If φ is a smooth function with compact support in R, $t_+^{-\alpha-1}$ is defined by analytic continuation for $\mathrm{Re}(\alpha)\geqslant 0$ and α not a positive integer. If $\mathrm{Re}(\alpha)<n+1$,

$$t_+^{-\alpha-1}(\varphi)=\int_0^1 t^{-\alpha-1}R_n(\varphi,t)\,dt+\int_1^\infty t^{-\alpha-1}\varphi(t)\,dt+\sum_{k=0}^{n}\frac{\varphi^{(k)}(0)}{k!\,(k-\alpha)}$$

when $R_n(\varphi,t)=\varphi(t)-\sum_{k=0}^{n}\frac{\varphi^{(k)}(0)\,t^k}{k!}$. If $n<\mathrm{Re}(\alpha)<n+1$,

$$t_+^{-\alpha-1}(\varphi)=\int_0^\infty t^{-\alpha-1}R_n(\varphi,t)\,dt.$$

The first formula shows that $t_+^{-\alpha-1}$ has a simple pole at $\alpha=n$ with residue $\frac{(-1)^n}{n!}\delta^{(n)}(t)$, where $\delta^{(n)}$ denotes the nth derivative of the δ-function. t_+^{-n-1} is defined to be

$$t_+^{-n-1}(\varphi)=\int_0^\infty t^{-n-1}\left[R_{n-1}(\varphi,t)-\frac{\varphi^{(n)}(0)\,t^n}{n!}\Theta(1-t)\right]dt$$
$$=\lim_{\alpha\to n}\frac{\partial}{\partial\alpha}(n-\alpha)\,t_+^{-\alpha-1}(\varphi)$$

where $\Theta(x)$ is the characteristic function of R^+ and where the derivative $\frac{\partial}{\partial\alpha}$ is computed in the sense of distributions. Similarly, one defines $t_-^{-\alpha-1}$ for $\mathrm{Re}(\alpha)\geqslant 0$, $\alpha\neq 0$; $t_-^{-\alpha-1}(\varphi)=t_+^{-\alpha-1}(\tilde\varphi)$ where $\tilde\varphi(t)=\varphi(-t)$. Then one defines $|t|^{-\alpha-1}=t_+^{-\alpha-1}+t_-^{-\alpha-1}$, $|t|^{-\alpha-1}\mathrm{sgn}(t)=t_+^{-\alpha-1}-t_-^{-\alpha-1}$, $\Gamma(-\alpha)^{-1}\times t_+^{-\alpha-1}$, $\Gamma(-\alpha)^{-1}t_-^{-\alpha-1}$, $\Gamma(-\alpha/2)^{-1}|t|^{-\alpha-1}$, and $\Gamma\big((1-\alpha)/2\big)^{-1}|t|^{-\alpha-1}\mathrm{sgn}(t)$ so that

$$\lim_{\alpha\to n}\Gamma(-\alpha)^{-1}t_+^{-\alpha-1}=\delta^{(n)}(t),$$
$$\lim_{\alpha\to n}\Gamma(-\alpha)^{-1}t_-^{-\alpha-1}=(-1)^n\delta^{(n)}(t),$$
$$\lim_{\alpha\to 2m}\Gamma(-\alpha/2)\,|t|^{-\alpha-1}=\big((2m)!\big)^{-1}(-1)^m\delta^{(2m)}(t)\,m!,$$
$$\lim_{\alpha\to 2m-1}\frac{|t|^{-\alpha-1}\mathrm{sgn}(t)}{\Gamma\big((1-\alpha)/2\big)}=\frac{(-1)^{m+1}\delta^{(2m+1)}(t)\,(m+1)!}{(2m+1)!}.$$

If f is a smooth function with compact support in E and if $\omega\in E$, $t_+^{-\alpha-1}(T_tf)(x)=\Gamma(-\alpha)Df^\alpha_\omega(x)$ where $-D_\omega$ is the infinitesmal generator of $T_tf(x)=f(x-t\omega)$; i.e. D_ω is the derivative of f in the direction ω, and

$$t_+^{-n-1}(T_tf)(x)$$
$$=\lim_{\alpha\to n}\frac{\partial}{\partial\alpha}(n-\alpha)\,t_+^{-\alpha-1}(T_tf)(x)=\lim_{\alpha\to n}\frac{\partial}{\partial\alpha}[(n-\alpha)\Gamma(-\alpha)D^\alpha_\omega f(x)].$$

The relationship between $t_+^{-\alpha-1}$ and D^α_ω follows from Komatsu's formula [6] for D^δ_ω when $0<\mathrm{Re}(\delta)<1$:

$$D^\delta_\omega f(x)=\Gamma(-\delta)^{-1}\int_{0+}^{\infty}\big(f(x-t\omega)-f(x)\big)t^{-\delta-1}\,dt.$$

Thus it is seen that at integers $n>0$ the operator $\int_\Sigma(\cdot)\,\Omega(\omega)\,d\sigma(\omega)$ with $\int_\Sigma\omega^\beta\,\Omega(\omega)\,d\sigma(\omega)=0$ for $|\beta|=n$ is needed to insure that $G^nJ^n=\lim_{\alpha\to n}G^\alpha J^\alpha$.

From [5] it is known that the only homogeneous distributions of degree $(-\alpha)$ are of the form $C_1t_+^{-\alpha-1}+C_2t_-^{-\alpha-1}$ when α is not a positive integer. If $\alpha=n$, the most general distribution which is homogeneous of degree $(-n)$ is of the form $C_1t^{-n-1}+C_2\delta^{(n)}(t)$ where $t^{-n-1}=t_+^{-n-1}+t_-^{-n-1}$ if n is odd and $t^{-n-1}=t_+^{-n-1}-t_-^{-n-1}$ if n is an even integer. For the homogeneous distributions there is

THEOREM 3.1. *Let n be a non-negative integer. If λ_α is a distribution over the real line which is homogeneous of degree $(-\alpha)$ with $n<\mathrm{Re}(\alpha)<n+1$, then if $y\neq 0$ is a vector in E and if $T_tf(x)=f(x-ty)$ for smooth functions f with compact support in E, then*

$$\lambda_\alpha(T_tf)(x)=\Gamma(-\alpha)[C_1D^\alpha_yf(x)+C_2(-D_y)^\alpha f(x)]$$

for complex constants C_1 and C_2. If λ_n is a distribution which is homogeneous of degree $(-n)$ over the real line, then when n is even or zero,

$$\lambda_n(T_tf)(x) = C_1 H_y D_y^n(f) + C_2 D_y^n(f)$$

where $H_y(g)(x) = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \big(T_tg(x) - T_{-t}g(x)\big)dt/t$ *is the Hilbert transform in the direction* y; *when* n *is an odd integer*

$$\lambda_n(T_tf) = C_1 \int_0^\infty t^{-n-1}\big(R_{n-1}(T_tf, t) + R_{n-1}(T_tf, -t)\big)dt + C_2 D_y^n f(x).$$

Proof. These formulas follow immediately from the observations in [5] regarding the general form of the homogeneous distributions over the real line. The case where n is an even integer contains a statement which requires explanation. Notice that

$$\begin{aligned} t^{-n-1}(T_tf) &= \int_{0+}^\infty t^{-n-1}\big(R_{n-1}(T_tf, t) - R_{n-1}(T_tf, -t)\big)dt \\ &= \big((n-1)!\big)^{-1}\int_{0+}^\infty t^{-n-1}\int_0^t D_y^n(T_uf - T_{-u}f)(t-u)^{n-1}du\,dt \\ &= \big((n-1)!\big)^{-1}\int_{0+}^\infty t^{-1}\int_0^1 D_y^n(T_{tu}f - T_{-tu}f)(1-u)^{n-1}du\,dt \\ &= (n!)^{-1} H_y D_y^n f(x) \end{aligned}$$

where $H_y g(x) = \int_{0+}^\infty t^{-1}(T_tg - T_{-t}g)\,dt$ is the Hilbert transform in the direction y, a bounded operator on $L_p(E)$.

This theorem suggests the following result.

COROLLARY 3.2. *If* Ω *is a complex valued function on* E *which is homogeneous of degree zero and integrable over* Ω *and satisfies* $\Omega(-\omega) = (-1)^{n+1}\Omega(\omega)$ *where* n *is a positive integer. Then the hypersingular integral operator with kernel* Ω *and order* n *satisfies*

$$G^n f(x) = K(n)\int_\Sigma D_\omega^n H_\omega f(x)\,\Omega(\omega)\,d\sigma(\omega)$$

where H_ω *is the Hilbert transform in the direction* ω.

Proof. Since $\Omega(-\omega) = (-1)^{n+1}\Omega(\omega)$,

$$\int_\Sigma R_n(f, r\omega)(x)\,\Omega(\omega)\,d\sigma(\omega) = \int_\Sigma R_{n-1}(f, r\omega)\,\Omega(\omega)\,d\sigma(\omega)$$

and

$$\int_\Sigma (-D_{r\omega})^n f(x - r\omega)\,\Omega(\omega)\,d\sigma(\omega) = -\int_\Sigma (-D_{r\omega})^n f(x + r\omega)\,\Omega(\omega)\,d\sigma(\omega).$$

Thus

$$2G^n f(x) = K\int_\Sigma\int_{0+}^\infty [(-D_\omega)^n f(x - r\omega) - (-D_\omega)^n f(x + r\omega)]r^{-1}dr\,\Omega(\omega)\,d\sigma(\omega),$$

which has the desired form for G^n.

Nelson [8] has studied a general class of distributions over R^+ which define functions of infinitesmal generators of semi-groups. Pertaining to the translation semi-group $T_tf(x) = f(x - t\omega)$, a contraction semi-group, Nelson studies the algebra of distributions $\bigcup_{n=0}^\infty \mathfrak{F}^n(1)$ where φ is in $\mathfrak{F}^n(1)$ if for smooth functions f with compact support,

$$\varphi(f) = \sum_{i=1}^n \int_0^\infty \left(\frac{\partial}{\partial t}\right)^i f(t)\,d\mu_i(t)$$

where the μ_i are finite Borel measures. The homogeneous distribution λ_a, a not a positive integer, which were considered above, have the form of a Nelson functional. One sees readily that distributions from $\mathfrak{F}^n(1)$ define operators from $L_p^a(E)$, $\operatorname{Re}(a) \geqslant n$, to $L_p(E)$. We shall examine another situation which is more closely related to the homogeneous distributions.

Many distributions, including derivatives of the ones listed above, which depend on a parameter a can be expressed in terms of t_+^{-a-1} and t_-^{-a-1}; e.g., $\left(\frac{\partial}{\partial a}\right)^m t_+^{-a-1} = t_+^{-a-1}(-\log t_+)^m$. For test functions φ,

$$t_+^{-a-1}(-\log t_+)^m(\varphi) = \int_{0+}^\infty t^{-a-1}(-\log t)^m R_n(\varphi, t)\,dt$$

if $n < \operatorname{Re}(a) < n+1$. Each of these distributions defines an operator $U(D_\omega)$ when D_ω is the negative of the infinitesmal generator of the semi-group $T_tf(x) = f(x - t\omega)$ and we expect these operators $U(D_\omega)$ to be continuous from some $L_p^\beta(E)$ to $L_p(E)$.

THEOREM 3.3. *If* a *is not a non-negative integer and if* m *is a non-negative integer, then for smooth functions* f *with compact support*

$$\left(\frac{\partial}{\partial a}\right)^m t_+^{-a-1}(T_tf)(x) = t_+^{-a-1}(-\log t_+)^m(T_tf)(x) = \left(\frac{\partial}{\partial a}\right)^m [\Gamma(-a)D_y^a f(x)].$$

If $\operatorname{Re}(\beta) > \operatorname{Re}(a)$ *and if* J^β *denotes the* β*th order Bessel potential, then* $V(a, \beta, m)f = J^\beta\left(\frac{\partial}{\partial a}\right)^m t_+^{-a-1}(T_tf)$ *extends to a bounded operator on* $L_p(E)$.

Proof. Since $S_a = D_y^a J^a$, $\operatorname{Re}(a) > 0$, and J^δ, $\operatorname{Re}(\delta) > 0$ are analytic semi-groups, $\left(\frac{\partial}{\partial a}\right)^m (D_y^a J^a)$ and $\left(\frac{\partial}{\partial \delta}\right)^n J^\delta$ are bounded operators on L_p (E) for all non-negative integers m and n and for positive $\operatorname{Re}(a)$ and $\operatorname{Re}(\delta)$. Since if $\beta = a + \delta$ with $\operatorname{Re}(\delta) > 0$.

$$J^\beta\left(\frac{\partial}{\partial a}\right)^m [\Gamma(-a)D_y^a(f)] = \sum_{k=0}^m \binom{m}{k}\Gamma^{(k)}(-a)J^{a+\delta}\left(\frac{\partial}{\partial a}\right)^{m-k} D_y^a(f),$$

it is sufficient to prove that for each integer $n \leqslant m$,

$$J^{\alpha+\delta}\left(\frac{\partial}{\partial\alpha}\right)^n D_y^\alpha(f) = \sum_{j=0}^{n}\sum_{k=0}^{n} A_{kj}\left(\frac{\partial}{\partial\alpha}\right)^k (D_y^\alpha J^\alpha)\left(\frac{\partial}{\partial\delta}\right)^j J^\delta(f)$$

where the A_{kj} are constants. This identity is verified by induction on n.

If $n = 1$,

$$J^{\alpha+\delta}\left(\frac{\partial}{\partial\alpha}\right) D_y^\alpha(f) = \frac{\partial}{\partial\alpha}(D_y^\alpha J^\alpha) J^\delta(f) - D_y^\alpha\left(\frac{\partial}{\partial\alpha}J^\alpha\right) J^\delta(f)$$
$$= \frac{\partial}{\partial\alpha}(D_y^\alpha J^\alpha) J^\delta(f) - D_y^\alpha J^\alpha \frac{\partial}{\partial\delta} J^\delta(f).$$

Assume that the identity holds for $n-1$. Then

$$J^{\alpha+\delta}\left(\frac{\partial}{\partial\alpha}\right)^n D_y^\alpha(f) = \frac{\partial}{\partial\alpha}\left(J^\alpha\left(\frac{\partial}{\partial\alpha}\right)^{n-1} D_y^\alpha(J^\delta f)\right) - \left(\frac{\partial}{\partial\alpha}J^\alpha\right)\left(\left(\frac{\partial}{\partial\alpha}\right)^{n-1} D_y^\alpha(J^\delta f)\right)$$
$$= \frac{\partial}{\partial\alpha}\left(J^\alpha\left(\frac{\partial}{\partial\alpha}\right)^{n-1} D_y^\alpha(J^\delta f)\right) - J^\alpha\left(\left(\frac{\partial}{\partial\alpha}\right)^{n-1} D_y^\alpha\left(\frac{\partial}{\partial\delta}J^\delta f\right)\right).$$

$J^\alpha\left(\left(\frac{\partial}{\partial\alpha}\right)^{n-1} D_y^\alpha\left(\frac{\partial}{\partial\delta}J^\delta\right)(f)\right)$ is a bounded operator as the representation in the induction assumption shows. By the induction assumption,

$$\frac{\partial}{\partial\alpha}\left(J^\alpha\left(\frac{\partial}{\partial\alpha}\right)^{n-1} D_y^\alpha(J^\delta f)\right) = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} A_{kj}\left(\frac{\partial}{\partial\alpha}\right)^{k+1}(D_y^\alpha J^\alpha)\left(\frac{\partial}{\partial\delta}\right)^j J^\delta(f).$$

Thus $J^{\alpha+\delta}\left(\frac{\partial}{\partial\alpha}\right)^n D_y^\alpha(f)$ has the desired representation and extends to a bounded operator on $L_p(E)$. This completes the proof.

COROLLARY 3.4. *If Ω is homogeneous of degree zero on E and integrable over the unit sphere Σ in E, if α is not a non-negative integer, and if m is a non-negative integer, then when $n < \mathrm{Re}(\alpha) < n+1$ and f is a smooth function with compact support in E*

$$U(\Omega, \alpha)f = \lim_{\varepsilon\to 0}\int_{\|y\|>\varepsilon} R_n(f, y)(x)\,\Omega(y)(\log\|y\|)^m \frac{dy}{\|y\|^{N+\alpha}}$$

extends to a bounded operator from $L_p^\beta(E)$ to $L_p(E)$ for each β with $\mathrm{Re}(\beta) > \mathrm{Re}(\alpha)$.

Proof. Rewrite the integral in polar coordinates as

$$U(\Omega, \alpha)f = \int_\Sigma\int_{0+}^{\infty} R_n(f, r\omega)(x)(\log r)^m \frac{dr}{r^{1+\alpha}}\,\Omega(\omega)\,d\sigma(\omega).$$

By Theorem 3.3,

$$\int_{0+}^{\infty} R_n(f, r\omega)(x)(\log r)^m \frac{dr}{r^{1+\alpha}} = (-1)^m\left(\frac{\partial}{\partial\alpha}\right)^m t_+^{-\alpha-1}(T_t f)(x)$$
$$= (-1)^m\left(\frac{\partial}{\partial\alpha}\right)^m \left(\Gamma(-\alpha)D_\omega^\alpha f\right)$$

extends to a bounded operator from $L_p^\beta(E)$ to $L_p(E)$ when $\mathrm{Re}(\beta) > \mathrm{Re}(\alpha)$. The norm of $J^\beta\left(\frac{\partial}{\partial\alpha}\right)^m\left(\Gamma(-\alpha)D_\omega^\alpha f\right)$ does not depend on ω. Thus

$$J^\beta U(\Omega, \alpha)f = \int_\Sigma J^\beta\left(\frac{\partial}{\partial\alpha}\right)^m\left(\Gamma(-\alpha)D_\omega^\alpha f\right)\Omega(\omega)\,d\sigma(\omega)$$

extends to a bounded operator on $L_p(E)$.

The distributions $t_+^{-n-1}(\log t_+)^m$ can also be considered, but these distributions are not related to the α-derivatives of $t_+^{-\alpha-1}$. They are related to the α-derivatives of the functional

$$F_{-n}(t_+, \alpha)(\varphi) = \int_0^\infty t^{-\alpha-1}\left[R_{n-1}(\varphi, t) - \frac{t^n\varphi^{(n)}(0)}{n!}\Theta(1-t)\right]dt$$

where Θ is the characteristic function of R^+.

4. Dependence on parameters. Preliminary to discussing the action of the hypersingular integrals on spaces with weighted norms, we shall study certain modifications of $\left(\frac{d}{dx}\right)^{i\gamma} = D^{i\gamma}$ on $L_p(R)$, R is the real line. From [4] we know that for real γ

$$D^{i\gamma}f(x) = \Gamma(-i\gamma)^{-1}\lim_{\varepsilon\to 0}\left[\int_\varepsilon^\infty f(x-t)t^{-1-i\gamma}\,dt - \frac{\varepsilon^{-i\gamma}f(x)}{i\gamma}\right]$$
$$= \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}\big(f(x-t)\big)$$

and

$$(-D)^{i\gamma}f(x) = \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}\big(f(x+t)\big).$$

$(-D)^{i\gamma}$ is the adjoint of $D^{i\gamma}$. For complex numbers δ and for non-negative integers m, define

$$D_{\delta m}^{i\gamma} f(x) = |x|^{\delta} D^{i\gamma}\big(|\cdot|^{-\delta}(\log|\cdot/x|)^m f(\cdot)\big)(x)$$

and define

$$(-D)_{\delta m}^{i\gamma} f(x) = |x|^{\delta}(-D)^{i\gamma}\big(|\cdot|^{-\delta}(\log|\cdot/x|)^m f(\cdot)\big)(x).$$

For $m = 0$, write $D_{\delta}^{i\gamma} = D_{\delta 0}^{i\gamma}$.

THEOREM 4.1. *$D_{\delta}^{i\gamma}$ is a γ-group of bounded operators on $L_p(R)$ for $-1/p < \mathrm{Re}(\delta) < 1/p'$; $\|D_{\delta}^{i\gamma}\|_p \leqslant |\Gamma(1+i\gamma)|^{-1}\big(|\gamma| M(p,\delta) + M(p)(|\gamma|+1)^2\big)$. For each non-negative integer m, $D_{\delta m}^{i\gamma}$ is a bounded operator on $L_p(R)$ if $-1/p < \mathrm{Re}(\delta) < 1/p'$.*

Proof. Assume first that $m \geqslant 1$ and write

$$D_{\delta m}^{i\gamma} f(x) = \Gamma(-i\gamma)^{-1} \int_{-\infty}^{x-} (x-t)^{-i\gamma-1} |x/t|^{\delta}(-\log|x/t|)^m f(t)\,dt$$

$$= \Gamma(-i\gamma)^{-1} \int_{-\infty}^{\infty} \Theta(x-t)(x-t)^{-i\gamma-1} |x/t|^{\delta}(-\log|x/t|)^m f(t)\,dt$$

where Θ is the characteristic function of R^+. Since

$$\big|\Theta(x-t)(x-t)^{-i\gamma-1}|x/t|^{\delta}(-\log|x/t|)^m\big| \leqslant |x-t|^{-1}|x/t|^{\mathrm{Re}(\delta)}\big|\log|x/t|\big|^m = K(x,t)$$

and $K(x,t)$ is homogeneous of degree -1,

$$|D_{\delta m}^{i\gamma} f(x)| < |\Gamma(-i\gamma)|^{-1} \int_{-\infty}^{\infty} K(x,t)|f(t)|\,dt = |\Gamma(i\gamma)|^{-1} \int_{-\infty}^{\infty} K(1,t)|f(xt)|\,dt$$

and Minkowski's integral inequality implies that

$$\|D_{\delta m}^{i\gamma}(f)\|_p \leqslant M(p,\gamma,\delta,m)\Big(\int_{-\infty}^{\infty} |t|^{-1/p-\mathrm{Re}(\delta)}|t-1|^{-1}\big|\log|t|\big|^m dt\Big)\|f\|_p.$$

The integral on the right is finite when $-1/p < \mathrm{Re}(\delta) < 1/p'$.

If $m = 0$ and if $\delta = \eta + i\mu$, consider $D_{\delta}^{i\gamma} = (D_{\delta}^{i\gamma} - D_{i\mu}^{i\gamma}) + D_{i\mu}^{i\gamma}$. $D_{i\mu}^{i\gamma}$ is a bounded operator since $\big||x|^{i\mu}\big| = 1$ and

$$\big(D_{\delta}^{i\gamma}(f) - D_{i\mu}^{i\gamma}(f)\big) = |\Gamma(-i\gamma)|^{-1} \int_{-\infty}^{\infty} \Theta(x-t)\,[|x/t|^{\delta} - |x/t|^{i\mu}](x-t)^{-i\gamma-1} f(t)\,dt.$$

Since $K_0(x,t) = |x-t|^{-1}\big||x/t|^{\mathrm{Re}(\delta)} - 1\big|$ is homogeneous of degree -1, the same argument used above on $D_{\delta m}^{i\gamma}$ shows that

$$\|D_{\delta}^{i\gamma}(f) - D_{i\mu}^{i\gamma}(f)\|_p \leqslant |\Gamma(-i\gamma)|^{-1} \int_{-\infty}^{\infty} |t-1|^{-1}\big||t|^{-\mathrm{Re}(\delta)} - 1\big|\,|t|^{-1/p}\,dt\,\|f\|_p$$

and the integral on the right has finite value $M(p,\delta)$ if $-1/p < \mathrm{Re}(\delta) < 1/p'$. From [4], we know that $\|D_{i\mu}^{\gamma i}\|_p \leqslant M(p)(|\gamma|+1)^2|\Gamma(1+i\gamma)|^{-1}$, so that $D_{\delta}^{i\gamma}$ has the desired estimate for the norm on $L_p(R)$. The group property for $D_{\delta}^{i\gamma}$ follows from the corresponding property for $D^{i\gamma}$ since $Df_{\delta}^{i\gamma} f(x) = |x|^{\delta} D^{i\gamma}\big(|\cdot|^{-\delta} f(\cdot)\big)(x)$.

THEOREM 4.2. *$D_{\delta m}^{i\gamma}$ is an analytic function of δ in the norm topology in the strip $-1/p < \mathrm{Re}(\delta) < 1/p'$ for each real δ and for each non-negative integer m. For non-negative integers n, $\left(\frac{\partial}{\partial\delta}\right)^n D_{\delta m}^{i\gamma} = (-1)^n D_{\delta(m+n)}^{i\gamma}$.*

Proof. Let $|h|$ be sufficiently small that $-1/p < \mathrm{Re}(\delta+h) < 1/p'$ and consider $h^{-1}(D_{(\delta+h)m}^{i\gamma} - D_{\delta m}^{i\gamma})$ which by an argument similar to that used in the proof of Theorem 4.1 has norm

$$\|h^{-1}(D_{(\delta+h)m}^{i\gamma} - D_{\delta m}^{i\gamma})\|_p \leqslant$$
$$|\Gamma(-i\gamma)|^{-1} \int_{-\infty}^{\infty} |t|^{-1/p}|h|^{-1}\big||t|^{-\delta-h} - |t|^{-\delta}\big||t-1|^{-1}\big|\log|t|\big|^m dt.$$

$h^{-1}(|t|^{-h} - 1)$ converges a.e. to $(-\log|t|)$ as h tends to zero. Set $z = h\log|t|$ so that $\big||t|^{-h} - 1\big|\big|h\log|t|\big|^{-1} = |e^{-z} - 1|\,|z|^{-1} \leqslant (e^{|z|} - 1)|z|^{-1}$ which is a bounded function for $|h| < \varepsilon$ and for $|t| \leqslant e^M$. Let M be large and suppose that $|t| > e^M$. Set $a = 1/p + \mathrm{Re}(\delta)$ and set $\log|t| = u$; then

$$|t|^{-1/p}\big||t|^{-\delta-h} - |t|^{-\delta}\big|\,|h(t-1)|^{-1}\big|\log|t|\big|^m \leqslant K|t|^{-1}e^{-au}\sum_{n=1}^{\infty}\frac{|h|^{n-1}u^{n+m}}{n!}.$$

The integral over $e^M \leqslant t < \infty$ of the right hand side of this inequality is dominated by

$$K\sum_{n=1}^{\infty}\frac{|h|^{n-1}\Gamma(n+m+1)}{n!\,a^{m+n+1}}.$$

If we take $|h| < a$, the ratio test implies convergence of the last series. A similar result holds for $-\infty < t < -e^M$. Thus the dominated convergence theorem implies that $\frac{\partial}{\partial\delta} D_{\delta m}^{i\gamma} = -D_{\delta(m+1)}^{i\gamma}$ in the norm topology. The proof of the theorem can now be completed by a simple induction argument.

Remark 1. The functions $|1-t|^{-1}(|t|^{-\delta} - 1)(\log|t|)^m$ are in $L_{p'}(R)$ if $-1/p < \mathrm{Re}(\delta) < 1/p'$ so that the difference quotients used in the proof of Theorem 4.2 to calculate $\frac{\partial}{\partial\delta} D_{\delta m}^{i\gamma} f(x)$ also converge almost everywhere to $-D_{\delta(m+1)}^{i\gamma} f(x)$.

Remark 2. It follows easily from the fact that the adjoint of $D^{i\gamma}$ is $(-D)^{i\gamma}$ that the adjoint of $D^{i\gamma}_{\delta m}$ is $(-1)^m(-D)^{i\gamma}_{-\delta m}$ for $-1/p < \operatorname{Re}(\delta) < 1/p'$ and non-negative integers m.

Remark 3. If $\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0$ and $|x| < 1$, the hypergeometric function is defined by

$$F(a, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-a}dt$$

and for all other values of β and γ except $\gamma = 0, -1, -2, \ldots$ by analytic continuation. In terms of the distributions discussed in Section 3,

$$\frac{x^{\gamma-1}}{\Gamma(\gamma)} F(a, \beta, \gamma; x) = \left(\frac{d}{dx}\right)^{\beta-\gamma} \left[\frac{x_+^{\beta-1}(1-x)_+^{-a}}{\Gamma(\beta)}\right]$$

for $\operatorname{Re}(\gamma) > \operatorname{Re}(\beta) > 0$. This distribution and the derivative on the right extend by analytic continuation to all values of β and γ except $\gamma = 0$, $-1, -2, \ldots$ Using these facts one can study an operator analogous to Okikiolu's operator $\tilde{H}^{\nu}$ for the operators $D^{i\gamma}$. Set $\tilde{D}^{i\gamma}_v(f)(x) = x_+^{i\gamma}D^{i\gamma}_v(f)(x)$. When $-1/p < \operatorname{Re}(w), \operatorname{Re}(v) < 1/p'$, the kernel for the operator $\tilde{D}^{i\delta}_w\tilde{D}^{i\gamma}_v$ can be calculated explicitly in terms of the hypergeometric function. $D^{i\delta}_w\tilde{D}^{i\gamma}_v$ is the singular integral operator $\int_{-\infty}^{x-} K_{vw}(x, u)f(u)\,du$ with

$$K_{vw}(x, u) = \frac{|u|^v x_+^{-i\delta-i\gamma-v-1}(x-u)_+^{-i\gamma-1}}{\Gamma(-i\gamma)\Gamma(-i\delta)} \times$$

$$\times F\left(1+i\gamma,\ -i\delta+w,\ -i\gamma-i\delta-v+w;\ \frac{x}{x-u}\right) \frac{\Gamma(-i\gamma-v)\Gamma(-i\delta+w)}{\Gamma(-i\gamma-i\delta-v+w)}$$

if $u < 0$ and

$$K_{vw}(x, u) = \frac{\Gamma(-i\delta+w)u^v x_+^{-w-i\gamma-v-1}(x-u)_+^{-i\gamma-i\delta+w-1}}{\Gamma(-i\delta)\Gamma(-i\gamma-i\delta+w)} F,$$

where $F = F\left(v+i\gamma-1,\ -i\delta+w,\ -i\delta-i\gamma+w;\ \frac{x-u}{x}\right)$, if $u > 0$. Note that $D^{i\delta}_w\tilde{D}^{i\gamma}_v(f)(x)$ has support in the positive half line. The formula for $K_{vw}(x, u)$ can be verified by using the definition of F as an analytic continuation, replacing $-i\delta$ by a with $\operatorname{Re}(a) > 0$, replacing $-i\gamma$ by β with $\operatorname{Re}(\beta) > 0$, and by using the classical integral for F which was mentioned above. See [5] and [12] for treatments of the hypergeometric function.

5. Weighted norms. In this section we shall estimate the norms of $G^{i\gamma}$ and $G^{a}J^{a}$ on the spaces $L_p(\delta)$. Let δ be a real number; $L_p(\delta)$ consists of those measurable functions (modulo null functions) f on E for which $\|f\|_{p\delta} = \|f|\cdot|^{\delta}\|_p = \left(\int_E |f(x)|^p|x|^{p\delta}dx\right)^{1/p} < \infty$. Because the kernel for the operator $G^{a}J^{a}$ has only minimal integrability properties [3], our results are essentially one dimensional and the most general theorems follow by the method of rotation. There is an N-dimensional theorem for the operators $G^{i\gamma}$ which we shall mention.

a) $G^{i\gamma}$ on $L_p(\delta)$. If μ is a finite Borel measure on E with $\mu(\{0\}) = 0$,

$$G^{i\gamma}f(x) = \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}\left[\int_E T_{ty}f\,d\mu(y)\right] = \int_E D^{i\gamma}_y f(x)\,d\mu(y)$$

where $D^{i\gamma}_y$ is the $(i\gamma)$-th power of the derivative D_y, the derivative in the direction y.

Before estimating $G^{i\gamma}$ on $L_p(\delta)$ we shall first consider the one-dimensional situation and a class of operators which is slightly larger than the class of $G^{i\gamma}$. We know that

$$\left(\frac{d}{dx}\right)^{i\gamma} f(x) = D^{i\gamma}f(x) = \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}\big(f(x-t)\big)$$

is a strongly continuous group of bounded operators on $L_p(R)$ with $\|D^{i\gamma}\|_p \leqslant A(p)(|\gamma|+1)^2|\Gamma(1+i\gamma)|^{-1}$. We shall consider the related operators

$$(iv+D)^{i\gamma}f(x) = \left(iv+\frac{d}{dx}\right)^{i\gamma} f(x) = \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}\big(f(x-t)e^{-itv}\big)$$

which admit the same estimate as $D^{i\gamma}$ for the norms on $L_p(R)$. To prove that $(iv+D)^{i\gamma}$ is a strongly continuous group of bounded operators on $L_p(R, \delta)$, it is sufficient to prove that $D(\gamma, \delta)f(x) = |x|^{\delta}(iv+D)^{i\gamma}(|\cdot|^{-\delta}f)(x)$ is a γ-strongly continuous group of bounded operators on $L_p(R)$.

THEOREM 5.1. *If $1 < p < \infty$ and if $-1/p < \delta < 1/p'$, $(iv+D)^{i\gamma}$ is a γ-strongly continuous group of bounded operators on $L_p(\delta)$ for each real number v.*

Proof. We shall show that $D(\gamma, \delta)$ is a γ-strongly continuous group of bounded operators on $L_p(R)$. Let $F(x) = f(x)e^{-ivx}$ and let $g_\varepsilon(t) = t^{-1-i\gamma}\Gamma(-i\gamma)^{-1}$ if $t > \varepsilon$ and $g_\varepsilon(t) = 0$ if $t < \varepsilon$. Then

$$(iv+D)^{i\gamma}f(x) = e^{ivx}\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{\infty} g_\varepsilon(x-t)F(t)\,dt - \frac{\varepsilon^{-i\gamma}F(x)}{i\gamma\Gamma(-i\gamma)}\right]$$

and since these operators are bounded on $L_p(R)$, we shall write $D(\gamma, \delta)f(x) = [D(\gamma, \delta)f(x) - (iv+D)^{i\gamma}f(x)] + (iv+D)^{i\gamma}f(x)$ and estimate the first term on the right. An argument which is nearly the same as that used in the proof of Theorem 4.1 shows that if $\|f\|_p \leqslant 1$

$$\|D(\gamma,\delta)f-(iv+D)^{i\gamma}f\|_p \leqslant |\Gamma(-i\gamma)|^{-1}\int_{-\infty}^{\infty}|t|^{-1/p}|t-1|^{-1}\big||t|^{-\delta}-1\big|\,dt$$

and the integral on the right has finite value $A(p,\delta)$ if $-1/p<\delta<1/p'$. Thus $\|D(\gamma,\delta)\|_p \leqslant \big(M(p,\delta)|\gamma|+M(p)(|\gamma|+1)^2\big)|\Gamma(1-i\gamma)|^{-1}$ the bound is uniform on a finite neighborhood of $\gamma=0$. Since $D(\gamma,\delta)f-f=\big(D(\gamma,\delta)f-(iv+D)^{i\gamma}f\big)+\big((iv+D)^{i\gamma}f-f\big)$, since $(iv+D)^{i\gamma}$ is strongly γ-continuous, and since $D(\gamma,f)-(iv+D)^{i\gamma}$ is norm γ-continuous because $\|D(\gamma,\delta)-(iv+D)^{i\gamma}\|_p \leqslant |\Gamma(-i\gamma)|^{-1}M(p,\delta)$, $D(\gamma,\delta)$ is strongly γ-continuous on $L_p(R)$. Thus $(iv+D)^{i\gamma}$ is strongly γ-continuous on $L_p(\delta)$.

It was mentioned in [4] that if f is a function with support in the positive half-line, then $\left(\frac{d}{dx}\right)^{-i\gamma}=I^{i\gamma}$ is the $(i\gamma)$-th power of the indefinite integral. The next corollary is an immediate consequence of Theorem 5.1.

Corollary 5.2. *$I^{i\gamma}$ is a strongly continuous group of bounded operators on $L_p(R^+,\delta)$ if $-1/p<\delta<1/p'$.*

Corollary 5.3. *If φ is the Fourier transform of a finite Borel measure ν on the line and if*

$$U^{\gamma}f(x)=\Gamma(-i\gamma)^{-1}\lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{\infty}f(x-t)\varphi(t)t^{-1-i\gamma}\,dt-\frac{\varepsilon^{-i\gamma}}{i\gamma}f(x)\nu(R)\right]$$

then U^{γ} is a bounded operator on $L_p(\delta)$ for $-1/p<\delta<1/p'$ with $\|U^{\gamma}\|_{p\delta} \leqslant \|D^{i\gamma}\|_{p\delta}\|\nu\|$.

Proof. Write $\varphi(t)=\int_{-\infty}^{\infty}e^{-itv}d\nu(v)$ so that $U^{\gamma}f(x)=\int_{-\infty}^{\infty}(iv+D)^{i\gamma}f(x)\,d\nu(v)$. Then Theorem 5.1 and Minkowski's integral inequality imply the desired estimate for $\|U^{\gamma}\|_{p\varrho}$.

Let E denote N-dimensional Euclidean space and let $\Omega(y)$ be homogeneous of degree zero and suppose that Ω is integrable over the unit sphere in E. It was shown in [4] that

$$\begin{aligned}T^{\gamma}f(x)&=\Gamma(-i\gamma)^{-1}\lim_{\varepsilon\to 0}\Big[\int_{\|y\|>\varepsilon}f(x-y)\,\Omega(y)\,\|y\|^{-N-i\gamma}\,dy-\\&\qquad-\frac{\varepsilon^{-i\gamma}}{i\gamma}f(x)\int_{\Sigma}\Omega(\omega)\,d\sigma(\omega)\Big].\\&=\int_{\Sigma}D_{\omega}^{i\gamma}f(x)\,\Omega(\omega)\,d\sigma(\omega)\end{aligned}$$

when D_ω is the derivative in the direction ω.

Theorem 5.4. *Let $-1/p<\delta<1/p'$ and let $T^{\gamma}f(x)=\int_{\Sigma}D_{\omega}^{i\gamma}f(x)\,\Omega(\omega)\,d\sigma(\omega)$ where Ω is integrable on the sphere in E. Then T^{γ} is a bounded operator on $L_p(E,\delta)$. T^{γ} is γ-strongly continuous on $L_p(E,\delta)$.*

Proof. First consider $D_{\omega}^{i\gamma}$ acting on $L_p(E,\delta)$. Let f be a smooth function with compact support and consider $\|D_{\omega}^{i\gamma}f(\cdot)|\cdot|^{\delta}\|_p^p$. Since Lebesgue measure is rotationally invariant, we may assume that $\omega=e_1,e_2,\ldots,e_N$ is an orthonormal basis for E. Write $x=(x_1,x_2,\ldots,x_N)$ with respect to this basis. Since $\varphi_u(x)=\|x\|^{-u}\sum_{j=1}^{N}|x_j|^u$ is a bounded function with bounded reciprocal on E for $u\geqslant 0$, take $\delta>0$ and write

$$\|D_{e_1}^{i\gamma}f(\cdot)|\cdot|^{\delta}\|_p^p \leqslant M(\delta)\sum_{j=1}^{N}\int_E|D_{e_1}^{i\gamma}f(x)|^p|x_j|^{p\delta}\,dx.$$

Write each of the integrals on the right as an iterated integral with the integral with respect to x_1 first. By Theorem 5.1 and Fubini's theorem,

$$\int_E|D_{e_1}^{i\gamma}f(x)|^p|x_j|^{p\delta}\,dx \leqslant A(p,\delta,\gamma)\int_E|f(x)|^p|x_j|^{p\delta}\,dx.$$

Since $\varphi_u(x)^{-1}$ is bounded, $\|D_{e_1}^{i\gamma}f\|_{p\delta}\leqslant A(p,\delta,\gamma)\|f\|_{p\delta}$ for $\delta>0$. The smooth functions with compact support are dense in $L_p(\delta)$, so the boundedness of $D_{\omega}^{i\gamma}$ on $L_p(\delta)$ is established for $0<\delta<1/p'$. Consider $D(\delta,\gamma,\omega)f(x)=\|x\|^{\delta}D_{\omega}^{i\gamma}(\|\cdot\|^{-\delta}f)(x)$; the adjoint of this operator is $D(-\delta,\gamma,-\omega)$ which must be bounded on $L_{p'}(E)$ since $D(\delta,\gamma,\omega)$ is bounded on $L_p(E)$. By the above argument, $D(-\delta,\gamma,-\omega)$ is bounded on $L_{p'}(E)$ for $0<-\delta<1/p$, so that $D(-\delta,\gamma,-\omega)$ is bounded on $L_{p'}(E)$ for $-1/p<\delta<1/p'$; this implies that $D(\delta,\gamma,\omega)$ is bounded on $L_p(E)$ for $-\frac{1}{p}<\delta<\frac{1}{p'}$ and $D_{\omega}^{i\gamma}$ is bounded on $L_p(\delta)$.

The operators $D(\delta,\gamma,\omega)$ form a γ-group on $L_p(E)$ and strong continuity in γ follows by an iterated integral argument similar to that used above for $\delta>0$. Since $L_p(\delta)$ is a reflexive space, the adjoint of $D(\delta,\gamma,\omega)$ is also a strongly continuous group in γ for $-1/p<\delta<0$. So $D_{\omega}^{i\gamma}$ is a strongly continuous group on $L_p(\delta)$ for $-1/p<\delta<1/p'$. The bound and strong continuity for T^{γ} now follow by Minkowski's integral inequality.

The next theorem treats a slightly more general operator, and a slight variant of the argument used in the proof of Theorem 5.4 can be used to prove it.

Theorem 5.5. *Let μ be a finite Borel measure on E with $\mu(\{0\})=0$ and let φ be the Fourier transform of a finite measure ν on R. Set*

$$V^{\gamma}f(x) = \Gamma(-i\gamma)^{-1}\lim_{\varepsilon\to 0}\Big[\int_{\varepsilon}^{\infty}\Big[\int_E f(x-ty)\,d\mu(y)\Big]\varphi(t)\,t^{-1i\gamma}\,dt - \frac{\varepsilon^{-i\gamma}}{i\gamma}f(x)\,\mu(E)\,\nu(R)\Big].$$

Then $V^{\gamma}f(x) = \int_R\int_E (iv+D_y)^{i\gamma}f(x)\,d\mu(y)\,d\nu(v)$ *and* V_γ *is a strongly continuous family of bounded operators on* $L_p(\delta)$ of $-1/p<\delta<1/p'$.

Proof. That V^γ can be written in the second form was proved in ([4], II). As in the proof of Theorem 5.4, is sufficient to study $(iv+D_y)^{i\gamma}$ and to use Minkowski's integral inequality to complete the proof. Write $(iv+D_y)^{i\gamma} = \|y\|^{i\gamma}(iv\|y\|^{-1}+D_\omega)^{i\gamma}$ where $\omega = y\|y\|^{-1}$ for $y\neq 0$. Then the same iterated integral argument which was used in the proof of Theorem 5.4 can be used to show that $(iv+D_y)^{i\gamma}$ is a strongly continuous group of bounded operators on $L_p(\delta)$. This completes the sketch of the proof.

The range of δ's for which Theorem 5.4 holds can be extended if we assume more integrability for Ω. The argument used by Walsh ([10], Proposition 8) for Calderón–Zygmund operators can be used to prove

THEOREM 5.6. *Suppose that* $1/q\leqslant 1-|\delta|/N$ *and that* $-N/p<\delta<N/p'$. *Suppose that* $\Omega\in L^{N,1}(\sum)\cap L^q(\sum)$, *then the operator* T^γ *in Theorem* 5.4 *is bounded on* $L_p(\delta)$. *Here* $L^{N,1}(\sum)$ *is the Lorentz space.*

b) G^aJ^a on $L_p(\delta)$. Now we can use Stein's interpolation theorem [14] to study the operators G^aJ^a of Section 1 on $L_p(\delta)$. The keys to considering G^aJ^a on $L_p(\delta)$ are the facts that $G^aJ^a(f) = \Gamma(-a)\int_\Sigma D^a_\omega J^a(f)\,\Omega(\omega)\,d\sigma(\omega)$ if a is not an integer and that $D^a_\omega J^a$ is an analytic semi-group of bounded operators which are very nearly Calderón–Zygmund operators. We shall sketch an argument parallel to our argument in [3] to prove that G^aJ^a is a bounded operator on $L_p(\delta)$ for $-1/p<\delta<1/p'$.

THEOREM 5.7. $D^a_\omega J^a$ *is an analytic semi-group of bounded operators on* $L_p(\delta)$ *for* $-1/p<\delta<1/p'$ *when* $|\arg(a)|<\pi/2$, *and* $D^a_\omega J^a$ *is a strongly continuous semi-group of bounded operators on* $L_p(\delta)$ *if* $\mathrm{Re}(a)\geqslant 0$.

Proof. We shall use Stein's interpolation theorem [14] to verify the boundedness of $D^a_\omega J^a$. Consider $D^{i\gamma}_\omega J^{i\gamma}$. Theorem 5.4 shows that $D^{i\gamma}_\omega$ is bounded and strongly continuous on $L_p(\delta)$. Since $J^{i\gamma}f(x) = \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}(e^{-t}P_tf(x))$ with P_tf being the Poisson integral of f, and since $e^{-|t|}$ is the Fourier transform of $\pi^{-1}(1+y^2)^{-1}$, Theorem 5.5 implies that $J^{i\gamma}$ is bounded and strongly continuous on $L_p(\delta)$. Since $J^{i\gamma} = (1+\Lambda)^{-i\gamma}$ where $\Lambda = (-\Delta)^{1/2}$ is the negative of the infinitesimal generator of P_t, $J^{i\gamma}$ is a group on $L_p(\delta)$ because it is a group on $L_p(E)$ as was shown in [4].

The Hilbert transform is a bounded operator on $L_p(\delta)$, $-1/p<\delta<1/p'$, by Okikiolu's theorem [9] and $H_\omega f(x) = P\int_{-\infty}^{\infty}f(x-t\omega)\,dt/t$, the Hilbert transform in the direction ω, is bounded on $L_p(E,\delta)$ for $-1/p<\delta<1/p'$ as an argument similar to that given in the proof of Theorem 5.4 shows. $D_\omega J$ can be expressed in terms of a double integral of the Hilbert transform in the directions η as in the proof of Theorem 4 of [3] and an argument similar to that used in the proof of Theorem 5.5 shows that $D_\omega J$ is a bounded operator on $L_p(\delta)$ for $-1/p<\delta<1/p'$.

If f is a smooth function with compact support in E, $D^a_\omega J^a(f)$ is an analytic function of a in $\mathrm{Re}(a)>0$ because D^a_ω and J^a are strongly analytic. Since $D^{k+\beta}_\omega J^{k+\beta} = D^k_\omega J^k D^\beta_\omega J^\beta$ for non-negative integers k and since $D^{i\gamma}J^{i\gamma}$ is a strongly continuous group of bounded operators on $L_p(\delta)$, Stein's interpolation theorem [14] applies and $D^t_\omega J^t D^{i\gamma}_\omega J^{i\gamma} = D^a_\omega J^a$ is a bounded operator for $t\geqslant 0$. If $|\arg(a)|\leqslant\Theta<\pi/2$, $\lim_{a\to 0}D^a_\omega J^a(f) = f$ for smooth functions f and since $\|D^a_\omega J^a\|_{p\delta}$ is bounded for $0\leqslant\mathrm{Re}(a)\leqslant 1$, $-1\leqslant\mathrm{Im}(a)\leqslant 1$, $D^a_\omega J^a$ has the required continuity properties; $D^a_\omega J^a$ is an analytic semi-group in $|\arg(a)|<\pi/2$ and a strongly continuous semi-group in $\mathrm{Re}(a)\geqslant 0$. This completes the proof.

COROLLARY 5.8. $\dfrac{\partial}{\partial a}(D^a_\omega J^a)$ *is a bounded operator on* $L_p(\delta)$ *for* $-1/p<\delta<1/p'$ *for each* $a>0$.

Proof. If T^a is an analytic semi-group with $\|T^a\|\leqslant Me^{|a|w}$, then $\left\|\dfrac{\partial}{\partial a}T^a\right\|\leqslant M(|a|+1)|a|^{-1}e^{|a|w}$; see [13]. By Theorem 5.7, $D^a_\omega J^a$ is an analytic semi-group on these $L_p(\delta)$.

THEOREM 5.9. G^aJ^a *is a bounded linear operator on* $L_p(\delta)$ *for* $-1/p<\delta<1/p'$ *for all* $\mathrm{Re}(a)\geqslant 0$, $a\neq 0$, *when* Ω *satisfies the assumptions of Theorem* 1.1.

Proof. If $k\leqslant\mathrm{Re}(a)<k+1$ and if $a\neq k$,

$$G^aJ^a(f) = \Gamma(-a)\int_\Sigma D^a_\omega J^a(f)\,\Omega(\omega)\,d\sigma(\omega)$$

and the conclusion follows from Theorem 5.7 and Minkowski's integral inequality. When $\mathrm{Re}(a) = k$ we need to assume that $\int_\Sigma\omega^\beta\,\Omega(\omega)\,d\sigma(\omega) = 0$ for all multi-indices β with $|\beta| = k$ as in Theorem 1.1. When $a = k>0$ and $\int_\Sigma\omega^\beta\,\Omega(\omega)\,d\sigma(\omega) = 0$ for all $|\beta| = k$, Corollary 5.8, the uniform boundedness principle, and Theorem 9 of [4] imply that

$$G^kJ^k(f) = \frac{(-1)^{k+1}}{k!}\int_\Sigma\frac{\partial}{\partial a}(D^a_\omega J^a)(f)|_{a=k}\,\Omega(\omega)\,d\sigma(\omega)$$

and Minkowski's integral inequality can be used to complete the proof of the boundedness on $L_p(\delta)$, $-1/p < \delta < 1/p'$.

Remark. The kernel Ω_a of $D^a_\omega J^a$ was calculated explicitly in [3] and shown to have even part in $L\log^+ L(\sum)$. More integrability for Ω_a seems to be difficult to establish. We do not expect a theorem of the form of Theorem 5.6 for the general hypersingular integral operators.

6. Commutators. Let $a(x)$ be a complex valued function on E and let A denote the linear operator $A(f)(x) = a(x)f(x)$. We shall consider the linear operators

$$\mathfrak{G}^\gamma(f, a) = G^{1+i\gamma}(Af) - AG^{1+i\gamma}(f)$$

and

$$\mathfrak{G}^\gamma(f, a) = \Lambda\big(G^{i\gamma}(Af) - AG^{i\gamma}(f)\big)$$

when $\Lambda = (-\Delta)^{1/2}$ on $L_p(E)$ for $1 < p < \infty$. When $a(x)$ is a sufficiently smooth function $\mathfrak{G}^\gamma$ and $\mathfrak{G}_\gamma$ are bounded operators on $L_p(E)$ in spite of the fact that each of these operators involves differentiation of order a with $\operatorname{Re}(a) = 1$.

Let μ be a finite Borel measure on E with $\mu(\{0\}) = 0$ and set $G^{i\gamma}(f)(x) = \Gamma(-i\gamma)^{-1} t_+^{-i\gamma-1}\big[\int_E T_{ty} f d\mu(y)\big]$ $T_{ty}f(x) = f(x-ty)$. From [4] it follows that if D_y denotes the derivative in the direction y, then

$$D_y^{i\gamma} f(x) = \Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0}\Big[\int_\varepsilon^\infty f(x-ty)t^{-i\gamma-1}dt - \frac{\varepsilon^{-i\gamma}}{i\gamma} f(x)\Big]$$
$$= \Gamma(-i\gamma)^{-1} t_+^{-i\gamma-1}\big(f(x-ty)\big)$$

for $y \neq 0$. If, in addition, we assume that $\gamma \neq 0$, $\int_E \|y\|\, d|\mu|(y) < \infty$, and that $\int_E \langle h, y\rangle d\mu(y) = 0$ for all $h \in E$, then

$$G^{1+i\gamma}(f) = \Gamma(-1-i\gamma)^{-1} t_+^{-2-i\gamma}\Big[\int_E f(x-ty)\, d\mu(y)\Big] = \int_E D_y^{1+i\gamma}(f)\, d\mu(y).$$

We shall begin by studying the operators $\mathfrak{D}^\gamma(f, a) = D_y^{1+i\gamma}(Af) - AD_y^{1+i\gamma}(f)$; the expressions for $G^{i\gamma}$ and for $G^{1+i\gamma}$ given above indicate that theorems for $\mathfrak{G}^\gamma(f, a)$ are corollaries of the corresponding theorems for $\mathfrak{D}^\gamma(f, a)$.

THEOREM 6.1. *Let $a(x)$ be a bounded function with bounded derivatives and let f be a smooth function with compact support. Then*

$$D_y^{1+i\gamma}(Af) - AD_y^{1+i\gamma}(f)$$
$$= \Gamma(-1-i\gamma)^{-1} \lim_{\varepsilon\to 0}\Big[\int_\varepsilon^\infty T_{ty}f(T_{ty}a - a)t^{-2-i\gamma}dt + \frac{\varepsilon^{-i\gamma}}{i\gamma} fD_y a\Big]$$

where $T_{ty}f(x) = f(x-ty)$.

Proof. $D_y^{1+i\gamma}(Af) - AD_y^{1+i\gamma}(f) = D_y^{i\gamma}(fD_y a + aD_y f) - AD_y^{i\gamma}(D_y f)$ since $D_y^a D_y^\beta = D_y^{a+\beta}$; ([6], I). Use the integral expression for $D_y^{i\gamma}$ to write the right hand side of the last equality as

$$\Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0}\Big[\int_\varepsilon^\infty T_{ty}\big((D_y a)f + aD_y f\big)t^{-1-i\gamma}dt - \frac{\varepsilon^{-i\gamma}}{i\gamma}\big((D_y a)f + aD_y f\big)\Big]$$
$$-A\Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0}\Big[\int_\varepsilon^\infty T_{ty}(D_y f)t^{-1-i\gamma}dt - \frac{\varepsilon^{-i\gamma}}{i\gamma} D_y f\Big]$$
$$= D_y^{i\gamma}\big((D_y a)f\big) + \Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0}\int_\varepsilon^\infty (T_{ty}a - a)T_{ty}D_y f t^{-1-i\gamma}dt.$$

Integrate by parts with $\dfrac{\partial}{\partial t} T_{ty} f = -T_{ty} D_y f$ to get

$$\mathfrak{D}^\gamma(f, a) = D_y^{i\gamma}\big((D_y a)f\big) + \Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0}\Big\{-\Big[\frac{(T_{ty}a - a)T_{ty}f}{t^{1+i\gamma}}\Big]_\varepsilon^\infty$$
$$-\int_\varepsilon^\infty T_{ty}(fD_y a)t^{-1-i\gamma}dt - (1+i\gamma)\int_\varepsilon^\infty (T_{ty}a - a)T_{ty}ft^{-2-i\gamma}dt\Big\}$$
$$= D_y^{i\gamma}\big((D_y a)f\big) - \Gamma(-i\gamma)^{-1} \lim_{\varepsilon\to 0}\Big[\int_\varepsilon^\infty T_{ty}(fD_y a)t^{-1-i\gamma}dt$$
$$+\varepsilon^{-i\gamma} fD_y a + (1+i\gamma)\int_\varepsilon^\infty (T_{ty}a - a)T_{ty}ft^{-2-i\gamma}dt\Big]$$

since $\lim_{\varepsilon\to 0}[\varepsilon^{-1-i\gamma}(T_{\varepsilon y}a - a)T_{\varepsilon y}f + \varepsilon^{-i\gamma}fD_y a] = 0$. Thus

$$\mathfrak{D}^\gamma(f, a) = \lim_{\varepsilon\to 0}\Gamma(-1-i\gamma)^{-1}\Big[\int_\varepsilon^\infty (T_{ty}a - a)T_{ty}ft^{-2-i\gamma}dt + \frac{\varepsilon^{-i\gamma}}{i\gamma} fD_y a\Big].$$

THEOREM 6.2. *Let a be a bounded function with bounded gradient and with $|\nabla a(x-h) - \nabla a(x)| \leqslant c\|h\|^\delta$ for some $\delta > 0$. Let μ be a finite Borel measure on E with $\mu(\{0\}) = 0$, with $\int_E \langle h, y\rangle d\mu(y) = 0$ for all h in E, and with $\int_E \|y\|\, d|\mu|(y) < \infty$. Then $\mathfrak{G}^\gamma(f, a) = G^{1+i\gamma}(Af) - AG^{1+i\gamma}(f)$ extends to a bounded operator on $L_p(E)$.*

Proof. The discussion preceding Theorem 6.1 shows that with our present assumptions on μ

$$\mathfrak{G}^\gamma(f, a) = \int_E \big(D_y^{1+i\gamma}(Af) - AD_y^{1+i\gamma}(f)\big)\, d\mu(y).$$

If f is a smooth function with compact support and if $\omega = y\|y\|^{-1}$, consider

$$D_\omega^{1+i\gamma}(Af) - AD_\omega^{1+i\gamma}(f) =$$

$$\Gamma(-1-i\gamma)^{-1}\lim_{\varepsilon\to 0}\Big[\int_\varepsilon^1 (T_{t\omega}a - a)T_{t\omega}ft^{-2-i\gamma}dt + \frac{\varepsilon^{-i\gamma}}{i\gamma}fD_\omega a\Big] +$$

$$+\Gamma(-1-i\gamma)^{-1}\int_1^\infty (T_{t\omega}a - a)T_{t\omega}ft^{-2-i\gamma}dt$$

from Theorem 6.1. The second expression on the right converges absolutely and has p-norm dominated by $2|\Gamma(-1-i\gamma)|^{-1}\|a\|_\infty\|f\|_p$. Write $a(x-t\omega) - a(x) = tD_\omega a(x-t\omega) + R(a, t\omega)(x)$ which because of the differentiability of a and the Lipschitz property of ∇a satisfies $|R(a, t\omega)(x)| \leqslant ct^{1+\eta}$ for $\eta < \delta$. From [7],

$$\lim_{\varepsilon\to 0}\Big[\int_\varepsilon^1 -D_\omega a(x-t\omega)f(x-t\omega)t^{-1-i\gamma}dt + \frac{\varepsilon^{+i\gamma}}{i\gamma}fD_\omega a\Big]$$

exists a.e. and in $L_p(E)$ and has p-norm dominated by $M(p)(|\gamma| + +1)^2|\gamma|^{-1}\|\nabla a\|_\infty\|f\|_p$. The integral $\int_{0+}^1 R(a, t\omega)(x)T_{t\omega}ft^{-2-i\gamma}dt$ converges absolutely and has p-norm dominated by $M(a)\|f\|_p$. Since $D_y^{1+i\gamma}F = \|y\|^{1+i\gamma}D_\omega^{1+i\gamma}F$, $\mathfrak{D}^\gamma(f, a)$ has p-norm dominated by $M(p, \gamma, a)\|y\|\|f\|_p$. Minkowski's integral inequality implies that $\|\mathfrak{G}^\gamma(f, a)\|_p \leqslant M_1(p, \gamma, a)\int_E \|y\|d|\mu|(y)\|f\|_p$. This completes the proof.

Let $\Lambda = (-\Delta)^{1/2}$; Λ is the infinitesimal generator of the Poisson integral

$$P_tf(x) = C_N\int_E f(x-z)t(t^2+\|z\|^2)^{-(N+1)/2}dz$$

$$= C_N\int_E f(x-tz)(1+\|z\|^2)^{-(N+1)/2}dz.$$

Furthermore, $\int_E \langle h, z\rangle(1+\|z\|^2)^{-(N+1)/2}dz = 0$, and $\Lambda^{i\gamma}(f) = C_N\int_E D_z^{i\gamma}f(1+ +\|z\|^2)^{-(N+1)/2}dz$, but $\int_E \|z\|(1+\|z\|^2)^{-(N+1)/2}dz = \infty$ so that Theorem 6.2 is inadequate for estimating $\Lambda^{1+i\gamma}(Af) - A\Lambda^{1+i\gamma}(f)$. The next theorem modifies Theorem 6.2 sufficiently so that we can estimate $\Lambda^{1+i\gamma}(Af) - -A\Lambda^{1+i\gamma}(f)$.

We begin with a lemma.

LEMMA 6.3. *As a closed operator on* $L_p(E)$, $(-\Delta)^{1/2} = \Lambda = i\sum_{j=1}^N R_j\frac{\partial}{\partial x_j} = i\sum_{j=1}^N \frac{\partial}{\partial x_j}R_j$, *where* R_j *is the j-th Riesz operator*

$$R_jf(x) = K_N\lim_{\varepsilon\to 0}\int_{\|y\|>\varepsilon} f(x-y)\frac{y_j\,dy}{\|y\|^{N+1}}$$

with Fourier transform $\|\xi\|^{-1}\xi_j$.

Proof. $\Lambda = (-\Delta)^{1/2}$ is the infinitesimal generator of the Poisson integral and $\frac{\partial}{\partial x_j}$ is the infinitesimal generator of the translation semigroup $T_tf(x) = f(x+te_j)$, $\{e_j\}_{j=1}^N$ is the standard basis for E. Thus Λ and $\frac{\partial}{\partial x_j}$ are closed and densely defined operators and $i\sum_{j=1}^N R_j\frac{\partial}{\partial x_j}$ is a closed and densely defined operator since the R_j are bounded. For smooth functions f with compact support, Fourier transformation shows that $\Lambda(f) = i\sum_{j=1}^N R_j\frac{\partial}{\partial x_j}f = \sum_{j=1}^N i\frac{\partial}{\partial x_j}R_jf$. Since all three operators are closed, the identity holds for all f in the domain of Λ. f is in the domain of each of the operators $\frac{\partial}{\partial x_j}$ if and only if $f\in L_p^1(E)$, the range of $(1+\Lambda)^{-1}$ with the inherited norm; [1]. $L_p^1(E)$ is easily seen to be equivalent to the domain of Λ when this domain is equipped with the graph norm. Thus the three operators in question have a common domain and each of them is closed.

THEOREM 6.4. *Let* $\Omega(y)$ *be homogeneous of degree* 0, *bounded, and have bounded gradient. Set*

$$U^{i\gamma}f = \Gamma(-i\gamma)^{-1}\lim_{\varepsilon\to 0}\Big[\int_{\|y\|>\varepsilon} f(x-y)\frac{\Omega(y)\,dy}{\|y\|^{N+i\gamma}} - \frac{\varepsilon^{-i\gamma}}{i\gamma}\int_E \Omega d\sigma(\omega)\Big]$$

$$= \int_E D_\omega^{i\gamma}f(x)\,\Omega(\omega)\,d\sigma(\omega).$$

Let $A(x)$ *be a bounded function on* E *which is differentiable and whose gradient is bounded and satisfies a Lipschitz condition of order* $\delta > 0$. *Then if* $\Lambda = (-\Delta)^{1/2}$, $\Lambda\big(AU^{i\gamma}(f) - U^{i\gamma}(Af)\big)$ *and* $AU^{i\gamma}(\Lambda f) - U^{i\gamma}(A\Lambda f)$ *extend to bounded operators on* $L_p(E)$.

Proof. Since the adjoint of $D_y^{i\gamma}$ is $(-D_y)^{i\gamma}$, and since Λ is self-adjoint, $\Lambda\big(AU^{i\gamma}(\cdot) - U^{i\gamma}(A\cdot)\big)$ and $AU^{i\gamma}(\Lambda\cdot) - U^{i\gamma}(A\Lambda\cdot)$ are essentially adjoints of each other. Thus it is sufficient to prove that $\Lambda\big(AU^{i\gamma}(\cdot) - U^{i\gamma}(A\cdot)\big)$

is a bounded operator on $L_p(E)$. By Lemma 6.3, $\Lambda = i\sum_{j=1}^{N}\frac{\partial}{\partial x_j}R_j$ and since the R_j are bounded operators on $L_p(E)$, we need only show that $\frac{\partial}{\partial x_j}\big(A U^{i\gamma}(\cdot)-U^{i\gamma}((A\cdot)\big)$ is a bounded operator of $j=1,2,\ldots,N$. If f is a smooth function with compact support,

$$A U^{i\gamma}(f)(x)-U^{i\gamma}(Af)(x)$$
$$=\Gamma(-i\gamma)^{-1}\lim_{\varepsilon\to 0}\int_{\|y\|>\varepsilon} f(x-y)\big(A(x)-A(x-y)\big)\frac{\Omega(y)\,dy}{\|y\|^{N+i\gamma}}.$$

Since f is smooth and has compact support, this last integral and its derivatives converge pointwise and in $L_p(E)$. Since $\frac{\partial}{\partial x_k}\big(A(x-y)-A(x)\big)$ is a bounded function,

$$P\int_E f(x-y)\frac{\partial}{\partial x_k}[A(x)-A(x-y)]\frac{\Omega(y)\,dy}{\|y\|^{N+i\gamma}}$$

is a bounded operator on L_p by [7] with norm at most $M(p,\gamma,\Omega)\|\nabla A\|_\infty$. Thus we need only consider the integral

$$\int_{\|y\|>\varepsilon}\Big[\frac{\partial}{\partial x_k}f(x-y)\Big]\big(A(x)-A(x-y)\big)\frac{\Omega(y)\,dy}{\|y\|^{N+i\gamma}}$$

for $\varepsilon>0$. Write $\frac{\partial}{\partial x_k}f(x-y) = -\frac{\partial}{\partial y_k}f(x-y)$ and use Green's formula [2] to integrate by parts. The last integral is

$$\int_{\|y\|=\varepsilon} f(x-y)\big(A(x)-A(x-y)\big)\frac{\Omega(y)}{\|y\|^{N+i\gamma}}\lambda_k\,d\sigma_\varepsilon(y)$$
$$+\int_{\|y\|>\varepsilon} f(x-y)\frac{\partial}{\partial y_k}\Big[[A(x)-A(x-y)]\frac{\Omega(y)}{\|y\|^{N+i\gamma}}\Big]dy$$

where λ_k is the direction cosine of the outward normal to the surface of the sphere $\|y\|=\varepsilon$, and where the first integral is the surface integral over the sphere of radius ε about $y=0$. The differentiability of A guarantees that the first integral tends to zero with ε. For the second integral, write

$$\int_{\|y\|>\varepsilon} = \int_{\varepsilon<\|y\|\leqslant 1} + \int_{\|y\|>1}.$$

Since A is differentiable with Lipschitz derivatives, the Taylor expansion of order 1 for A is of the form $A(x)-A(x-y) = \langle\nabla A(x-y), y\rangle + R(x,y)$ with $|R(x,y)|\leqslant M\|y\|^{1+\eta}$ for $\eta<\delta$. Thus

$$\int_{\varepsilon<\|y\|\leqslant 1} f(x-y)R(x,y)\frac{\partial}{\partial y_k}\Big[\frac{\Omega(y)}{\|y\|^{N+i\gamma}}\Big]dy$$

converges absolutely as does

$$\int_{1\leqslant\|y\|\leqslant\infty} f(x-y)\big(A(x)-A(x-y)\big)\frac{\partial}{\partial y_k}\Big[\frac{\Omega(y)}{\|y\|^{N+i\gamma}}\Big]dy.$$

$\int_{1\leqslant\|y\|>\infty} f(x-y)\frac{\partial}{\partial y_k}A(x-y)\frac{\Omega(y)}{\|y\|^{N+i\gamma}}dy$ is a bounded operator by [7] with norm at most $M(p,\gamma)\|\Omega\|_1\|\nabla A\|_\infty$, $\|\Omega\|_1 = \int_\Sigma|\Omega(\omega)|\,d\sigma(\omega)$. For each $j=1,2,\ldots,N$,

$$\int_{\varepsilon<\|y\|\leqslant 1} f(x-y)\frac{\partial}{\partial x_j}A(x-y)y_j\frac{\partial}{\partial y_k}\Big[\frac{\Omega(y)}{\|y\|^{N+i\gamma}}\Big]dy$$

is a bounded operator on L_p by [7] since $y_j\frac{\partial}{\partial y_k}\Big[\frac{\Omega(y)}{\|y\|^{N+i\gamma}}\Big]$ is homogeneous of degree $-(N+i\gamma)$. Since smooth functions with compact support are dense in $L_p(E)$, this completes the proof of the theorem.

COROLLARY 6.5. *If A and $U^{i\gamma}$ are as in Theorem 6.4, if h is a non-zero vector in E and if D_h denotes the derivative in the direction h, then $D_h\big(AU^{i\gamma}(\cdot)-U^{i\gamma}(A\cdot)\big)$ and $(AU^{i\gamma}(D_h\cdot)-U^{i\gamma}(AD_h\cdot)$ extend to bounded operators on $L_p(E)$.*

Proof. Since the operators in question are essentially adjoints of each other, it is sufficient to consider the second operator. For smooth functions f with compact support, $D_hf = K\Lambda R_hf = KR_h\Lambda f$ with $R_h=\sum_{j=1}^{N}h_jR_j$ where R_j is the jth Riesz operator and h_j is the jth coordinate of h with respect to the standard basis in E. Since R_h is a bounded operator on $L_p(E)$, Theorem 6.4 implies that the operators in the statement of the corollary are bounded on $L_p(E)$.

COROLLARY 6.6. *Let A satisfy the assumptions of Theorem 6.4 and let $\Lambda=(-\Delta)^{1/2}$. Then $\alpha_A^\gamma(f) = \Lambda^{1+i\gamma}(Af)-A\Lambda^{1+i\gamma}(f)$ extends to a bounded operator on $L_p(E)$.*

Proof. Since $\Lambda(Af) = (\Lambda f)A+(\Lambda A)f$ and since $\Lambda^{i\gamma}\big((\Lambda A)f\big)$ is a bounded operator on $L_p(E)$, [4], we need only consider $\Lambda^{i\gamma}(A\Lambda f)-A\Lambda^{i\gamma}(\Lambda f)$. For smooth functions g with compact support $\Lambda^{i\gamma}(g) = \Gamma(-i\gamma)^{-1}t_+^{-i\gamma-1}(P_tg)$

where $P_t g$ is the Poisson integral of g. As was mentioned above, $\Lambda^{i\gamma}(g)$ can be rewritten as

$$\Lambda^{i\gamma}(g) = C_N \int_E D_y^{i\gamma}(g)(1+\|y\|^2)^{-(N+1)/2} dy.$$

Write $D_y^{i\gamma}(g) = \|y\|^{i\gamma} D_\omega^{i\gamma}(g)$ and rewrite the last integral in polar coordinates so that

$$\Lambda^{i\gamma}(g) = K(n,\gamma)\int_\Sigma D_\omega^{i\gamma}(g)\,d\sigma(\omega).$$

If we set $\Omega(\omega) = 1$, $\Lambda^{i\gamma}$ has the form of $U^{i\gamma}$ of Theorem 6.4 and $\Lambda^{i\gamma}(A\Lambda\cdot)-A\Lambda^{i\gamma}(\Lambda\cdot)$ extends to a bounded operator on $L_p(E)$ by Theorem 6.4.

Remark. It should be possible to extend Theorem 6.4 by assuming only that A is in $L_r^1(E)$ for some r in $1 \leqslant r \leqslant \infty$. The conclusion should read that $\|\mathfrak{G}_\gamma(f,a)\|_q \leqslant M(p,\gamma,r)\|\Lambda A\|_r \|f\|_p$ when $1/p+1/r = 1/q$. This conjecture is suggested by the fact that $\Lambda(Af)-A\Lambda f = (\Lambda A)f$ extends to a bounded operator from L_p to L_q with $\|\Lambda(Af)-A\Lambda f\|_q \leqslant \|\Lambda A\|_r \|f\|_p$ and by the fact that imaginary powers of derivatives do not decrease the smoothness of a function. No assumption on Ω beyond integrability over $\sum$ should be necessary. Imaginary powers of derivatives do alter the support of a function, however, and this may adversely influence the conjectured result. To verify the last statement, calculate $\left(\frac{d}{dx}\right)^{i\gamma}(X_{ab})(x)$ where X_{ab} is the characteristic function of a compact interval $[a,b]$ in the real line.

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