Let H be dense in L^p. By the theorem, $\hat{H}$ separates K $\hat{E}$-essentially. Hence there exists an $\hat{E}$-null set N in K such that $\hat{H}$ separates $M_1 \setminus N$ and $M_2 \setminus N$. Using the condition (b) of Proposition 1 in Section 1 for $\varepsilon = 1/n$ and applying the usual compactness argument, we can construct the sequences of sets which are completely separated by $\hat{H}$ and whose union is equal to $M_1 \setminus N$ resp. $M_2 \setminus N$. Conversely, let M_1, M_2 be two disjoint closed subsets of K. Since the L^p-norm is order continuous, K is extremely disconnected and every rare subset is an $\hat{E}$-null set in K. Hence we can assume that M_1 and M_2 are open-and-closed. It is now clear that the above condition implies that $\hat{H}$ separates M_1 and M_2 except for an $\hat{E}$-null set.

Remark. First, it is clear that the proposition holds also if H is an algebra contained in L^∞. Moreover, the case of a σ-finite measure space can be treated similarly. This shows that the result of R. H. Farrell [8] is included. If E is a Banach function space on (X, Γ, μ) with absolutely continuous norm (see [7]), H. Nakano [6] has shown that the norm of E is even order continuous. Hence nothing essential has to be changed and Theorem 2.2 of M. M. Rao [7] follows also.

COROLLARY. *Let X be a compact space, μ a finite Borel measure on X such that every non-empty open set in X has positive measure. $C(X)$ can be (canonically) identified with a dense sublattice of $L^p(X, \mu)$, $1 \leqslant p < \infty$, if and only if μ is regular.*

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Existence of some special bases in Banach spaces

by

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Abstract. The main result of the paper is that if X is a Banach space with a basis and Y has a normalized basis which is weakly convergent to zero and satisfies a certain condition, then $X+Y$ has a normalized basis which is weakly convergent to zero. A few similar results for other classes of bases are stated. New bases in $C[0,1]$ and $L_1[0,1]$ are constructed. A few results about universal bases are stated.

0. Introduction. In this paper we consider the following problem: Suppose we have a Banach space X with a basis and a Banach space Y with a basis possessing some additional properties. Can we construct a basis possessing some additional properties in the space $X+Y$? We solve this problem for wc_0-bases and for p-Hilbertian and p-Besselian bases (for the definitions see below).

Section 1 contains the definitions, notations and some known facts which are used later.

The central section of the present paper is Section 2. In this section we prove one fact on bases in the finite-dimensional Banach space (Proposition 2.1). This proposition is our main tool in Sections 3 and 4.

In Section 3, Proposition 3.1, we prove that if X has a basis and Y has a wc_0-basis satisfying some technical conditions, then $X+Y$ has a wc_0-basis. In particular, from our results it follows that if Y has a shrinking basis, then $X+Y$ has a wc_0-basis.

In Section 4 we prove some analogous theorems for Besselian and Hilbertian bases. As an application we obtain the existence of some interesting bases in $C[0,1]$ and $L_1[0,1]$. Those examples answer certain questions of A. Pełczyński [8] (cf. also [10] Problem 11.1).

Section 5 is devoted to universal bases. We prove the non-existence of wc_0-basis universal for all wc_0-bases. We obtain some information about bases universal for all shrinking bases. Since the proof of this result is a simple modification of the proof of Szlenk [12], we only point out the necessary changes in his proof.

The author is greatly obliged to prof. A. Pełczyński for suggesting the problem and many useful comments during the preparation of the present paper. In particular, the possibility of applying Proposition 2.1

to Besselian and Hilbertian bases, that is in fact the whole Section 4, was observed by him.

1. Preliminaries. A sequence (x_n) of elements of a Banach space X is said to be *basis* iff there exists a sequence (x_n^*) of bounded linear functionals on X such that $x = \sum_{n=1}^{\infty} x_n^*(x)x_n$ and $x_n^*(x_m) = \delta_{n,m}$ for $n, m = 1, 2, \ldots$ The functionals (x_n^*) are called the *coefficient functionals* of the basis (x_n). They are a basis for the subspace of the dual X^* which they span. It is well known that the sequence (x_n) of elements of a Banach space X is a basis for X iff there exists a constant K such that $\|\sum_{i=1}^{n} a_i x_i\| \leqslant K\|\sum_{i=1}^{n+k} a_i x_i\|$ for each sequence of scalars $(a_i)_{i=1}^{n+k}$ and each n and k. The smallest such constant K is called the *norm of the basis*. A basis is called *normalized* iff $\|x_n\| = 1$ for $n = 1, 2, \ldots$, and it is called *seminormalized* iff $0 < \inf\|x_n\| \leqslant \sup\|x_n\| < \infty$. A basis is called a *$wc_0$-basis* iff it is seminormalized and weakly convergent to zero. A basis (x_n) in X is called *shrinking* iff for every $f \in X^*$ we have $\lim_{k\to\infty}\|f|_{\mathrm{span}\{x_n\}_{n>k}}\| = 0$. A sequence (X_n) of finite-dimensional subspaces of X is called a *finite-dimensional decomposition* iff for each $x \in X$ there exists a unique sequence of vectors $x_n \in X_n$, $n = 1, 2, \ldots$ such that $x = \sum x_n$. If we have a basis (x_n) and an increasing sequence of integers (n_k), then the sequence of subspaces $X_k = \mathrm{span}\{x_{n_k+1}, \ldots, x_{n_{k+1}}\}$ forms a finite-dimensional decomposition. The following Lemma is a partial converse of this fact.

LEMMA 1.1. *Let (X_n) be a finite-dimensional decomposition in X and let each X_n have a basis $(x_i^n)_{i=1}^{k_n}$ of norm less than or equal to K. Then the sequence*

$$x_1^1, \ldots, x_{k_1}^1, x_1^2, \ldots, x_{k_2}^2, x_1^3, \ldots$$

is a basis for X.

This lemma is well known and goes back to Grinbylum [2]. In this form it is stated in [4].

The symbol $X+Y$ means the direct sum of Banach spaces X and Y. The elements of $X+Y$ will be denoted by $x+y$ where $x \in X$ and $y \in Y$. We will identify the space X (resp. Y) with the subspace $\{x+0\colon x \in X\} \subset X+Y$ (resp. $\{0+y\colon y \in Y\} \subset X+Y$). We can introduce the equivalent norm on $X+Y$ by $\|x+y\| = \max(\|x\|, \|y\|)$. The space $X+Y$ with this particular norm will be denoted by $(X+Y)_\infty$.

The reader is referred to [10] for general information about bases considered in this paper and for the proofs of all the facts about bases used without specific reference.

All the considerations are valid both for the real and for the complex case.

2. In this section we prove Proposition 2.1 concerning bases in finite-dimensional spaces. This proposition is our main tool in the rest of the present paper.

PROPOSITION 2.1. *Let X be an n-dimensional Banach space, $n \geqslant 2$, with basis $(e_i)_{i=1}^n$ of norm K and coefficient functionals $(e_i^*)_{i=1}^n$. Let R be the one-dimensional Banach space and let $e \in R$ be a vector of norm one. Then the sequence*

$$y_0 = \frac{e}{c} + \sum_{i=1}^{n} e_i, \quad y_i = \frac{e}{c} + e_i \quad (i = 1, 2, \ldots, n)$$

where $c = \|\sum_{i=1}^{n} e_i^*\|$ is a basis in the space $(R+X)_\infty$ of norm less than or equal to

$$K\left(1+\frac{2c}{n-1}\left\|\sum_{i=1}^{n} e_i\right\|\right) + \frac{2c}{n-1}\left\|\sum_{i=1}^{n} e_i\right\|.$$

Proof. It is easily seen that $(y_i)_{i=0}^n$ is a basis in $(R+X)_\infty$. We shall estimate the norm of the new basis. For this purpose let us take

$$z = \sum_{i=0}^{n} \zeta_i y_i \in (R+X)_\infty \quad \text{with } \|z\| \leqslant 1.$$

Using the definition of y_i we have

$$z = \left(\sum_{i=0}^{n} \zeta_i\right)\frac{1}{c}e + \sum_{i=1}^{n} (\zeta_0+\zeta_i)e_i.$$

Since $\|z\| \leqslant 1$, by the definition of the norm in $(R+X)_\infty$ we get

$$\left|\sum_{i=0}^{n} \zeta_i\right| \leqslant c \quad \text{and} \quad \left\|\sum_{i=1}^{n} (\zeta_0+\zeta_i)e_i\right\| \leqslant 1.$$

Thus

$$\left|\sum_{i=0}^{n} \zeta_i + (n-1)\zeta_0\right| = \left|\sum_{i=1}^{n} (\zeta_0+\zeta_i)\right| = \left|\left(\sum_{i=1}^{n} e_i^*\right)\left(\sum_{i=1}^{n} (\zeta_0+\zeta_i)e_i\right)\right| \leqslant c,$$

which implies $|\zeta_0| \leqslant 2c(n-1)^{-1}$.

Next we estimate the norm of the vector $\sum_{i=0}^{k} \zeta_i y_i$ for $1 \leqslant k < n$.

$$\left\|\sum_{i=0}^{k} \zeta_i y_i\right\| = \left\|\left(\sum_{i=0}^{k} \zeta_i\right)\frac{e}{c} + \sum_{i=1}^{k} (\zeta_0+\zeta_i)e_i + \left(\sum_{i=k+1}^{n} e_i\right)\zeta_0\right\|$$

$$= \max\left(\left|\frac{1}{c}\left(\sum_{i=0}^{k} \zeta_i\right)\right|, \left\|\sum_{i=1}^{k} (\zeta_0+\zeta_i)e_i + \left(\sum_{i=k+1}^{n} e_i\right)\zeta_0\right\|\right).$$

We will estimate each quantity separately. For the first one we have

$$\left|\frac{1}{c}\Big(\sum_{i=0}^{k}\zeta_i\Big)\right| \leqslant \frac{1}{c}|\zeta_0|+\frac{1}{c}\Big|\sum_{i=1}^{k}\zeta_i\Big| = \frac{1}{c}|\zeta_0|+\frac{1}{c}\Big|\Big(\sum_{i=1}^{n}e_i^*\Big)\Big(\sum_{i=1}^{k}\zeta_i e_i\Big)\Big|$$

$$\leqslant \frac{2}{n-1}+\Big\|\sum_{i=1}^{k}\zeta_i e_i\Big\| \leqslant \frac{2}{n-1}+K\Big(1+\frac{2c}{n-1}\Big\|\sum_{i=1}^{n}e_i\Big\|\Big)$$

because $\|\sum_{i=1}^{k}\zeta_i e_i\| \leqslant K\|\sum_{i=1}^{n}\zeta_i e_i\|$ and

$$\Big\|\sum_{i=1}^{n}\zeta_i e_i\Big\| \leqslant \Big\|\sum_{i=1}^{n}(\zeta_0+\zeta_i)e_i\Big\|+|\zeta_0|\,\Big\|\sum_{i=1}^{n}e_i\Big\| \leqslant 1+\frac{2c}{n-1}\Big\|\sum_{i=1}^{n}e_i\Big\|.$$

For the second quantity we have

$$\Big\|\sum_{i=1}^{k}(\zeta_0+\zeta_i)e_i+\Big(\sum_{i=k+1}^{n}e_i\Big)\zeta_0\Big\| \leqslant \Big\|\sum_{i=1}^{k}(\zeta_0+\zeta_i)e_i\Big\|+|\zeta_0|\,\Big\|\sum_{i=k+1}^{n}e_i\Big\|$$

$$\leqslant K+\frac{2c}{n-1}(K+1)\Big\|\sum_{i=1}^{n}e_i\Big\|.$$

So we obtain

$$\Big\|\sum_{i=0}^{k}\zeta_i y_i\Big\|$$

$$\leqslant \max\Big(\frac{2}{n-1}+K\Big(1+\frac{2c}{n-1}\Big\|\sum_{i=1}^{n}e_i\Big\|\Big),\ K+\frac{2c}{n-1}(K+1)\Big\|\sum_{i=1}^{n}e_i\Big\|\Big)$$

$$= K\Big(1+\frac{2c}{n-1}\Big\|\sum_{i=1}^{n}e_i\Big\|\Big)+2\max\Big(\frac{1}{n-1},\ \frac{1}{n-1}c\Big\|\sum_{i=1}^{n}e_i\Big\|\Big)$$

$$= K\Big(1+\frac{2c}{n-1}\Big\|\sum_{i=1}^{n}e_i\Big\|\Big)+\frac{2c}{n-1}\Big\|\sum_{i=1}^{n}e_i\Big\|.$$

This completes the proof.

3. In this section we consider the question of the existence of wc_0-bases in some Banach spaces.

We begin with the following

PROPOSITION 3.1. *Let X be a Banach space with a basis and let Y be a Banach space with the wc_0-basis (y_n) satisfying the following condition:*

(*) *there exist two sequences of natural numbers $(p_r)_{r=1}^{\infty}$ and $(k_r)_{r=1}^{\infty}$ such that $k_r < p_{r+1} < k_{r+1}-1$ and $\lim_{r\to\infty}(k_r-p_r) = \infty$ and the sequence $\|\sum_{p_r}^{k_r}y_n\|^{-1}(\sum_{p_r}^{k_r}y_n)$ is weakly convergent to zero.*

Then $X+Y$ has a wc_0-basis.

We shall need the following essentially known fact (cf. e.g. [10] Chapter II § 9).

LEMMA 3.2. *Let Y be an n-dimensional Banach space and let $(e_k)_{k=1}^{n}$ be a normalized basis in Y of norm K. Then the sequence $f_1 = e_1$, $f_i = e_i-e_{i-1}$, $i = 2, 3, \ldots, n$ is a basis in Y of norm less than or equal to $K+\sup_k\|\sum_{i=1}^{k}e_i\|$. Moreover, we have $2 \geqslant \sup\|f_i\| \geqslant \inf\|f_i\| \geqslant \sup\|e_i^*\|^{-1}$ for $i = 1, 2, \ldots, n$ and $\|\sum_{i=1}^{n}f_i^*\| \geqslant n$, where (e_i^*) and (f_i^*) denote the sequences of coefficient functionals of the bases (e_i) and (f_i), respectively.*

Proof. The sequence (f_i) is obviously a basis in Y. Pick an $y = \sum_{i=1}^{n}f_i^*(y)f_i \in Y$ with $\|y\| \leqslant 1$. Since $y = \sum_{i=1}^{n-1}(f_i^*(y)-f_{i+1}^*(y))e_i+f_n^*(y)e_n$, we have

$$\Big\|\sum_{i=1}^{k}f_i^*(y)f_i\Big\| = \Big\|\sum_{i=1}^{k-1}(f_i^*(y)-f_{i+1}^*(y))e_i+f_k^*(y)e_k\Big\| \leqslant K+\|f_k^*\|$$

$$\leqslant K+\sup_k\Big\|\sum_{i=1}^{k}e_i^*\Big\|$$

because $f_k^* = \sum_{i=1}^{k}e_i^*$ for $k = 1, 2, \ldots, n$.

Since $\|f_i\| \leqslant 2$ for $i = 1, 2, \ldots, n$, we get

$$1 = e_i^*(f) \leqslant \|e_i^*\|\,\|f_i\| \leqslant 2\|e_i^*\|,$$

which implies the second of the desired inequalites. Finally, we have

$$n = \Big|\Big(\sum_{i=1}^{n}f_i^*\Big)\Big(\sum_{i=1}^{n}f_i\Big)\Big| \leqslant \Big\|\sum_{i=1}^{n}f_i^*\Big\|\cdot\Big\|\sum_{i=1}^{n}f_i\Big\| = \Big\|\sum_{i=1}^{n}f_i^*\Big\|\,\|e_n\| = \Big\|\sum_{i=1}^{n}f_i^*\Big\|.$$

Proof of Proposition 3.1. Let (y_n^*) be the coefficient functionals of the basis (y_n). We can assume without loss of generality that $\lim_{r\to\infty}\|\sum_{p_r}^{k_r}y_n^*\| = \infty$.

Let us prove this claim.

If we have $\limsup_r\|\sum_{p_r}^{k_r}y_n^*\| = \infty$, we can easily pass to a subsequence in the sequences (p_r) and (k_r). So assume that $\|\sum_{p_r}^{k_r}y_n^*\| \leqslant M$. We apply Lemma 3.2 for each r to a basis $(y_{p_r+i})_{i=0}^{k_r-p_r}$ in the space $Y_r = \operatorname{span}\{y_{p_r+i}\}_{i=0}^{k_r-p_r}$. Since (y_n^*) is a basic sequence, we have $\sup_r\sup_{k\leqslant k_r-p_r}\|\sum_{p_r}^{p_r+k}y_n^*\| < \infty$. Moreover, since (y_n) is a basis, the norms of bases $(y_{p_r+i})_{i=0}^{k_r-p_r}$ are uniformily bounded. So we can apply Lemma 1.1 to the sequence of spaces

$$\operatorname{span}\{y_i\}_{i=1}^{p_1-1},\quad \operatorname{span}\{y_i\}_{i=p_1}^{k_1},\quad \operatorname{span}\{y_i\}_{i=k_1+1}^{p_2-1},\quad \operatorname{span}\{y_i\}_{i=p_2}^{k_2},\ \ldots$$

and to natural bases in odd spaces and to bases constructed in Lemma 3.2 in even spaces in order to obtain a seminormalized basis in Y. It is easily seen that this basis has the desired property.

Let us consider the space $(X+Y)_\infty$. Let $(x_n)_{n=1}^\infty$ be a normalized basis in X. In the space $(X+Y)_\infty$ we consider the finite-dimensional decomposition $(Z_n)_{n=1}^\infty$ where

$$Z_{2r-1} = \operatorname{span}\{y_i\}_{i=k_{r-1}+1}^{p_r-1} \quad \text{and} \quad Z_{2r} = \operatorname{span}\{x_r, \{y_i\}_{i=p_r}^{k_r}\}.$$

To each space Z_{2r} we apply Proposition 2.1 in order to obtain the new basis in it. Since the sequence $(\sum_{p_r}^{k_r} y_i)_{r=1}^\infty$ is a basic sequence, by Theorem 3.1 of [10] we have

$$(k_r-p_r)^{-1}\Big\|\sum_{p_r}^{k_r} y_i\Big\| \ \Big\|\sum_{p_r}^{k_r} y_i^*\Big\| \leqslant M_1.$$

Thus by Proposition 2.1 the norms of bases in the spaces of the decomposition are uniformily bounded. So we can apply Lemma 1.1 in order to find that the sequence

$$y_1, \ldots, y_{p_1-1}, \Big\|\sum_{p_1}^{k_1} y_i\Big\|^{-1}\Big(\frac{x_1}{A_1}+\sum_{p_1}^{k_1} y_i\Big), \frac{x_1}{A_1}+y_{p_1}, \frac{x_1}{A_1}+y_{p_1+1}, \ldots, \frac{x_1}{A_1}+y_{k_1},$$

$$y_{k_1+1}, \ldots, y_{p_2-1}, \Big\|\sum_{p_2}^{k_2} y_i\Big\|^{-1}\Big(\frac{x_2}{A_2}+\sum_{p_2}^{k_2} y_i\Big), \frac{x_2}{A_2}+y_{p_2}, \ldots,$$

where $A_r = \|\sum_{p_r}^{k_r} y_i^*\|$, is a seminormalized basis in $(X+Y)_\infty$. Since $A_r \to \infty$, we see that the summands in which x_r appears are norm-convergent to zero. The other summands are by our assumptions weakly convergent to zero, and so this basis is a wc_0-basis.

Since $X+Y \sim (X+Y)_\infty$, the Proposition follows.

Remark 3.3. We suspect that the condition $(*)$ is superflows. We do not know an example of wc_0-basis which does not satisfies this condition.

COROLLARY 3.4. *If X has a basis and contains a complemented subspace Y such that $Y+Y \sim Y$ and Y has a wc_0-basis satisfying $(*)$, then X has a wc_0-basis.*

Proof. $X \sim Y+Z \sim Y+Y+Z \sim Y+X$ and we apply Proposition 3.1.

THEOREM 3.5. *Let X have a basis and Y a shrinking basis, more specific if Y is reflexive. Then $X+Y$ has a wc_0-basis.*

Proof. It is well known that any shrinking basis is a wc_0-basis. So in view of Proposition 3.1 it is enough to prove that any shrinking basis satisfies the condition $(*)$. Let (y_i) be a shrinking basis in Y. Then, for any sequence (p_r), (k_r) of natural numbers such that $p_r < k_r < p_{r+1}-1$ and $(k_r-p_r) \to \infty$, we find that $\|\sum_{p_r}^{k_r} y_i\|^{-1}\sum_{p_r}^{k_r} y_i$ is weakly convergent to zero by the definition of the shrinking basis.

In particular, since the space l_p, $1<p<\infty$ has a shrinking basis and is isomorphic to its square, by Theorem 3.1 and Corollary 3.4 we obtain.

COROLLARY 3.6. *Let a Banach space X have a basis and contain a complemented subspace isomorphic to l_p for some p with $1<p<\infty$. Then X has a wc_0-basis.*

Since by Theorem of Sobczyk [11] (cf. also [7]) c_0 is always complemented in a separable Banach space and the unite vector basis in c_0 is shrinking and $c_0+c_0 \sim c_0$, we have

COROLLARY 3.7. *If a Banach space X has a basis and contains a subspace isomorphic to c_0, then X has a wc_0-basis.*

A predual of L_1 is a Banach space X such that X^* is linearly isometric to $L_1(\mu)$ for some measure μ. Combining our Corollary 3.7 with the result of Zippin [16], we infer that every separable predual of L_1 has a wc_0-basis. In particular, the space $C[0,1]$ has a wc_0-basis. This fact has recently been proved by Warren [13].

Let us recall that a Banach space X has the bounded approximation property iff there exists a sequence of finite-dimensional operators $T_n\colon X \to X$ such that for every $x \in X$ we have $T_n(x) \to x$.

COROLLARY 3.8. *If a Banach space X has a basis and Y^* has a basis (or Y has a basis and Y^* has a bounded approximation property), then $X+Y$ has a wc_0-basis.*

Proof. By Theorem 1.4 of [4] Y has a shrinking basis, and so we apply Theorem 3.5.

Remark 3.9. The spaces $C[0,1]+l_p$ where $1<p<\infty$ give an example of continuum non-isomorphic spaces with wc_0-bases. No basis in $C[0,1]+l_p$ is shrinking or boundedly complete or unconditional. A slightly weaker fact was established by Holub [3] Theorem 4.4.

4. In this section we consider p-Besselian and p-Hilbertian bases. Let us recall the definitions.

A seminormalized basis (x_n) in a Banach space X is said to be

p-Hilbertian if $\sum_{n=1}^\infty |a_n|^p < \infty \Rightarrow \sum_{n=1}^\infty a_n x_n$ is convergent;

p-Besselian if $\sum_{n=1}^\infty a_n x_n$ is convergent $\Rightarrow \sum_{n=1}^\infty |a_n|^p < \infty$.

We assume $1<p<\infty$.

It is obvious that if $p<p_1$ and the basis (x_n) is p_1-Hilbertian, then (x_n) is p-Hilbertian, and if (x_n) is p-Besselian, then it is p_1-Besselian.

THEOREM 4.1. *Let X have a basis and contain a complemented subspace isomorphic either to l_p, $1 < p < \infty$, or to c_0. Then X has a basis which is p'-Hilbertian for every $p' < p$; for every p if X contains c_0.*

Proof. We will prove the theorem in the case where $p < \infty$. In the case of c_0 the proof is the same. Since $X \sim X + l_p$ it is enough to prove the assertion of the Theorem for the space $(X+l_p)_\infty$. Let $(x_n)_{n=1}^\infty$ be a normalized basis in X and let $(e_n)_{n=2}^\infty$ be the unit vector basis in the space l_p. Let $q = p(p-1)^{-1}$. We put

$$z_n^0 = \Big(2^{-\frac{n}{q}} x_n + \sum_{2^n}^{2^{n+1}-1} e_r\Big) 2^{-\frac{n}{q}}, \qquad z_n^i = 2^{-\frac{n}{q}} x_n + e_{2^n+i-1},$$

$$n = 1, 2, \ldots, i = 1, 2, \ldots, 2^n$$

Since $2^{\frac{n}{q}} = \|\sum_{2^n}^{2^{n+1}-1} e_r^*\|$, it follows from Lemma 1.1 and Proposition 2.1 (in an analogous way as in proof of Proposition 3.1) that the sequence $z_1^0, z_1^1, z_1^2, z_2^0, \ldots, z_2^4, z_3^0, \ldots$ is a seminormalized basis in the space $(X+l_p)_\infty$.

Let us take a sequence c_n^k, $n = 1, 2, \ldots$, $k = 0, 1, \ldots, 2^n$, such that $\sum_{n=1}^\infty \sum_{k=0}^{2^n} |c_n^k|^{p'} < \infty$ for some $p' < p$. We have to prove that $\sum_{n=1}^\infty \sum_{k=0}^{2^n} c_n^k z_n^k$ is convergent. We have

$$\sum_{n=1}^\infty \sum_{k=0}^{2^n} c_n^k z_n^k = \sum_{n=1}^\infty \Big(c_n^0 2^{-n} + 2^{-\frac{n}{q}} \sum_{k=1}^{2^n} c_n^k\Big) x_n + \sum_{n=1}^\infty \sum_{k=1}^{2^n} (c_n^k + c_n^0 2^{-\frac{n}{p}}) e_{2^n+k-1}.$$

We will consider each sum separately. The first one is absolutly convergent. Indeed

$$\sum_{n=1}^\infty \Big| c_n^0 2^{-n} + 2^{-\frac{n}{q}} \sum_{n=1}^{2^n} c_n^k \Big| \leqslant \sum_{n=1}^\infty 2^{-n} |c_n^0| + \sum_{n=1}^\infty \sum_{k=1}^{2^n} 2^{-\frac{n}{q}} |c_n^k|$$

$$\leqslant \sum_{n=1}^\infty 2^{-n} |c_n^0| + \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} |c_n^k|^{p'}\Big)^{\frac{1}{p'}} \cdot \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} 2^{-\frac{n}{q}q'}\Big)^{\frac{1}{q'}}$$

$$= \sum_{n=1}^\infty 2^{-n} |c_n^0| + \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} |c_n^k|^{p'}\Big)^{\frac{1}{p'}} \cdot \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} 2^{n\left(1-\frac{q'}{q}\right)}\Big)^{\frac{1}{q'}} < \infty \qquad \text{for } p' < p.$$

The second sum is convergent because for $p' < p$ we have

$$\Big(\sum_{n=1}^\infty \sum_{k=0}^{2^n} |c_n^k + c_n^0 2^{-\frac{n}{p}}|^p\Big)^{\frac{1}{p}} \leqslant \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} |c_n^k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} |c_n^0 2^{-\frac{n}{p}}|^p\Big)^{\frac{1}{p}}$$

$$= \Big(\sum_{n=1}^\infty \sum_{k=1}^{2^n} |c_n^k|^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^\infty |c_n^0|^p\Big)^{\frac{1}{p}} < \infty.$$

This completes the proof.

Remark 4.2. A proof analogous to that of Theorem 4.1 gives the following result:

Let X have a complemented subspace isomorphic either to l_p, $1 < p < \infty$, or to c_0. Let N be an Orlicz function such that $M(x)x^{-q} \underset{x\to 0}{\to} 0$ where $p^{-1} + q^{-1} = 1$ and M is an Orlicz function complementary to N. Then X has a normalized basis (x_n) such that $\sum_{n=1}^\infty a_n x_n$ is convergent whenever $\sum_{n=1}^\infty N(|a_n|) < \infty$.

For the definitions and properties of Orlicz functions we refer to [5].

COROLLARY 4.3. *The space $C[0,1]$ has a basis which is p-Hilbertian for all $p < \infty$.*

Remark 4.4. Since a p-Hilbertian basis for $1 < p < \infty$ is a wc_0-basis, Corollary 4.3. improves the result of Warren [13].

THEOREM 4.5. *If X has a basis and contains a complemented subspace isomorphic to l_p, $1 \leqslant p < \infty$, then X has a basis which is p'-Besselian for all $p' > p$.*

Outline of the proof. Let (x_n^*) be the sequence of coefficient functionals of the basis in X and let (e_n^*) be the sequence of coefficient functionals of the unit vector basis in l_p. We apply the construction used in the proof of Theorem 4.1 to bases (x_n^*) and (e_n^*) in order to obtain a basis (y_n^*) in the space $\mathrm{span}\{x_n^*\} + \mathrm{span}\{e_n^*\}$ which is q-Hilbertian for all $q < p/p-1$. It follows from the construction that the coefficient functionals of this basis form a basis in $X + l_p$. Thus it is a basis in X which is p'-Besselian for all $p' > p$.

The dual version of Remark 4.2 is also true.

COROLLARY 4.6. *The space $L_1[0,1]$ has a basis which is p-Besselian for all $p > 1$.*

Remark 4.7. Let (x_n) be a basis in $L_1[0,1]$ given by Corollary 4.6. This basis has the additional property that if $\sum_{n=1}^\infty a_n x_n$ is unconditionally convergent, then it is absolutly convergent. To see it let us consider an operator $T\colon L_1[0,1] \to l_2$ defined by $T\big(\sum_{n=1}^\infty a_n x_n\big) = (a_n)$. This operator is continuous and by the Grothendiesk inequality (cf. [6]) is absolutly summing. But this means that T transforms unconditionally convergent series into absolutely convergent ones. So we have $\sum |a_n| < \infty$.

Remark 4.8. Corollaries 4.3 and 4.6 answer the question of Pełczyński [8] (cf. also [10] Problem 11.1).

5. This section is devoted to universal bases.

DEFINITION (cf. [9]). Let $\mathscr{A}$ be a class of bases. A basis (x_n) is said to be *universal for* $\mathscr{A}$ iff every basis $(y_n)\in\mathscr{A}$ is equivalent to a suitable subbasis of (x_n).

Let (x_n) be a wc_0-basis in X. Denote by K_{X^*} the closed unit ball in X^* equipped with the w^*-topology. K_{X^*} is a compact, metric space. The sequence (x_n) can be regarded as a sequence of elements of $C(K_{X^*})$, the space of all continuous scalar-valued functions on K_{X^*}. The sequence (x_n) is convergent to zero in the weak topology of $C(K_{X^*})$.

Let us define the index $\eta(x_n)$ of a basis (x_n).

First, by transfinite induction, we define for $\varepsilon > 0$ sets $P_\alpha(\varepsilon, (x_n))$ as follows:

$P_0(\varepsilon, (x_n)) = K_{X^*}$,

$P_{\alpha+1}(\varepsilon, (x_n)) = \{x^*\in K_{X^*}$: there exists a sequence (z_n^*), $z_n^* \xrightarrow[w^*]{} x^*$ and an increasing sequence of indices k_n such that $\liminf |x_{k_n}(z_n^*)| \geqslant \varepsilon\}$,

$P_\alpha(\varepsilon, (x_n)) = \bigcap_{\beta<\alpha} P_\beta(\varepsilon, (x_n))$ when α is a limit ordinal number.

The index $\eta(x_n)$ is defined by

$$\eta(x_n) = \sup_{\varepsilon>0} \sup\{\alpha\colon P_\alpha(\varepsilon, (x_n)) \neq \emptyset\}.$$

It was proved by Zalcwasser [15] and by Gillespie and Hurwitz [1] that for each sequence of functions (x_n) weakly convergent to zero in $C(Q)$, (Q-compact metric) we have $\eta(x_n) < \omega_1$ where ω_1 is the first uncountable ordinal number. Thus we have

LEMMA 5.1. *If (x_n) is a wc_0-basis, then $\eta(x_n) < \omega_1$.*

Modifying the proofs of Proposition 2.3 and Proposition 3.2 of [12] by using the set $\{x_n\}$ instead of the unit ball of X we obtain the following:

LEMMA 5.2. *If (x_n) is equivalent to a subbasis of a wc_0-basis (y_n), then $\eta(x_n) \leqslant \eta(y_n)$.*

LEMMA 5.3. *For each countable ordinal number α there exists a normalised basis $(x_n^\alpha)_{n=1}^\infty$ in a reflexive space such that $\eta(x_n^\alpha) \geqslant \alpha$. Since $(x_n^\alpha)_{n=1}^\infty$ is a basis in a reflexive space, it is shrinking and boundedly complete.*

Using Lemmas 5.1, 5.2, 5.3 one easily obtain (cf. [9] Th. 4 and [14] Corollary 2) the following:

THEOREM 5.4. *There is no wc_0-basis which is universal for all wc_0-bases.*

The same argument show

THEOREM 5.5. *If (x_n) is a basis universal for all normalised shrinking bases, then (x_n) is not a wc_0-basis.*

This Theorem extends Theorem 4 of [9].

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