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A Stone-Weierstrass theorem for Banach lattices

by

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Abstract. Let E be a Banach lattice with a quasi-interior point $u \in E_+$, and let H be a sublattice of E containing u. Using the representation theory for such Banach lattices we give a necessary and sufficient condition for H to be dense in E. The classical Stone–Weierstrass theorem for $E = C(X)$ and other known results for L^p-spaces and Banach function spaces are easy consequences of the main theorem.

Recently, E. B. Davies [2] and H. P. Lotz [5] (see also the paper of H. H. Schaefer [9]) developed a representation theory for Banach lattices with quasi-interior points in the positive cone. We restate the main theorem:

Let E be a Banach lattice with quasi-interior elements in the positive cone. Then E is isomorphic to a vector lattice $\hat{E}$ of continuous numerical functions on a compact space K (the "structure space" of E), each of which is infinite only on some rare subset of K. $\hat{E}$ is a Banach lattice for the norm transferred from E and contains $C(K)$ as a dense ideal. This property determines K to within homeomorphism.

This result contains as special cases the Kakutani representation theorems for AM-spaces with order unit and AL-spaces with weak order unit. Moreover, the structure space K seems to be quite appropriate for further research on the structure of Banach lattices. In this paper we shall use it for the investigation of dense sublattices, and we will prove a theorem which includes the classical Stone-Weierstrass theorem for Banach lattices $C(K)$ as well as the results of R. H. Farrell [3] for L^p-spaces and of M. M. Rao [7] for function spaces.

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We start in Section 1 with a more detailed study of the structure space K taking into account the specific norm stucture of E. In doing so, we observe that certain rare subsets of K are in some way neglected by the norm of E. We call these sets "E-null sets". Using this concept we are able to formulate a necessary and sufficient condition for a sublattice H to be dense in E (see Section 2). It is easily seen that the classical Stone–Weierstrass theorem is subsumed. Further applications are discussed in Section 3.

This paper relies strongly on [9], whose terminology is used.

1. Null sets in the structure space. For the following, E denotes always a Banach lattice. Let u be a quasi-interior point of the positive cone E_+ (1). We assume now that E is represented according to the above theorem, such that each $x \in E$ is identified with a continuous numerical (2) function on the compact structure space K, u being the unit function. Consequently, the ideal E_u can be identified with the space $C(K)$ of all continuous real-valued functions on K.

DEFINITION. A subset N of K is called an *E-null set* if the ideal $I_N := \{x \in E: x(N) = \{0\}\}$ is dense in E.

From $I_N = I_{\bar{N}}$ it follows that the closure of an E-null set is still an E-null set. If the interior of $\bar{N}$ is not empty, there exists $0 \neq y \in C(K)$ orthogonal to every $x \in I_N$. Hence, I_N cannot be dense in E, and we see that every E-null set is a rare subset of K. But for additional information on E-null sets we have to consider the specific norm structure of E.

EXAMPLE 1. Let $E = C(K)$. For every $\varphi \in K$, $I_{\{\varphi\}}$ is a closed proper ideal in $C(K)$. It follows that the empty set is the only $C(K)$-null set in K.

EXAMPLE 2. Let X be a completely regular space and denote by $C_b(X)$ the Banach lattice of all bounded continuous real-valued functions on X endowed with the sup-norm. For a strictly positive function $v \in C_b(X)$, let F be the closed ideal generated by v in $C_b(X)$, i.e. $F = \overline{\bigcup_{n \in N} n[-v, v]} \subset C_b(X)$. Since the vector lattice $F_v = \bigcup_{n \in N} n[-v, v]$ is isomorphic to $C_b(X)$ we conclude that the structure space K of F is homeomorphic to the structure space of $C_b(X)$, hence to the Stone-Čech compactification βX of X. Denote by $\bar{v}$ the continuous extension of v to βX and set $N_0 := \{\varphi \in \beta X: \bar{v}(\varphi) = 0\}$. N_0 is contained in $\beta X \setminus X$, and one can show that N_0 is a F-null set containing all other F-null sets.

If v is the unit function, we retrieve example 1 and get $N_0 = \emptyset$. If X is locally compact and v vanishes at infinity, we have $F = C_0(X)$ (the Banach lattice of all continuous functions vanishing at infinity) and $N_0 = \beta X \setminus X$.

EXAMPLE 3. Every AL-space F having a weak order unit can be represented according to the above representation theorem. Since F is always order complete, the structure space K is extremaly disconnected. Moreover, the closed ideals in F correspond precisely to the open-and-closed subsets of K (see [9]: § 3, prop. 7 and § 2, prop. 3, corollary). From this it follows that, for a rare subset N of K, I_N is not contained in a proper closed ideal of F. Hence every rare subset of K is a F-null set.

(1) This means that $E_u := \bigcup_{n \in N} n[-u, u]$ is dense in E.

(2) By a numerical function we understand a map into the two-point compactification of $\mathbf{R}$.

PROPOSITION 1. *Let N be a closed subset of K. The following conditions are equivalent:*

(a) *N is an E-null set.*

(b) *For all $\varepsilon > 0$ there exists a neighborhood U of N satisfying: $\|x\| < \varepsilon$ for all $0 \leqslant x \leqslant u$ and $x(\complement U) = \{0\}$.*

(c) *There exists $y \in E$ such that $y(N) = \{\infty\}$.*

(d) *N is a μ-null set for all $0 \leqslant \mu \in E'$* (3).

Proof. "(a) ⇒ (b)". By assumption, u can be approximated by elements in I_N. I_N is an ideal in E, hence we can find $0 \leqslant z \leqslant u$, $z \in I_N$ such that $\|u - z\| \leqslant \varepsilon/2$ for a given $\varepsilon > 0$. Set $y = 2(u - z)$ and $U = \{\varphi \in K: y(\varphi) > 1\}$. If $x \in E$ satisfies $0 \leqslant x \leqslant u$, $x(\complement U) = \{0\}$, then $0 \leqslant x \leqslant y$ and consequently $\|x\| \leqslant \|y\| \leqslant \varepsilon$.

"(b) ⇒ (c)". Choose neighborhoods U_n of N satisfying condition (b) for $\varepsilon_n = 2^{-n}$, $n \in \mathbf{N}$. Then we can find $x_n \in E$, $0 \leqslant x_n \leqslant u$, such that $x_n(N) = \{1\}$, $x_n(\complement U_n) = \{0\}$. For $y := \sum_{n \in N} x_n$ we get $y(N) = \{\infty\}$.

"(c) ⇒ (d)". This is obvious.

"(d) ⇒ (a)". Assume that N is a μ-null set for all $0 \leqslant \mu \in E'$. Then N has to be a rare subset of K, and hence, $\sup G = u$ for $G := \{x \in E: x \in I_N \cap [0, u]\}$. In addition, since μ is a regular Borel measure on K it follows that $\sup\{\langle x, \mu\rangle: x \in G\} = \langle \sup G, \mu\rangle = \langle u, \mu\rangle$, N being a μ-null set. The corollary 1 of ([8], V. 4.3) shows that $u \in \bar{G}$, hence $\bar{I}_N = E$.

COROLLARY 1. *The countable union of an E-null sets is an E-null set.*

Proof. The assertion follows easily by applying condition (d) of the above proposition.

COROLLARY 2. *Let N be a closed E-null set in K. Then $K = \beta(K \setminus N)$, i.e. every bounded continuous real-valued function on $K \setminus N$ has a unique continuous extension to K.*

Proof. Let z be a bounded continuous real-valued function on $K \setminus N$. For every $n \in N$ choose an open neighborhood U_n of N which satisfies the condition of (b) for $\varepsilon_n = 2^{-n}$. Then we can find $y_n \in C(K)$ which coincide with z on $\complement U_n$ and form a Cauchy sequence in E. Denote its limit by $y \in E$. y has to coincide with z on the complement of every U_n. Again from condition (b) it follows that $M = \bigcap_{n \in N} U_n$ is an E-null set, hence is a rare subset of K. This shows that the extension y of z restricted to $\complement M$ is unique. A fortiori, y is the unique extension to K of z.

We are now able to characterize the two extreme cases occuring in the Examples 1 and 3 by properties involving the norm structure of the Banach lattice E. Recall for this that u is an interior point of E_+ iff it

(3) E' can be identified with a subspace of $C(K)'$.

is an order unit. In this case E is isomorphic to an AM-space with order unit ([8], V. 6.2.C2). On the other hand, the norm of E is said to be *order continuous* if $x_\lambda \in E$, $x_\lambda \downarrow 0$ implies $\|x_\lambda\| \downarrow 0$.

PROPOSITION 2. *Let E be a Banach lattice, u a quasi-interior point of E_+ and let K be the structure space of E. The empty set is the only E-null set in K iff u is an interior point of E_+. Every rare subset of K is an E-null set iff the norm of E is order continuous.*

Proof. If u is an interior point of E_+, the assertion follows from Example 1 and from the remark preceding the proposition. Conversely, let $\emptyset$ be the only E-null set in K. Then, condition (c) above shows that every $x \in E$ takes only finite values on K. Since K is compact, we can conclude that u is an order unit and hence an interior point of E_+.

For the proof of the second part of the proposition assume first that the norm of E is order continuous. Then it is well known that E is order complete and that the closed ideals in E (which are always bands) correspond precisely to the open-and-closed subsets of K. Repeating the argument of Example 3 yields the assertion. Conversely, denote by $\{x_\lambda: \lambda \in \Lambda\}$ a downwards directed family in E_+ with $\inf_\lambda x_\lambda = 0$. This implies $\inf_\lambda x_\lambda(\varphi) = 0$ for all $\varphi \in K \setminus N$, N a subset of first category of K. By assumption and Corollary 1, N is an E-null set. Choose an open neighborhood U of N satisfying the condition of (b) for $\varepsilon > 0$. $\{x_\lambda\}$ converges uniformly to 0 on the complement of U. Hence, there is a $x_\varepsilon \in \{x_\lambda\}$ such that $\|x_\varepsilon\| < 2\varepsilon$.

2. A Stone-Weierstrass theorem. As in Section 1, E denotes a Banach lattice with quasi-interior points in the positive cone, represented as a Banach lattice of continuous numerical functions on a compact space K such that $C(K)$ is dense in E. We suppose now that a (linear) sublattice H of E be given containing a quasi-interior point u of E_+. Without loss of generality we will assume that u is the unit function on K. Under which (necessary and sufficient) conditions is H dense in E?

For $E = C(K)$ the classical Stone-Weierstrass theorem asserts that H is dense in $C(K)$ iff H separates the points of K. In general, E is the completion of $C(K)$ with respect to a lattice norm coarser than the sup-norm on $C(K)$. Hence, if H separates the points of K it is certainly dense in E. But as the following example shows, this is far from being necessary.

Take the Banach lattice $l^1(\mathbf{N})$. The sequence $u = (2^{-n})$ is a quasi-interior point of the positive cone, and the sublattice H spanned by u and all finite sequences is dense in $l^1(\mathbf{N})$. It is easy to see that the ideal generated by u in $l^1(\mathbf{N})$ is isomorphic to $l^\infty(\mathbf{N})$. Hence, the structure space K of $l^1(\mathbf{N})$ is homeomorphic to $\beta\mathbf{N}$ and under the canonical representation, the elements of $l^1(\mathbf{N})$ become continuous numerical functions on $\beta\mathbf{N}$. In particular, H can be identified with all continuous functions on $\beta\mathbf{N}$ which are constant on $\beta\mathbf{N}\setminus\mathbf{N}$. This shows that H does not separate the points of the structure space of $l^1(\mathbf{N})$.

But by recalling the results of Section 1, we observe that $\beta\mathbf{N}\setminus\mathbf{N}$ is a rare subset in $K = \beta\mathbf{N}$, hence a l^1-null set. This suggests that an appropriate criterion for the denseness of H in E has to take into account the existence of E-null sets in K. In particular, separation of points in K seems to be inadequate. Instead, we will say that *the subsets X_1, X_2 of K are separated by F*, $F \subset E$, if for every $\varphi_1 \in X_1$, $\varphi_2 \in X_2$ there is a $x \in F$ satisfying $x(\varphi_1) \neq x(\varphi_2)$.

DEFINITION. A subset F of E is said to *separate the structure space K E-essentially* if for every pair M_1, M_2 of disjoint closed subsets of K, there exists an E-null set N such that F separates $M_1 \setminus N$ from $M_2 \setminus N$.

For $E = C(K)$, the empty set is the only $C(K)$-null set. Hence a subset of $C(K)$ separates the structure space K $C(K)$-essentially iff it separates the points of K. For the Banach lattice $l^1(\mathbf{N})$ it is clear that $\beta\mathbf{N}\setminus\mathbf{N}$ is a $l^1(\mathbf{N})$-null set containing all $l^1(\mathbf{N})$-null sets. Since the sublattice H in the above example separates the points of $\mathbf{N}$ it separates $\beta\mathbf{N}$ $l^1(\mathbf{N})$-essentially.

THEOREM. *Let E be a Banach lattice and let H be a sublattice containing the quasi-interior point $u \in E_+$. H is dense in E if and only if H separates the structure space K of E E-essentially.*

The following lemma is essential for the first half of the proof of the theorem and it is also interesting in its own right.

LEMMA. *Every (norm)-convergent sequence in E contains a subsequence which converges pointwise on K except for an E-null set.*

Proof.[4] It is sufficient to suppose that (x_n) is a positive sequence in E converging to zero. Choose a subsequence (x_{n_k}) of (x_n) for which $\|x_{n_k}\| \leqslant 2^{-k}$, $k \in \mathbf{N}$. Set now $y := \sum_{k\in\mathbf{N}} x_{n_k}$ and $M := \{\varphi \in K: y(\varphi) = \infty\}$. By condition (c) in Section 1, M is an E-null set. For $\varphi \notin M$ we have $\sum_{k\in\mathbf{N}} x_{n_k}(\varphi) \leqslant y(\varphi) < \infty$. This proves that $x_{n_k}(\varphi)$ converges to zero for $\varphi \notin M$.

Proof of the theorem: "$\Rightarrow$". If M_1, M_2 are disjoint closed subsets of K, then exists a continuous function $0 \leqslant x \leqslant u$, $x \in E$, such that $x(M_1) = \{0\}$, $x(M_2) = \{1\}$. Assume that H is dense in E. By the above lemma we can find a sequence in H which converges pointwise to x except on an E-null set N. This already shows that H separates $M_1 \setminus N$ and $M_2 \setminus N$.

"$\Leftarrow$": Let $\varepsilon > 0$ and $x \in E$, $0 \leqslant x \leqslant u$ be fixed. Choose $\delta_i \in \mathbf{R}$, $0 \leqslant \delta_i \leqslant 1$, for $i = 1, \ldots, n$ such that $K = \bigcup_{i=1}^{n} V_i$ for $V_i = x^{-1}([\delta - \varepsilon, \delta_i + \varepsilon])$. Take now a fixed index $i \in \{1, \ldots, n\}$ and set $V := V_i$. The complement $\complement V$ of V in K is a countable union of closed sets. Hence it follows from Corollary 1 in Section 1 that H separates V and $\complement V$ except for an E-null set N. For N we can find an open neighborhood U satisfying the condition (b)

[4] This simple proof is due to M. Wolff.

of Proposition 1 for ε/n. $\complement U$ is a compact space and the elements of H restricted to $\complement U$ separate $V \cap \complement U$ and $\complement V \cap \complement U$.

We proceed now by using the method known from the proof of the lattice theoretical Stone–Weierstrass theorem (see [8], V. 8.1): For every $\varphi \in V \cap \complement U$ and $\psi \in \complement V \cap \complement U$ there exists a $z_{\varphi,\psi} \in H$ such that $z_{\varphi,\psi}(\varphi) = x(\varphi)$ and $z_{\varphi,\psi}(\psi) = x(\psi)$. Using the compactness of $\complement U$ we can find $\psi_1, \ldots, \psi_k \in \complement V \cap \complement U$ such that for $z_\varphi := \sup(z_{\varphi,\psi_1}, \ldots, z_{\varphi,\psi_k}, x(\varphi)u)$ we have $z_\varphi(\varphi) = x(\varphi)$ and $z_\varphi(\psi) > x(\psi) - 2\varepsilon$ for all $\psi \in \complement U$. Repeat this construction for all $\varphi \in V \cap \complement U$. Again, $V \cap \complement U$ is compact and we can take the infimum over $z_{\varphi_1}, \ldots, z_{\varphi_m} \in H$ such that $z := \inf(z_{\varphi_1}, \ldots, z_{\varphi_m}) \in H$ and $z(\psi) > x(\psi) - 2\varepsilon$ for all $\psi \in \complement U$ and $z(\psi) < x(\psi) + 2\varepsilon$ for all $\psi \in V \cap \complement U$.

Construct such functions z_i for each $V := V_i$, $i = 1, \ldots, n$ and define $y := \inf(z_1, \ldots, z_n) \in H$. Set now $U_0 := \bigcup_{j=1}^{n} U_j$. We can conclude that $|x(\varphi) - y(\varphi)| < 4\varepsilon$ for all $\varphi \notin U_0$. But U_0 satisfies the condition (b) for $n\frac{\varepsilon}{n} = \varepsilon$. Since we can assume that $|y| < u$, it follows that $\|x - y\| < 4\varepsilon + \varepsilon = 5\varepsilon$. This finishes the proof.

COROLLARY. *Let E be a Banach lattice of continuous numerical functions on a compact space K such that $C(K)$ is dense in E and let H be a subalgebra of $C(K)$ containing the constants. Then H is dense in E iff H separates K E-essentially.*

Proof. Let H_0 be the closure of H with respect to the sup-norm. Evidently, H_0 separates the same sets as H does. But H_0 is a sublattice of E (see [1], § 4, no. 2). Since the norm of E is coarser than the sup-norm, the closure of H in E contains H_0.

Remark. Since there are no non-trivial $C(K)$-null sets in K, the classical Stone–Weierstrass theorem is retrieved.

3. Applications. a) Let X be a completely regular space and let $v \in C_b(X)$ be a strictly positive function on X. We consider the Banach lattice $E := \overline{\bigcup_{n \in N} n[-v, v]} \subset C_b(X)$ as in Example 2 of Section 1. Let H be a sublattice (or subalgebra) of E containing v. The structure space of E is (homeomorphic to) βX and the E-null sets are all contained in the greatest E-null set N_0 (consisting of all $\varphi \in \beta X$ on which the continuous extension of v vanishes). Hence we can apply our theorem. But it would be interesting to conclude denseness of H in E from the behaviour of the elements of H on X alone (and not on the whole of βX). For this purpose we adopt the following diction: two subsets X_1, X_2 of X are *completely separated* by F, $F \subset E$, if there is a $z \in F$ such that $z(\varphi) \leqslant -1$ on X_1, $z(\varphi) \geqslant 1$ on X_2.

PROPOSITION. *H is dense in E iff every pair $X_1, X_2 \subset X$, completely separated by E, is already completely separated by H.*

Proof. Every function $x \in E$ can be extended continuously to βX. We denote this extension also by x. On the other hand, let $\hat{E}$ be the Banach lattice of continuous numerical functions on βX which is isomorphic to E. If we choose v to become the unit function $\hat{v}$ on βX, the isomorphism from E onto $\hat{E}$ is the multiplication of the functions in E by the numerical function v^{-1}.

From our main theorem and from the characterization of the $\hat{E}$-null sets in βX it follows that H is dense in E iff $\hat{H}$ separates the points in $\beta X \setminus N_0$. The above consideration implies that we can take H instead of $\hat{H}$. Consequently, we have to prove that H separates the points of $\beta X \setminus N_0$ iff the condition of the proposition is satisfied.

Let $X_1, X_2 \subset X$ be completely separated by E. Then $\bar{X}_1, \bar{X}_2 \subset \beta X$ are disjoint, compact and contained in $\beta X \setminus N_0$. Hence H separates $\bar{X}_1$ and $\bar{X}_2$. By the standard argument, we can construct a $z \in H$ separating X_1 and X_2 completely. Conversely, let φ_1, φ_2 be two points in $\beta X \setminus N_0$. One can find neighborhoods of φ_1, φ_2 which are completely separated by E. The function $y \in H$, which separates these sets completely, separates also φ_1 and φ_2.

Remark. For v the unit function on X and $E = C_b(X)$ the above proposition contains the result of E. Hewitt [4] as a special case. If X is locally compact, v vanishes at infinity and $E = C_0(X)$, the subsets of X completely separated by E are exactly the relatively compact sets in X. Hence, it follows from the proposition that H is dense in $C_0(X)$ iff H separates X.

b) Let (X, Γ, μ) be a finite measure space and set $E = L^p(X, \Gamma, \mu)$ for $1 \leqslant p < \infty$. We assume that H is a sublattice of L^p containing the constants. In analogy to the previous example, we say that $f \in L^p$ *separates completely* the measurable sets $X_1, X_2 \subset X$ if $f(t) \leqslant -1$ a.e. on X_1, $f(t) \geqslant 1$ a.e. on X_2.([5])

PROPOSITION. *H is dense in L^p iff the following condition is satisfied: For every pair of disjoint measurable subsets X_1, X_2 of X there exist measurable sets X_{1n}, X_{2n} which are completely separated by H for all $n \in N$ and for which X_i is equal to $\bigcup_{n \in N} X_{in}$ up to a μ-null set.*

Proof. $E = L^p$ can be represented as a Banach lattice $\hat{E}$ according to the representation theorem and in such a way that L^∞ is isomorphic to $C(K)$, K the structure space of E. Denote by $\hat{H}$ the isomorphic image of H in $\hat{E}$. It turns out that H satisfies the above condition iff for every pair of disjoint open-and-closed subsets M_1, M_2 of K there exist open-and-closed sets M_{1n}, M_{2n} which are completely separated by $\hat{H}$ and for which M_i is up to an $\hat{E}$-null set equal to $\bigcup_{n \in N} M_{in}$, $i = 1, 2$.

([5]) Here, f denotes simultaneously a function and its equivalence class in L^p.

Let H be dense in L^p. By the theorem, $\hat{H}$ separates K $\hat{E}$-essentially. Hence there exists an $\hat{E}$-null set N in K such that $\hat{H}$ separates $M_1 \setminus N$ and $M_2 \setminus N$. Using the condition (b) of Proposition 1 in Section 1 for $\varepsilon = 1/n$ and applying the usual compactness argument, we can construct the sequences of sets which are completely separated by $\hat{H}$ and whose union is equal to $M_1 \setminus N$ resp. $M_2 \setminus N$. Conversely, let M_1, M_2 be two disjoint closed subsets of K. Since the L^p-norm is order continuous, K is extremely disconnected and every rare subset is an $\hat{E}$-null set in K. Hence we can assume that M_1 and M_2 are open-and-closed. It is now clear that the above condition implies that $\hat{H}$ separates M_1 and M_2 except for an $\hat{E}$-null set.

Remark. First, it is clear that the proposition holds also if H is an algebra contained in L^∞. Moreover, the case of a σ-finite measure space can be treated similarly. This shows that the result of R. H. Farrell [8] is included. If E is a Banach function space on (X, Γ, μ) with absolutely continuous norm (see [7]), H. Nakano [6] has shown that the norm of E is even order continuous. Hence nothing essential has to be changed and Theorem 2.2 of M. M. Rao [7] follows also.

COROLLARY. *Let X be a compact space, μ a finite Borel measure on X such that every non-empty open set in X has positive measure. $C(X)$ can be (canonically) identified with a dense sublattice of $L^p(X, \mu)$, $1 \leqslant p < \infty$, if and only if μ is regular.*

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Existence of some special bases in Banach spaces

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Abstract. The main result of the paper is that if X is a Banach space with a basis and Y has a normalized basis which is weakly convergent to zero and satisfies a certain condition, then $X+Y$ has a normalized basis which is weakly convergent to zero. A few similar results for other classes of bases are stated. New bases in $C[0,1]$ and $L_1[0,1]$ are constructed. A few results about universal bases are stated.

0. Introduction. In this paper we consider the following problem: Suppose we have a Banach space X with a basis and a Banach space Y with a basis possessing some additional properties. Can we construct a basis possessing some additional properties in the space $X+Y$? We solve this problem for wc_0-bases and for p-Hilbertian and p-Besselian bases (for the definitions see below).

Section 1 contains the definitions, notations and some known facts which are used later.

The central section of the present paper is Section 2. In this section we prove one fact on bases in the finite-dimensional Banach space (Proposition 2.1). This proposition is our main tool in Sections 3 and 4.

In Section 3, Proposition 3.1, we prove that if X has a basis and Y has a wc_0-basis satisfying some technical conditions, then $X+Y$ has a wc_0-basis. In particular, from our results it follows that if Y has a shrinking basis, then $X+Y$ has a wc_0-basis.

In Section 4 we prove some analogous theorems for Besselian and Hilbertian bases. As an application we obtain the existence of some interesting bases in $C[0,1]$ and $L_1[0,1]$. Those examples answer certain questions of A. Pełczyński [8] (cf. also [10] Problem 11.1).

Section 5 is devoted to universal bases. We prove the non-existence of wc_0-basis universal for all wc_0-bases. We obtain some information about bases universal for all shrinking bases. Since the proof of this result is a simple modification of the proof of Szlenk [12], we only point out the necessary changes in his proof.

The author is greatly obliged to prof. A. Pełczyński for suggesting the problem and many useful comments during the preparation of the present paper. In particular, the possibility of applying Proposition 2.1