THEOREM 4.3. *Let f and φ be essentially countable to one bounded complex valued Borel function on X and Z respectively. Then T_f and T_φ are unitarily equivalent if and only if the corresponding first kind Hellinger–Hahn decompositions of X and Z are equivalent, i.e., if and only if f and φ are equivalent.*

Remark. It can happen that f is essentially uncountable to one, φ is countable to one and T_f and T_φ are unitarily equivalent. Indeed any bounded normal operator on a separable Hilbert space is unitarily equivalent to $T_{(\psi)}$ where (ψ) is the function on $I \times C$ (I = Set of positive integer) given by $(\psi)(n, x) = x$, and where a measure on $I \times C$ is determined by the operator in question. Note that (ψ) is always countable to one.

Acknowledgement (September 9, 1972). I would like to acknowledge here that Proposition 1.1 together with the first part of Theorem 2.2 are contained in Rohlin's beautiful study of Lebesgue Spaces in his paper "*On fundamental ideas of measure theory*" (Amer. Math. Soc. Trans. Series 1, 10, page 45). Theorem 3.1 also follows from his result on the "existence of independent complement for measurable decompositions which are not one sheeted on any set of positive measure". I am grateful to D. Ramachandran for pointing this out to me and for acquainting me with the contents of Rohlin's paper. Rohlin's proofs of the results mentioned rely on the existence of canonical system of measures and they are obtained in the process of giving a complete classifications of measurable decompositions of a Lebesgue Space. Our proof of Theorem 2.2 is directly in the spirit of classical Hellinger–Hahn theorem for spectral measures. Theorem 3.1 also does not depend in anyway on canonical system of measures.

References

[1] P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York 1951.

[2] F. Hausdorff, *Set theory*, Chelsea New York, 1957.

[3] A. I. Plessner and V. A. Rohlin, *Spectral theory of linear operators*, Uspehi Mat Nauk (N. S) 1 (1946), pp. 71–191 (in Russian).

[4] M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloquium Pub., Vol. 15, New York 1932.

Received February 15. 1972 (473)

STUDIA MATHEMATICA, T. XLVII. (1973)

Linear topologies which are suprema of dual-less topologies*

by

N. T. PECK and HORACIO PORTA (Urbana Ill.)

Abstract. The first result of this paper is that every topological linear space of algebraic dimension at least the continuum is linearly homeomorphic to a subspace of a dual-less space (i.e., a topological linear space with zero dual) in such a way that the dimension and codimension of the image are equal. Using this result, it is then proved that the norm topology of many of the classical separable Banach spaces can be written as the supremum of a finite number of dual-less topologies. Some extensions of this are given for the non-separable case and for other topological linear spaces.

0. INTRODUCTION

It is well known that the topology of convergence in measure is one of the weakest topologies on a function space; for example, on the space of all Lebesgue measurable functions on $[0, 1]$ the only linear functional which is continuous for convergence in measure is the zero functional. In view of this it may be somewhat surprising that the norm topology on the classical Banach spaces can be expressed as simultaneous convergence in three topologies, each of which is an inverse image of a topology of convergence in measure. This is proved below as a consequence of more general results concerning the following problems:

a) which linear topologies on a vector space are restrictions of "very weak" topologies on a larger space?

b) which linear topologies on a vector space can be expressed as suprema of families of "very weak" topologies on it?

By a "very weak" topology we mean a linear topology that is at least dual-less in the sense that it does not have any non-trivial continuous linear functional. Theorems A, B, C below provide some answers to these problems.

Questions of this sort were investigated by Klee in [5], to which we refer the reader for background. In this paper, Klee proved that the supre-

* Research supported by NSF Grant GP 28577

mum of dual-less topologies need not be dual-less and asked whether every linear topology is the restriction of a type *se* dual-less topology (to be defined below). Theorem A below answers this question.

We shall use $E, F, X, \ldots$ to denote real vector spaces, $\sigma, \tau, \mu, \ldots$ for linear topologies on them and shall talk about the "linear topological space (E, σ)", etc. As we said above, a topology σ on E is *dual-less* if (E, σ) has no non-trivial continuous linear functionals. This will surely be the case if all absorbing convex (resp. semi-convex) sets are everywhere dense. Such a topology will be called dual-less of type *e* (resp., type *se*) (1).

In the first section we prove

THEOREM A. *Every topological linear space (E, σ) with* $\dim E \geqslant 2^{\aleph_0}$ *is linearly homeomorphic under a linear map η_E to a subspace of a dual-less space $(S(E), \mu(\sigma))$ of type se such that* $\dim E = \operatorname{codim} \eta_E E$ *in $S(E)$ and such that the density characters of E and $S(E)$ are the same.*

If E is a linear space, let $\sigma_a E$ (respectively, $\sigma_c E$) denote the finest linear topology (respectively, the finest locally convex linear topology) on E. Using Theorem A we prove

THEOREM B. 1) *For an infinite-dimensional linear space E, $\sigma_a E$ and $\sigma_c E$ can each be expressed as the supremum of three linearly homeomorphic dual-less Hausdorff topologies on E;*

2) *If* $\dim E \geqslant 2^{\aleph_0}$, *then $\sigma_a E$ is the supremum of three linearly homeomorphic type se dual-less Hausdorff topologies on E.*

The main result of the second section is

THEOREM C. *Let X be a separable infinite-dimensional normed space. If $n \geqslant 2$, the norm topology on $X \times \ldots \times X$ (n times) is the supremum of $n+1$ linearly homeomorphic dual-less topologies.*

In particular, if X is isomorphic to $X \times X$, the norm topology on X is the supremum of three dual-less topologies. This is the case of the classical Banach spaces L_p, c_0, C, etc.

In the third section, we consider problem b) for locally convex topologies on infinite-dimensional vector spaces. We provide some answers in a few cases and consider some related problems.

A corollary of Theorem B (Corollary 1.3) answers some other questions of Klee [5]; Theorem A generalizes Theorem 1.1 in [6].

Throughout, we use the following notation: if E is a linear space, $\dim E$ is the cardinality of a Hamel basis for E; if E, F are linear spaces with $E \subset F$, then "$\operatorname{codim} E$ (in F)" is the codimension of E in F. For linear spaces E and F, $E \times F$ is the algebraic direct product of E and F; if, in addition, E and F are topological linear spaces, $E \oplus F$ is the direct

(1) These topologies were called "nearly exotic", "exotic" and "strongly exotic" by Klee in [5], which explains the choice of the symbols *e* and *se* used here.

product $E \times F$ with the product topology. If A is a set, χ_A will denote the characteristic function of A. A *pseudo-norm* on a vector space E is a non-negative, positive-homogeneous function p on E such that $p(x+y) \leqslant c(p(x)+p(y))$ for all x, y in E and a fixed constant c. Finally, a set C is *semi-convex* if $C+C \subset kC$ for some $k > 0$.

1. EMBEDDINGS IN DUAL-LESS SPACES OF TYPE *se*

Let E be a vector space. We will say that a function $f\colon [0,1] \to E$ is *measurable* if the values of f all lie in a finite dimensional subspace $F \subset E$ (which depends on f), and f is Lebesgue measurable as a function from $[0,1]$ into F, where F is provided with its unique Hausdorff linear topology. As usual, operations are defined pointwise and functions agreeing almost everywhere are identified. The resulting set of classes of equivalent measurable functions is a vector space $S(E)$. Observe that two constant functions agree almost everywhere if and only if they are identical. This shows that there is a one-to-one linear map $\eta_E\colon E \to S(E)$ that assigns to each $x \in E$ the (class of the) constant function $f(t) = x$, $0 \leqslant t \leqslant 1$.

Assume that E is expressed as an algebraic direct sum $E = \sum_{i \in I} E_i$, where $\dim E_i = 1$ for all i. Then it is easy to see that $S(E)$ can be identified with the algebraic direct sum $\sum_{i \in I} S(E_i)$, that is,

$$S\Big(\sum_{i \in I} E_i\Big) = \sum_{i \in I} S(E_i). \tag{1.1}$$

LEMMA 1.1. *The dimension of $S(E)$ and the codimension of $\eta_E E$ in $S(E)$ coincide, and they are equal to* $2^{\aleph_0} \dim E$.

Proof. Let us represent E as $E = \sum_{i \in I} E_i$ with $\dim E_i = 1$ for each i. Then $\dim E = \operatorname{Card} I$. Now using (1.1) we have $\dim S(E) = \dim \sum_{i \in I} S(E_i) = \dim S(\boldsymbol{R}) \cdot \operatorname{Card} I = 2^{\aleph_0} \cdot \dim E$, since $\dim S(\boldsymbol{R}) = 2^{\aleph_0}$. For $f \in S(E)$ let now $f^* \in S(E)$ be the function $f^*(t) = f(2t)$, $0 \leqslant t \leqslant 1/2$ and $f^*(t) = 0$ for $1/2 < t \leqslant 1$. Clearly $f \to f^*$ is linear and one-to-one. It is also clear that the quotient map $S(E) \to S(E)/\eta_E E$ is one-to-one on the subspace $\{f^*\colon f \in S(E)\}$. Thus $\dim\big(S(E)/\eta_E E\big) \geqslant \dim S(E)$, and the opposite inequality being obvious, the proof is complete.

Assume now that σ is a linear topology on E. We recall that the topology on $S(E)$ of σ-convergence in measure, which, we denote here by $\mu(\sigma)$, is defined as the linear topology on $S(E)$ for which a neighborhood base at 0 is given by the family of sets $\{f \in S(E)\colon m\{t;\, f(t) \notin V\} \leqslant \varepsilon\}$ where m denotes Lebesgue measure, V runs over all σ-neighborhoods of 0 and $\varepsilon > 0$. In Theorem 1.1 of [6] it is proved that $\mu(\sigma)$ is always dual-less. Actually the following stronger result holds:

LEMMA 1.2. *The topology $\mu(\sigma)$ is always dual-less of type se. Moreover $\mu(\sigma)$ is Hausdorff if and only if σ is Hausdorff.*

Proof. It is known (see [5]) that $S(\boldsymbol{R})$ is dual-less of type *se*; the same proof gives that $(S(F), \mu(\sigma))$ is dual-less of type *se* for all finite dimensional F. Consider now the general case. Let (E, σ) be given, and let $C \subset S(E)$ be an absorbing semi-convex set. Let $f_0 \in S(E)$ and let V be a $\mu(\sigma)$ neighborhood of f_0. Consider a finite dimensional subspace $F \subset E$ containing the range of f_0. We can identify $S(F)$ with the subspace of $S(E)$ of all functions with range in F, and it is easy to see that the relative topology of $\mu(\sigma)$ on $S(F)$ coincides with $\mu(\tau)$, where τ is the relative topology of σ on F. Thus, by the preceding remark, $S(F)$ is of type *se* and therefore, by definition, $C \cap S(F)$ is dense in $S(F)$. In particular $C \cap S(F)$ intersects $V \cap S(F)$ (which is not empty since $f_0 \in V \cap S(F)$) and a fortiori C intersects V. This shows that C is dense, and the proof is complete.

Proofs of Theorems A and B. For the proof of Theorem A, it is easy to see that $\eta_E\colon (E, \sigma) \to (S(E), \mu(\sigma))$ is a linear homeomorphism, and Lemma 1.1 and Lemma 1.2 complete the proof.

We now prove Theorem B in the order 2), 1).

Proof of B.2). Let $\dim E$ be at least $2^{\aleph_0}$ and let $E' \subset S(E)$ be the range of η_E. According to Theorem A, we can decompose $S(E)$ into an algebraic direct sum $S(E) = E' + E''$ with $\dim E' = \dim E'' = \dim S(E)$. Consider now the linear maps T_1, T_2 from $S(E)$ into $S(E)$ defined by $T_1(e'+e'') = e'-e''$ $(e' \in E', e'' \in E'')$ and T_2 is any linear map satisfying $TE' = E''$ and $T_2^2 = \mathrm{Id}$. Clearly T_1, T_2 are one-to-one and onto. Denote by τ_1, τ_2 and τ_3 the topologies on $S(E)$ defined as: τ_1 and τ_2 are the images of $\mu(\sigma_a E)$ under T_1 and T_2, respectively and $\tau_3 = \mu(\sigma_a E)$. These topologies are linearly homeomorphic and from Theorem A it follows that they are dual-less of type *se*. Let $\tau = \operatorname*{Sup}_{1\leqslant j\leqslant 3} \tau_j$ and assume that the net $\{e'_a + e''_a\}$ converges to $0 \in S(E)$ for τ. Then also $e'_a - e''_a = T_1(e'_a + e''_a)$ converges to 0 for τ and therefore $e'_a \xrightarrow{\tau} 0$. Since η_E is a homeomorphism, τ_3 induces the finest linear topology on E', and τ being finer than τ_3 we have $\tau_3|_{E'} = \tau|_{E'} = \sigma_a E'$, whence $e'_a \to 0$ for $\sigma_a E'$. If T_2 is applied to $e'_a + e''_a$, the above procedure leads to $e''_a \to 0$ for $\sigma_a E''$. Thus $e'_a + e''_a \to 0$ for $(\sigma_a E') \times (\sigma_a E'')$. However, it is easy to see that $\sigma_a(E' \times E'') = (\sigma_a E') \times (\sigma_a E'')$, which implies that τ (is finer than, and therefore) coincides with $\sigma_a(E'+E'') = \sigma_a S(E)$. This shows that B.2) holds for $S(E)$ and since $\dim S(E) = \dim E$, it also holds for E. This completes the proof of B.2).

Proof of B.1). Let E be infinite dimensional and let σ be either of $\sigma_a E$ or $\sigma_c E$. Let $S_s(E)$ be the linear span in $S(E)$ of the functions $t \to \chi_{(r,s)}(t)x$ $(0 \leqslant t \leqslant 1)$, where r and s are rational, $r < s$, and $x \in E$; it is not hard to see that $\dim E = \dim S_s(E)$ and that $S_s(E)$ is $\mu(\sigma)$ – dense in $S(E)$, hence dual-less. The argument in the proof of Lemma 1.1 shows that the codimension of $\eta_E E$ in $S_s(E)$ equals $\dim E$.

For $\sigma = \sigma_a E$, the proof is completed exactly as in the proof of B.2). In order to apply these arguments when $\sigma = \sigma_c E$, it is necessary to know that the topologies τ_1, τ_2 and τ_3 are weaker than $\sigma_c S_s(E)$. This is clear since $\mu(\sigma_c E)$ on $S_s(E)$ is weaker than the locally convex topology of uniform convergence (functions in $S_s(E)$ are bounded). It is also necessary to know that $\sigma_c(E' \times E'') = (\sigma_c E') \times (\sigma_c E'')$. This is easily verified, and the proof of Theorem B is complete.

If $\dim E = \aleph_0$, a category argument shows that there are no non-trivial type *se* dual-less topologies on E, and therefore the conclusion in B.2) does not hold in this case. Without using the continuum hypothesis, we do not have a version of B.2 for $\aleph_0 < \dim E < 2^{\aleph_0}$. Since no non-trivial type *se* dual-less topology can be weaker than a locally convex topology, there is no form of B.2) for $\sigma_c E$.

Given a vector space E, we denote by $\sigma_{ne} E$ (resp., $\sigma_e E$, $\sigma_{se} E$) the supremum of all dual-less (resp., of type *e*, of type *se*) linear topologies on E (this agrees with the notation in [5]). Theorem B has the following immediate

COROLLARY 1.3. *For all infinite dimensional vector spaces E, we have $\sigma_{ne} E = \sigma_a E$; if $\dim E \geqslant 2^{\aleph_0}$, then also $\sigma_{se} E = \sigma_e E = \sigma_{ne} E = \sigma_a E$.*

Since $\sigma_a E$ is complete [3], Cor. 1.3 trivially implies that $\sigma_{ne} E$ is always complete and that $\sigma_e E$ and $\sigma_{se} E$ are complete for $\dim E \geqslant 2^{\aleph_0}$. This settles some conjectures in [5] (paragraphs preceding 2.3 and 3.7 in [5]), but in the absence of the continuum hypothesis, it remains unknown whether $\sigma_e E$ and $\sigma_{se} E$ are complete when $\aleph_0 < \dim E < 2^{\aleph_0}$.

2. THE NORMED SPACE CASE

If $U\colon Y \to (Z, \sigma)$ is a linear map, the *inverse topology* $U^{-1}(\sigma)$ is defined to be the linear topology on Y having $\{U^{-1}(V)\}$ as a neighborhood base at 0, where V runs through a σ-neighborhood base at 0 in Z. It is clear that a net (y_a) converges to y in $U^{-1}(\sigma)$ if and only if $U(y_a)$ converges to $U(y)$ in σ. The proof of the following statement is routine:

LEMMA 2.1. *Let $U\colon Y \to (Z, \sigma)$ be a linear map.*

(i) *If σ is a pseudo-norm topology, then $U^{-1}(\sigma)$ is also a pseudo-norm topology;*

(ii) *if σ is dual-less and the range of U is dense in Z, then $U^{-1}(\sigma)$ is also dual-less.*

Now we prove:

LEMMA 2.2. *Let (X, τ) and (Z, σ) be linear topological spaces and let S and T be continuous linear maps from X to Z. Assume that*

(i) *T is a homeomorphism of X with $T(X)$;*

(ii) *The range of S is dense in Z.*

Assume further that σ is the supremum of n dual-less (resp., pseudo-norm) topologies on Z. Then the product topology on $X \oplus \ldots \oplus X$ (k times, $k \geqslant 2$) is the supremum of $n(k+1)$ dual-less (resp., pseudo-norm) topologies.

Proof. Assume that σ is the supremum of the dual-less topologies $\sigma_1, \sigma_2, \ldots \sigma_n$ on Z. Define $U\colon X \oplus \ldots \oplus X \to Z$ by $U(x_1, \ldots, x_k) = T(x_1) + S(x_2)$, and let $\tau_j = U^{-1}(\sigma_j)$. Since the range of U is σ-dense (because it contains the range of S), it is also σ_j-dense, and by Lemma 2.1, (ii) applied to $Y = X \oplus \ldots \oplus X$, τ_j is dual-less for each j. Clearly τ_j is weaker than the product topology $\tau \times \ldots \times \tau$. Now let $L_1\colon Y \to Y$ be defined by $L_1(x_1, x_2, \ldots, x_k) = (x_1, -x_2, \ldots, -x_k)$, and for $j \geqslant 2$ let $L_j\colon Y \to Y$ be defined by $L_j(x_1, x_2, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k) = (x_j, x_2, \ldots, x_{j-1}, x_1, x_{j+1}, \ldots, x_k)$. The topologies $\tau_{jn+l} = L_j^{-1}(\tau_l)$, $1 \leqslant j \leqslant k$, $1 \leqslant l \leqslant n$ are all dual-less and weaker than $\tau \times \ldots \times \tau$. Thus, the supremum of all τ_h, $1 \leqslant h \leqslant n(k+1)$ is also weaker than $\tau \times \ldots \times \tau$.

We shall prove that the reverse is true as well. Assume a net $x^\alpha = (x_1^\alpha, \ldots, x_k^\alpha)$, $\alpha \in A$, converges to 0 for each τ_h, $1 \leqslant h \leqslant n(k+1)$. Then also $L_1(x^\alpha) \to 0$ for each τ_h, and therefore $(x_1^\alpha, 0, \ldots, 0) = \frac{1}{2}\left(x^\alpha + L_1(x^\alpha)\right) \to 0$. This means that $U(x_1^\alpha, 0, \ldots, 0) = T(x_1^\alpha) \to 0$ for σ_j, $1 \leqslant j \leqslant n$, and hypothesis (i) implies that $x_1^\alpha \to 0$ for τ. Similarly, since $L_j(x^\alpha) \to 0$ for $2 \leqslant j \leqslant k$, we also get $x_j^\alpha \to 0$ for τ, $(2 \leqslant j \leqslant k)$, and therefore $x^\alpha \to 0$ for $\tau \times \ldots \times \tau$, as claimed. It follows that $\tau \times \ldots \times \tau$ coincides with the supremum of $\{\tau_h\}$, $1 \leqslant h \leqslant n(k+1)$. It is clear that if all σ_j are pseudo-norm topologies, then so are the topologies τ_h, and this completes the proof of the lemma.

Proof of Theorem C. Let $\hat{X}$ be the completion of the normed space X, and let $L^{\frac12}(\hat{X})$ be the completion of $C([0,1], \hat{X})$ under the pseudo-norm $\|f\|_{\frac12} = \left(\int_0^1 \|f(t)\|^{\frac12} dt\right)^2$. It is not hard to see that $L^{\frac12}(\hat{X})$ is dual-less. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of elements in $L^{\frac12}(\hat{X})$ satisfying $\|\varphi_n(t)\| \leqslant 1$, $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$, and such that the linear span of the φ_n is dense in $L^{\frac12}(\hat{X})$; this is possible, since $\hat{X}$ is separable. Let $\{x_n, f_n\}$ be a biorthogonal family for X (i.e., $f_n(x_m) = 0$ if $n \neq m$, $f_n(x_n) \neq 0$ — see [4]) such that $\|f_n\| \leqslant 1$ for all n. Now for $x \in X$ and $t \in [0,1]$, we have

$$\Big\| \sum_{n=p}^{p+s} 2^{-n} f_n(x)\varphi_n(t) \Big\| \leqslant \|x\| \sum_{n=p}^{p+s} 2^{-n};$$

thus

$$\Big\| \sum_{n=p}^{p+s} 2^{-n} f_n(x)\varphi_n \Big\|_{\frac12} \leqslant \|x\| \sum_{n=p}^{p+s} 2^{-n}.$$

This shows that the series $S(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)\varphi_n$ is convergent in $L^{\frac12}(\hat{X})$, and moreover $\|Sx\|_{\frac12} \leqslant \|x\|$; therefore $S\colon X \to L^{\frac12}(\hat{X})$ is continuous. Since the range of S contains all linear combinations of the φ_n (by the biorthogonality), it is clearly dense. Finally, for $x \in X$, define $f_x \in L^{\frac12}(\hat{X})$ by $f_x(t) = x$, $0 \leqslant t \leqslant 1$; if T is the map $x \to f_x$, T is clearly an isometry. Lemma 2.2 applies to complete the proof.

COROLLARY 2.3. *The norm topology on each of the following Banach spaces: $L^p[0,1]$, $l^p (1 \leqslant p < \infty)$, $C[0,1]$, and c_0 is the supremum of three dual-less topologies.*

Proof. Just observe that for each of the separable Banach spaces X listed above, X is isomorphic to $X \oplus X$.

Since any locally convex topology is a supremum of semi-norm topologies and the proof of Theorem C carries over to semi-normed spaces, we have

COROLLARY 2.4. *If X is a separable locally convex space, the product topology on $X \oplus \ldots \oplus X$ (k times, $k \geqslant 2$) is the supremum of a family of dual-less topologies.*

For an infinite-dimensional normed space which is not isomorphic to the square of a Banach space, we have a slightly weaker version of Corollary 2.3:

PROPOSITION 2.5. *Let X be an infinite-dimensional separable normed space which has a decomposition $X = X_1 \oplus X_2$ with $\dim X_1 = \dim X_2 = \infty$. Then the norm topology on X is the supremum of four dual-less pseudo-norm topologies.*

Proof. We proceed as in the proof of Theorem C. Let $\hat{X}$ be the completion of X. For $i = 1, 2$ let $T_i\colon X_i \to L^{\frac12}(\hat{X})$ be defined by $T_i(z)(t) = z$, $0 \leqslant t \leqslant 1$, and let S_i be a continuous linear map of X_i onto a dense subspace of $L^{\frac12}(\hat{X})$ (constructed as in the proof of Theorem C). Let $V_1(x, y) = T_1(x) + S_2(y)$, $V_2(x, y) = T_1(x) - S_2(y)$, $V_3(x, y) = S_1(x) + T_2(y)$, $V_4(x, y) = S_1(x) - T_2(y)$. Let ν be the pseudo-norm topology of $L^{\frac12}(\hat{X})$ and let $\tau_i = V_i^{-1}(\nu)$. Each τ_i is a pseudo-norm dual-less topology on X. If $\tau = \sup_{1 \leqslant i \leqslant 4} \tau_i$, τ is weaker than the norm topology on X; we show that it coincides with the norm topology.

If $(x^\alpha, y^\alpha) \to 0$ for τ, then $T_1(x^\alpha) + S_2(y^\alpha) \to 0$ and $T_1(x^\alpha) - S_2(y^\alpha) \to 0$ in $L^{\frac12}(\hat{X})$. Hence $T_1(x^\alpha) \to 0$, so since T_1 is a homeomorphism, $x^\alpha \to 0$ in the norm topology of X_1. Similarly $y^\alpha \to 0$, and the proof is complete.

Remark. Observe that the conclusion of Theorem C is true, in particular, for $L\oplus L$, $L\oplus L\oplus L$, etc., where L is the space considered by James in [2]. We do not know whether the norm topology on L itself is a supremum of dual-less topologies. Also, observe that if B is any separable infinite-dimensional reflexive Banach space, and $Y = L\oplus B$, Theorem C does not apply to Y (Y is not isomorphic to $X\oplus \ldots \oplus X$ (n times, $n \geq 2$) for any Banach space X) but Proposition 2.5 does apply.

THEOREM 2.6. *For every infinite set Γ, the norm topology on $c_0(\Gamma)$ and $l^p(\Gamma)$, $1 \leq p \leq \infty$, is the supremum of three dual-less topologies.*

Proof. For $l^p(\Gamma)$, $1 \leq p \leq \infty$, let Z be the space $l^p(\Gamma, L^{\frac{1}{2}})$ consisting of all families $f = (f_\gamma)_{\gamma\in\Gamma}$ with $f_\gamma \in L^{\frac{1}{2}}[0,1]$ for each γ and $\|f\|_Z = \big(\sum_{\gamma\in\Gamma} \|f_\gamma\|_{\frac{1}{2}}^p\big)^{1/p}$ finite (if $p = \infty$ we require $\|f\|_Z = \sup_{\gamma\in\Gamma} \|f_\gamma\|_{\frac{1}{2}} < \infty$). It is easy to see that $\| \ \|_Z$ is a pseudo-norm on Z, and when $p < \infty$, the elements f with only finitely many non-zero coordinates are dense in Z, so Z is dual-less. When $p = \infty$, Z is dual-less by Proposition 2.1 of [6].

Choose a sequence $\{\varphi_n\}_{n=1}^\infty$ in $L^{\frac{1}{2}}$ satisfying $|\varphi_n(t)| \leq 1$ for all t and whose linear span is dense in $L^{\frac{1}{2}}$, and define $Q\colon l^p \to L^{\frac{1}{2}}$ by $Q(x) = \sum_{n=1}^\infty 2^{-2n} x_n \varphi_n$. The map Q has dense range, and we easily obtain $\|Q(x)\|_{\frac{1}{2}} \leq \|x\|_p$; Q then induces a map $S\colon l^p(\Gamma, l^p) \to Z$, again continuous and with dense range. But since $l^p(\Gamma, l^p)$ is isometric to $l^p(\Gamma)$, we have a map satisfying (ii) of Lemma 2.2. If $\varphi(t) = 1$ for all $t\in[0,1]$, the map $T\colon l^p(\Gamma) \to Z$ defined by $T(x) = (x_\gamma \varphi)_{\gamma\in\Gamma}$ is an isometry of $l^p(\Gamma)$ into Z. Thus Lemma 2.2 applies and the norm topology on $l^p(\Gamma)$ ($\approx l^p(\Gamma) + l^p(\Gamma)$) is the supremum of three dual-less topologies. The proof for $c_0(\Gamma)$ follows from the preceding proof for $l^\infty(\Gamma)$ by observing that the image of c_0 under Q is still dense in $L^{\frac{1}{2}}$.

For topological linear spaces X, Y let $X \approx Y$ mean "X is isomorphic to Y". If α is a cardinal number, let $I^\alpha = \prod_{a\in A} I$, where $I = [0,1]$ and card $A = \alpha$.

LEMMA 2.7. *For any cube I^α and any Banach space B, $C(I^\alpha, B) \approx C(I^\alpha, B)\oplus C(I^\alpha, B)$.*

Proof. We first take $B = R$, so that $C(I^\alpha, B) = C(I^\alpha)$. If α is finite, the result follows from ([7], Theorem 8.5). If α is infinite, I^α is homeomorphic to $I^\alpha \times I$. But for any compact Hausdorff space K, $C(K\times I) \approx C(K\times I)\oplus C(K\times I)$. In fact, $C(K\times I) \approx C(K, C(I)) \approx C(K, C(I)\oplus C(I)) \approx C(K, C(I)) \oplus C(K, C(I)) \approx C(K\times I)\oplus C(K\times I)$.

Now let B be an arbitrary Banach space. Denoting by $E\hat{\otimes} F$ the completion of $E\otimes F$ under the topology of bi-equicontinuous convergence, we have $(E_1\oplus E_2)\hat{\otimes} F \approx (E_1\hat{\otimes} F)\oplus(E_2\hat{\otimes} F)$ for locally convex spaces E_1, E_2, F — see [1] for definitions. By ([1], 1.3.3), $C(I^\alpha, B) \approx C(I^\alpha)\hat{\otimes} B$; the foregoing implies $C(I^\alpha)\hat{\otimes} B \approx \big(C(I^\alpha)\oplus C(I^\alpha)\big)\hat{\otimes} B \approx \big(C(I^\alpha)\hat{\otimes} B\big)\oplus\big(C(I^\alpha)\hat{\otimes} B\big) \approx C(I^\alpha, B)\oplus C(I^\alpha, B)$, as claimed.

THEOREM 2.8. *For any cube I^α and any Banach space B, the norm topology on $C(I^\alpha, B)$ is the supremum of three dual-less topologies.*

Proof. If α is finite, $C(I^\alpha, B) \approx C(I, B) \approx C(I)\hat{\otimes} B \approx C(I\times I)\hat{\otimes} B \approx C(I\times I, B) \approx C(I, C(I, B))$. If α is infinite, I^α is homeomorphic to $I^\alpha \times I$, whence $C(I^\alpha, B) \approx C(I\times I^\alpha, B) \approx C(I, C(I^\alpha, B))$. Thus in either case, if $X = C(I^\alpha, B)$, $X \approx C(I, X)$. Let μ be the topology of convergence in measure on $C(I, X)$ introduced in Section 1 above; μ is weaker than the norm topology and $(C(I, X), \mu)$ is dual-less. Since the constant functions in $C(I, X)$ under the topology of measure convergence are a subspace linearly homeomorphic to X, Lemma 2.2 applies and the proof is complete.

3. MISCELLANEOUS RESULTS

In this section we prove some results which are related to those of the preceding section but do not conveniently fit into the earlier discussion.

PROPOSITION 3.1. *If $(E, \| \ \|)$ is an infinite dimensional normed linear space, the norm topology on E is weaker than the supremum of three pseudo-norm, dual-less topologies on E.*

Proof. Let $\{e_a\}$ be a Hamel basis for E consisting of elements in the norm unit ball. Define a new norm p on E by setting $p(x) = \Sigma|\lambda_a|$ if $x = \Sigma \lambda_a e_a$; the p-topology on E is clearly finer than the $\| \ \|$-topology. By 2.6, the norm topology on $l_1(\{e_a\})$ is the supremum of three dual-less pseudo-norm topologies; since (E, p) is linearly isometric to a dense subspace of $l_1(\{e_a\})$, the same is true for (E, p), and the proof of 3.1 is complete.

Since a symmetric linearly bounded convex absorbing set is the unit ball for a suitable norm, from 3.1 we obtain

COROLLARY 3.2. *If E is an infinite dimensional linear space, each locally convex topology on E is weaker than the supremum of a family of pseudo-norm, dual-less topologies on E.*

We do not know whether in Proposition 3.1 "dual-less" can always be replaced by "type c". The following particular case may be of interest:

PROPOSITION 3.3. *The norm topology on a separable Hilbert space is weaker than the supremum of three linearly homeomorphic, pseudo-norm, dual-less topologies of type c.*

Proof. Consider the space $L^{1/2}$ and write it as $L^{1/2} = E + F$ where E is the closed linear span of the Rademacher functions and F is an algebraic complement of E ($E \cap F = \{0\}$). The $L^{1/2}$-pseudo-norm on E is equivalent to the L^2-norm, because of the Khintchin inequalities ([8], Theorem V. 8.4]. The argument given in § 1 to prove Theorem B shows that the supremum of the $L^{1/2}$-pseudo-norm topology (which is of type e) and its images under suitable maps, is finer than the product of two separable Hilbert space topologies, and the proof is finished.

THEOREM 3.4. *Let (E, σ) be a l.t.s. with σ Hausdorff. Let $f_1, \ldots, f_n$ be linearly independent linear functionals on E. Then there is a linear topology σ' on E with (E, σ) and (E, σ') linearly homeomorphic such that $f_1, f_2, \ldots, \ldots, f_n$ are all continuous for the supremum of σ and σ'.*

Proof. This is a reworking of the construction on p. 243 of [5]. We begin by picking $x_1, \ldots, x_n$ in E such that $f_j(x_i) = 2\delta_{ij}$ (Kronecker delta). Define $T: E \to E$ by $Tx = x - \sum_{i=1}^{n} f_i(x)x_i$. Clearly T is linear, and it is easy to see that $T^2 = \mathrm{Id}$. Thus T is invertible. Define σ' as the image of σ under T. In order to show that f_j is continuous for the supremum of σ and σ', pick a σ-neighborhood of 0 in E such that $\sum_{i=1}^{n} \lambda_i x_i \notin U - U$ if at least one $|\lambda_i|$ exceeds 1. This can be done, since σ is Hausdorff. Now assume $x \in U \cap T(U)$. Then $x = z - \sum f_i(z)x_i$ for some $z \in U$, or $\sum f_i(z)x_i = z - x \in U - U$. This implies that $|f_j(z)| \leqslant 1$ for all $j = 1, \ldots, n$. But since $f_j(x) = -f_j(z)$, it follows that $|f_j| \leqslant 1$ on $U \cap T(U)$, and therefore each f_j is continuous for the supremum of σ and σ'.

Theorem 3.4 shows in particular that the supremum of two type e or type se dual-less topologies may fail to be dual-less. This contradicts Proposition 2.2 in [5].

Observe that in the proof of $\sigma_a E = \sigma_{ne} E$ given in § 1, we exploited the fact that the family of dual-less topologies is invariant under linear invertible maps. Theorem 3.4 can be applied to such families, as follows:

COROLLARY 3.5. *Let Σ be a family of linear topologies on a vector space E such that Σ is invariant under invertible linear maps and contains some non-trivial topology. Then the supremum σ_Σ of the family Σ is finer than the finest weak topology $\sigma_w E$.*

Proof. It is easy to see that σ_Σ is Hausdorff and that the image of σ_Σ under any invertible linear map coincides with σ_Σ. Theorem 3.4 applies to show that all linear functionals are σ_Σ-continuous and this means that σ_Σ is finer than $\sigma_w E$.

Acknowledgments. We thank A. Pełczyński for a conversation concerning Section 1. The research for Sections 2 and 3 was stimulated by a question raised by J. Hampson, communicated to us by V. Klee.

References

[1] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs A. M. S., no. 16, 1955.

[2] R. C. James, *A non-reflexive Banach space isometric with its second conjugate*, Proc. Nat. Acad. Sci. U. S. 37 (1951), pp. 174–177.

[3] J. Kelley and I. Namioka et al., *Linear topological spaces*, 1963.

[4] V. Klee, *On the borelian and projective types of linear subspaces*, Math. Scand. 6 (1958), pp. 189–199.

[5] —, *Exotic topologies for linear spaces*, Proc. Symp. General Topology and its Relations to Modern Analysis and Algebra, Prague (1961), pp. 238–249.

[6] N. T. Peck, and H. Porta, *Subspaces and m-products of nearly exotic spaces*, Math. Annalen 199 (1972), pp. 83–90.

[7] A. Pełczyński, *Linear extensions, linear averagings and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Mat.) 68, 1968.

[8] A. Zygmund, *Trigonometric series*, (2nd Ed.), 1968.

Received February 18, 1972 (475)