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On group algebras of nilpotent Lie groups

by

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Abstract. It is shown that N. Wieners Tauber-theorem holds for all connected nilpotent Lie groups N of class 2:

If $\mathscr{J}$ is a proper closed twosided ideal in $L^1(N)$ then there exists a primitive ideal M, i.e. a kernel of non trivial irreducible continuous unitary representation of $L^1(N)$, containing $\mathscr{J}$. The quotient algebras $L^1(N)/M$ are explicitely determined.

It would be a not to extreme standpoint of view abstract harmonic analysis as the attempt to generalize N. Wieners famous theorem from 1932 as far as possible within the realm of general locally compact groups, see e.g. the remarks in [15], esp. ch. I, § 4, ch. VI, § 1. After that had been done successfully for abelian groups it was inevitable that investigations would turn towards noncommutative groups. But it turns out immediately that in the nonabelian case there are two essentially different possible directions of research corresponding to the fact that for a commutative locally compact group G one can work either with the algebra $L^1(G)$ of Haar-integrable functions or with the algebra $A(\hat{G})$ of functions on the dual group $\hat{G}$ of G, which are Fourier-transforms of functions from $L^1(G)$. An extension of the theory to noncommutative groups, based on the $A(G)$-algebra approach, was given by P. Eymard in his thesis in 1964 [2], with so much success that other workers on $A(G)$ for noncommutative G seemed to have been paralyzed for some time. A more recent survey about the A_p-algebras can be found in [3].

Left open seems to be the question wether some kinds of Wiener-type theorems hold for $L^1(G)$ for noncommutative and — to avoid trivialities — noncompact groups G. A very natural question in this context seems to me the following one:

Under what conditions on G is it true, that every proper closed twosided (or $*$-invariant) ideal of $L^1(G)$ is annihilated by a nontrivial continuous unitary representation of $L^1(G)$, or in other words, is contained in an ideal from $\mathrm{Prim}(G)$, the set of primitive ideals of $L^1(G)$, defined by irreducible continuous unitary representations of $L^1(G)$.

Wieners theorem states that this is true for $G = \boldsymbol{R}$. Of course it is always true for abelian and for compact G. But apparently very few is known in more general cases. Among these the most interesting case, where the answer is affirmative, seems to be the connected affine group G of the real line, i.e. the solvable twodimensional Lie group of all pairs of real numbers (x, y) with $x > 0$ and multiplication $(x, y)(x', y') =(xx', x'y+y')$. For this group P. Müller–Römer proved a very surprising and interesting theorem [14], which implies that for this group the answer to the question raised above is *yes*.

The main purpose of the present paper is to prove that the same answer holds for all connected nilpotent Lie groups of class 2. Moreover we determine completely $\mathrm{Prim}(G)$ for such groups G and describe explicitely the quotient algebras $L^1(G)/M$ for all ideals $M \in \mathrm{Prim}(G)$. These quotient algebras are simple and symmetric and are closely related to the "imprimitivity algebras" in the sense of [4], [5] and [11].

The first part of the paper contains a discussion of the interrelations between some properties closely related to the one stated in our problem. It is easily proved that symmetry of $L^1(G)$ implies the validity of a version of Wieners theorem slightly weaker than the one stated above. Hence according to results of Hulanicki [6] and Z. Anusiak [1] the theorem holds for all discrete nilpotent and the weaker version holds for all class compact groups. This first part, by the way, deals not only just with algebras $L^1(G)$ but with general involutive Banach algebras. Clearly our problem can be stated for these algebras.

In part II we define involutive Banach algebras $\Gamma(G)$ and $\Gamma_*(G)$ for an arbitrary locally compact group G. Here $\Gamma_*(G)$ is exactly the imprimitivity algebra of G with respect to the trivial subgroup $\{e\}$, as defined in [5]. While $\Gamma_*(G)$ is always simple, we can prove this for $\Gamma(G)$ only if G is abelian.

The third part contains the investigation of the ideal-theory of $L^1(G)$ for connected nilpotent Lie groups of class 2 and the proof of the main result already mentioned above. It is easy to see that it suffices to prove the theorem for a special type of "universal" nilpotent Lie groups N of class 2. The main idea then is that these N *as manifolds* can be identified with real vector spaces A in such a way that the Haar measure on N coincides with the Lebesgue measure on A and that moreover in the corresponding identification of the Banach *spaces* $L^1(N)$ and $L^1(A)$ the twosided ideals in $L^1(N)$ are at the same time ideals in the commutative algebra $L^1(A)$. So we can apply Wieners original theorem. Unfortunately this method does not work for arbitrary nilpotent Lie groups of classes greater than two.

My investigations had been stimulated by interesting conversations with Professor P. Porcelli in Baton Rouge, especially on the subjects discussed in part I, and with Professor A. Hulanicki. To both of them I wish to express my gratitude.

I

Let $\mathscr{A}$ be an involutive Banach algebra over the complex number field C, with an isometric involution $x \to x^*$. Let $\tilde{\mathscr{A}} = \mathscr{A}$, if $\mathscr{A}$ has an identity, (which will be always denoted by 1), and let as usually $\tilde{\mathscr{A}} = C \oplus \mathscr{A}$ if $\mathscr{A}$ does not have an identity, with $(\alpha\oplus x)(\beta\oplus y) = \alpha\beta\oplus(\alpha y+\beta x+xy)$, $|\alpha\oplus x| = |\alpha|+|x|$. We write $\alpha\oplus 0 = \alpha$, $0\oplus x = x$ and identify $\mathscr{A}$ with the ideal of the elements $0\oplus x = x$. With $(\alpha+x)^* = \bar{\alpha}+x^*$, $\tilde{\mathscr{A}}$ is an involutive Banach algebra with unit.

Let $P(\mathscr{A})$ be the cone of non zero continuous positive linear functionals on $\mathscr{A}$, let $\mathrm{Sp}\,x$ be the spectrum of x, i.e. the set of all $\lambda \in C$ such that $\lambda 1 - x$ is not invertible in $\tilde{\mathscr{A}}$, and let $\varrho(x)$ be the spectral radius, i.e. $\varrho(x) = \sup\{|\lambda|;\ \lambda \in \mathrm{Sp}\,x\}$.

For $\mathscr{A}$ we state the following properties:

(S) $\mathscr{A}$ is symmetric, i.e. $\mathrm{Sp}(x^*x) \geqslant 0$ for all $x \in \mathscr{A}$,

($\tilde{\mathrm{S}}$) $\tilde{\mathscr{A}}$ is symmetric,

(U) $\mathscr{A}$ is unitary, i.e. for every proper modular left ideal $\mathscr{L} \subset \mathscr{A}$ there exists a non zero positiv definite linear function f with $f(\mathscr{L}) = 0$,

($\tilde{\mathrm{U}}$) $\tilde{\mathscr{A}}$ is unitary.

(T) If $\mathscr{J}$ is a closed twosided ideal in $\mathscr{A}$ such that the quotient algebra $\mathscr{A}/\mathscr{J}$ is not a radical ring then there exist non trivial continuous unitary representations of $\mathscr{A}$, which annihilate $\mathscr{J}$.

We have

PROPOSITION 1 (S), ($\tilde{\mathrm{S}}$), (U) *and* ($\tilde{\mathrm{U}}$) *are equivalent and imply* (T).

Proof. For (S) $\Leftrightarrow$ ($\tilde{\mathrm{S}}$) $\Rightarrow$ ($\tilde{\mathrm{U}}$) see [16].

($\tilde{\mathrm{U}}$) $\Rightarrow$ (U): Let $\mathscr{L}$ be a proper modular left ideal in $\mathscr{A}$. There exists a left ideal $\tilde{\mathscr{L}}$ in $\tilde{\mathscr{A}}$ with $\tilde{\mathscr{L}} \cap \mathscr{A} = \mathscr{L} \neq \tilde{\mathscr{L}}$. Let $\tilde{f} \in P(\tilde{\mathscr{A}})$ with $\tilde{f}(\tilde{\mathscr{L}}) = 0$. The restriction f from $\tilde{f}$ on $\mathscr{A}$ is non zero, because $\tilde{\mathscr{A}} = \tilde{\mathscr{L}}+\mathscr{A}$ and $\tilde{f} \neq 0$. Hence $f \in P(\mathscr{A}), f(\mathscr{L}) = 0$.

(U) $\Rightarrow$ ($\tilde{\mathrm{U}}$): Let $\mathscr{A}$ be unitary and take a proper left ideal $\tilde{\mathscr{L}}$ in $\tilde{\mathscr{A}}$ with $\mathscr{A} \not\subset \tilde{\mathscr{L}}$. Then there exists $f \in P(\mathscr{A})$ with $f(\tilde{\mathscr{L}} \cap \mathscr{A}) = 0$. Because $\tilde{\mathscr{L}} \cap \mathscr{A}$ is modular in $\mathscr{A}$, the functional f is extendable to $\tilde{\mathscr{A}}$, hence let $\tilde{f}$ from $P(\tilde{\mathscr{A}})$ be an extension of f. Let π be a cyclic unitary representation of $\tilde{\mathscr{A}}$ in a Hilbert space $\mathfrak{H}$ and let ε be a cyclic vector with $\tilde{f}(x) = (\pi(x)\varepsilon|\varepsilon)$, $x \in \tilde{\mathscr{A}}$. Then $\mathfrak{H}_0 = \overline{\pi(\mathscr{A})\varepsilon}$, the closed hull of the span of all $\pi(x)\varepsilon$, $x \in \mathscr{A}$, is $\pi(\mathscr{A})$ — invariant and consequently also $\pi(\tilde{\mathscr{A}})$ — invariant. Let $\varepsilon = \varepsilon_0+\varepsilon_1$ with $\varepsilon_0 \in \mathfrak{H}_0$ and $(\mathfrak{H}_0|\varepsilon_1) = 0$. For $x \in \mathscr{A}$ it follows that

$$\pi(x)^*\mathfrak{H} = \overline{\pi(x^*\tilde{\mathscr{A}})\varepsilon} \subset \overline{\pi(\mathscr{A})\varepsilon} = \mathfrak{H}_0$$

hence $(\mathfrak{H}|\pi(x)\varepsilon_1) = (\pi(x)^*\mathfrak{H}|\varepsilon_1) = 0$ and $\pi(x)\varepsilon_1 = 0$. This implies $f(x)$

$= (\pi(x)\varepsilon_0|\varepsilon_0)$ for $x\in\mathscr{A}$. Thus $x\to(\pi(x)\varepsilon_0|\varepsilon_0)$ for $x\in\tilde{\mathscr{A}}$ also extends f and consequently we may assume that $\mathfrak{H}=\mathfrak{H}_0$. Then π is non degenerate on $\mathscr{A}$, for $\pi(\mathscr{A})\xi=0$ implies

$$(\mathfrak{H}|\xi) = (\pi(\mathscr{A})\varepsilon|\xi) = (\varepsilon|\pi(\mathscr{A})\xi)=0,$$

hence $\xi=0$. Because $\dim(\tilde{\mathscr{L}}/\mathscr{L})=1$ there exists $q\in\tilde{\mathscr{A}}$ with $\tilde{\mathscr{L}}=(q)+\mathscr{L}$. It follows that

$$\pi(\mathscr{A})\pi(q)\varepsilon\subset\pi(\mathscr{A}q)\varepsilon\subset\pi(\mathscr{L})\varepsilon=0,$$

hence $\pi(q)\varepsilon=0$, $\pi(\tilde{\mathscr{L}})=0$, $\tilde{f}(\tilde{\mathscr{L}})=0$.

(U) ⇒ (S). Let $z\in\mathscr{A}$ and $h=z^*z$. If h would not be quasiregular, then $\mathscr{A}(1+h)$ would be a proper modular left ideal in $\mathscr{A}$ and there would exist an $f\in P(\mathscr{A})$ with $f(\mathscr{A}(1+h))=0$. It follows that $f(x+xh)=f(x)+f(xh)=0$ for all $x\in\mathscr{A}$. Because f is positive and $h=z^*z$ this implies $f(h)+f(h^2)=0$, $f(h)=0=f(h^2)$. From the Cauchy–Schwarz-inequality we conclude $f(xh)=0$ for all $x\in\mathscr{A}$ and hence $f(x)\equiv 0$, $f=0$, which is not the case. Therefore $\mathscr{A}(1+h)=\mathscr{A}$ and so $\mathscr{A}$ is symmetric.

(U) ⇒ (T) Let $\mathscr{J}$ be a proper closed twosided ideal in $\mathscr{A}$ such that $\mathscr{A}/\mathscr{J}$ is not a radical ring. Then $\mathscr{J}_0=\mathscr{J}\cap\mathscr{J}^*$ is $*$-invariant and $\mathscr{A}/\mathscr{J}_0$ is not a radical ring. If π is a unitary representation of $\mathscr{A}$ and $\pi(\mathscr{J}_0)=0$, then also $\pi(\mathscr{J})=0$, for if $x\in\mathscr{J}$, then $x^*x\subset\mathscr{J}_0$, hence $\pi(x^*x)=\pi(x)^*\pi(x)=0$ and therefore $\pi(x)=0$.

So we may assume that $\mathscr{J}=\mathscr{J}^*$. Because $\mathscr{A}'=\mathscr{A}/\mathscr{J}$ is not a radical ring there exists $u\in\mathscr{A}'$ with $\mathscr{A}'(1+u)\neq\mathscr{A}'$. Hence $\mathscr{A}$ contains a proper modular left ideal $\mathscr{L}$ with $\mathscr{J}\subset\mathscr{L}$. $\mathscr{A}'(1+u)\subset\mathscr{L}/\mathscr{J}$. By (U) there exists $f\subset P(\mathscr{A})$ with $f(\mathscr{L})=0$. Considering f as a functional on $\mathscr{A}'$ we conclude that $P(\mathscr{A}')$ is non empty and consequently has non trivial unitary representations. It follows that $\mathscr{A}$ satisfies (T).

COROLLARY *If $\mathscr{A}$ is commutative, then* (S), (T) *and* (U) *are equivalent.*

Proof. If $\mathscr{L}$ is a proper modular left ideal in $\mathscr{A}$, then it is twosided and $\mathscr{A}/\mathscr{L}$ is not a radical ring. Hence by (T) we have a representation $\pi\neq 0$ annihilating $\mathscr{L}$. Then of course all states belonging to π vanish on $\mathscr{L}$, i.e. (T) implies (U).

Remarks. 1) Without the restriction on the structure of $\mathscr{A}/\mathscr{J}$ in (T) the implication (S) ⇒ (T) would not hold, in other words there exists symmetric Banach algebras $\mathscr{A}$ containing twosided closed and $*$-closed proper ideals $\mathscr{J}$ such that $\mathscr{A}/\mathscr{J}$ is radical:

By Malliavins theorem the real line $\mathbf{R}$ (or any other non discrete locally compact abelian group) contains a set M which is not a set of spectral synthesis, i.e. $\mathscr{A}=L^1(\mathbf{R})$ contains a closed ideal $\mathscr{J}$ with $h(\mathscr{J})=\{x\in\mathbf{R};\hat{f}(x)=0,\ f\in\mathscr{J}\}=M$, but $\mathscr{J}\neq k(M)=\{g\in L^1(\mathbf{R});\ \hat{g}|M=0\}$. By [16], (4.7.7), $k(M)$ is symmetric, but $k(M)/\mathscr{J}$ is a radical ring.

2) Let $\tilde{\mathscr{A}}\neq\mathscr{A}$. Because any representation π of $\mathscr{A}$ can be extended to $\tilde{\mathscr{A}}$ by $\tilde{\pi}(\lambda\oplus x)=\lambda 1+\pi(x)$ it is immediate that $\tilde{\mathscr{A}}$ satisfies (T), if $\mathscr{A}$ does.

3) The equivalence (U) ⇔ ($\tilde{\text{U}}$) of course also follows from (U) ⇔ (S) and (S) ⇔ ($\tilde{\text{S}}$).

4) As the referee has pointed out to me, Mrs. Anusiak, using approximate units, proved the equivalence (S) ↔ (U) for L^1-group algebras (On generalized Beurling theorem and symmetry of L_1-group algebras, Colloq. Math. 23, pp. 287–297 (1971), esp. p. 290).

II

Let G be a locally compact group with left-Haar measure dx, let $A(G)=A_2(G)$ denote the Fourier–Eymard algebra of G and let $C_\infty(G)$ be the C^*-algebra of all complex valued continuous functions on G vanishing at infinity. Then $A(G)\subset C_\infty(G)$ and for the respective norms we have $|a|\geqslant|a|_\infty$ for $a\in A(G)$. The left translations in G define a strongly continuous monomorphism T of G into the automorphismgroups of $A(G)$, respectively $C_\infty(G)$. The continuity of T with respect to $C_\infty(G)$ is clear, with respect to $A(G)$ it follows easiest from the fact that $a\in A(G)$ has the form $a=f*\tilde{g}$, $\tilde{g}(x)=\overline{g(x^{-1})}$, with $f,g\subset L^2(G)$ and $|a|\leqslant|f|_2|g|_2$. The Gelfand spaces both of $A(G)$ and $C_\infty(G)$ can be canonically identified with G so that the action of G on G, induced by T, is the left translation.

Now we can form the generalized L^1-algebras

$$\Gamma(G)=L^1(G,A(G);T,E),\qquad \Gamma_*(G)=L^1(G,C_\infty(G);T,E)$$

in the sense of [9], [10] with trivial factor system E. We have a natural normdecreasing injection

$$\Gamma(G)\to\Gamma_*(G)$$

and the C^*-hulls of both algebras coincide.

Evidently $\Gamma_*(G)$ is the imprimitivity algebra of G with respect to the trivial subgroup $\{e\}$, see [5], [13]. It is the simplest possible case in the sense of [4], p. 110 ff, or [11], p. 277 ff, with $H=\{e\}$, $\mathscr{A}=\mathbf{C}$ and trivial factor system. It follows that up to equivalence the only irreducible representation of $\Gamma_*(G)$ respectively $\Gamma(G)$ is the one induced from the only onedimensional non zero representation of $L^1(H,\mathbf{C})=\mathbf{C}$. If we identify elements from $\Gamma_*(G)$ with functions on $G\times G$ in the obvious way, then this representation ω acts in $L^2(G)$ and is given by

$$(\omega(f)\xi)(x)=\int_G f(t,t^{-1}x)\,\xi(t^{-1}x)\,dt=\int_G f(xt,t^{-1})\,\xi(t^{-1})\,dt$$

for $\xi \in L^2(G)$ and $f \in \Gamma_*(G)$ (or $f \in \Gamma(G)$). It follows that if f is continuous on $G\times G$ and has compact support then $\omega(f)$ is a Hilbert–Schmidt operator. Consequently the C^*-hulls of $\Gamma(G)$ and $\Gamma_*(G)$ may be identified with the algebra $\mathscr{K}\big(L^2(G)\big)$ of compact operators of $L^2(G)$. Hence we have partially proved:

THEOREM 1. *For an arbitrary locally compact group G the C^*-hulls of $\Gamma(G)$ and $\Gamma_*(G)$ are isomorphic to the algebra $\mathscr{K}\big(L^2(G)\big)$ of all compact bounded operators of $L^2(G)$, especially these algebras are liminal with only one point in their dual spaces. Furthermore $\Gamma_*(G)$ is a simple involutive Banach algebra. If G is abelian then also $\Gamma(G)$ is simple.*

Proof. Let $\mathscr{J}$ be a twosided closed ideal in $\Gamma(G)$, respectively $\Gamma_*(G)$ and let $\mathscr{A} = A(G)$, respectively $\mathscr{A} = C_\infty(G)$. Then $\mathscr{J}$ is invariant under left – and right – multiplication with the operators D_z, $a^{\#} \in \Gamma(G)^b$, respectively $\in \Gamma_*(G)^b$ (see [9], § 4) for every $z \in G$, $a \in \mathscr{A}$, at least if $\mathscr{A}$ has an approximate identity, which is always the case for $\mathscr{A} = C_\infty(G)$ and for $\mathscr{A} = A(G)$ if G is amenable [11].

For any $f \in \Gamma_{(*)}(G)$ we find

$$(\mathrm{i}_1)\qquad (D_z f)(x, y) = f(z^{-1}x, y),$$

$$(\mathrm{i}_2)\qquad (fD_z)(x, y) = f(xz^{-1}, zy),$$

$$(\mathrm{i}_3)\qquad (a^{\#}f)(x, y) = a(xy)f(x, y),$$

$$(\mathrm{i}_4)\qquad (fa^{\#})(x, y) = a(y)f(x, y).$$

Now let us consider first the case $\mathscr{A} = C_\infty(G)$.

In this case $\Gamma_*(G)$ is a module over $C_b(G\times G)$ (= complex-valued continuous bounded functions on $G\times G$). For $q \in C_b(G\times G)$ and $f \in \Gamma_*(G)$ the element qf is defined by $(qf)(x, y) = q(x, y)f(x, y)$. It follows that $|qf| \leqslant |q|_\infty |f|$ for the $\Gamma_*(G)$-norm. Now let a_i, b_i be arbitrary elements from $C_\infty(G)$, $i = 1, \ldots, n$, and let $q(x, y) = \sum a_i(xy)b_i(y)$. Then from $(\mathrm{i}_{3,4})$ we see that $qf = \sum a_i^{\#} f b_i^{\#}$. Hence $q\mathscr{J} \subset \mathscr{J}$ for all these q and because these functions form a dense subalgebra in $C_\infty(G\times G)$ it follows that the ideal $\mathscr{J}$ is invariant under multiplication with functions from $C_\infty(G\times G)$. For every compact K in G take $q_K \in C_\infty(G\times G)$ with $0 \leqslant q_K(x, y) \leqslant 1$ for all $x, y \in G$, $q_K(x, y) = 1$ for $x, y \in K$ and such that q_K has compact support. The family $\{q_K\}_K$ is naturally directed and satisfies

$$\lim_K q_K f = f$$

in $\Gamma_*(G)$ for every $f \in \Gamma_*(G)$: This is trivial for f with compact support and follows for arbitrary f from $|q_K f| \leqslant |f|$ and because functions with compact support are dense in $\Gamma_*(G)$. It follows that $qf = \lim q_K q f \in \mathscr{J}$ for every $f \in \mathscr{J}$ and every $q \in C_b(G\times G)$, because $q_K q \in C_\infty(G\times G)$.

Now we can apply Satz 8 and Satz 9 from [10]: With the notation of these theorems we see that $\mathscr{J}$ is $C(G)$-invariant and because of (i_1)

also λ-invariant, hence (Satz 9) $\mathscr{J}$ is of the form $l(\mathscr{K})$ with a closed ideal $\mathscr{K} \subset \mathscr{A}$. From Satz 8 (iv), we see that $\mathscr{K}$ must be G-invariant. But 0 and $C_\infty(G)$ are the only closed G-invariant ideals in $C_\infty(G)$. Hence $\mathscr{J} = l(\mathscr{K}) = 0$ or $= \Gamma_*(G)$.

Now let G be abelian with dual group $\hat{G}$. Then we can identify $A(G)$ with $L^1(\hat{G})$, the automorphisms T_z given by

$$(T_z f)(\psi) = (z\,|\,\psi)f(\psi), \qquad \psi \in \hat{G},$$

with the notation $\psi\colon x \to (x\,|\,\psi)$ for the character $\psi \in \hat{G}$.

Now as Banach spaces we have a natural isomorphism $L^1\big(G, L^1(\hat{G})\big) \cong L^1(G\times\hat{G})$, the latter L^1-space formed with the Haar-measure $dx\,d\psi$ on the product group $G\times\hat{G}$. The formula for T_z shows that this isomorphism becomes an isomorphism of involutive Banach algebras if we define

$$(f*g)(x, \psi) = \int (y\,|\,\psi\chi)f\big((x, \psi)(y, \chi)\big)g\big((y, \chi)^{-1}\big)\,dy\,d\chi$$

$$f^*(x, \psi) = \overline{(x\,|\,\psi)}\,\overline{f\big((x, \psi)^{-1}\big)}.$$

But with these definitions $L^1(G\times\hat{G})$ is exactly the generalized L^1-algebra $L^1(G\times\hat{G}, \boldsymbol{C}; P)$ with trivial action of $G\times\hat{G}$ on $\boldsymbol{C}$ and with the factor system

$$P_{(x,\psi),(y,\chi)} = \overline{(y\,|\,\psi)}.$$

Hence $\Gamma(G) \cong L^1(G\times\hat{G}, \boldsymbol{C}; P)$.

Instead of $(\mathrm{i}_1), \ldots, (\mathrm{i}_4)$ we now have

$$(\mathrm{j}_1)\qquad (D_{(z,\zeta)}f)(x, \psi) = (x^{-1}z\,|\,\zeta)f(z^{-1}x, \zeta^{-1}\psi),$$

$$(\mathrm{j}_2)\qquad (fD_{(z,\zeta)})(x, \psi) = (z\,|\,\zeta\psi^{-1})f(xz^{-1}, \psi\zeta^{-1}).$$

Now let $\mathscr{J}$ again be a closed twosided ideal in $L^1(G\times\hat{G}; P)$ and let $f \in \mathscr{J}$. Then also $D_{(z,\varepsilon)}f\colon (x, \psi) \to f(z^{-1}x, \psi)$ (ε the identity in G) and $fD_{(e,\zeta)}\colon (x, \psi) \to f(x, \psi\zeta^{-1})$ are contained in $\mathscr{J}$ which means that $\mathscr{J}$ is an ideal in the *commutative* ordinary L^1-algebra $L^1(G\times\hat{G})$. Moreover (j_1) and (j_2) imply that $\mathscr{J}$ is also invariant under multiplication with the functions $(x, \psi) \to (x\,|\,\zeta)(z\,|\,\psi)$ for arbitrary $\zeta \in \hat{G}$, $z \in G$. But because $\widehat{G\times\hat{G}} = \hat{G}\times G$ these are exactly the characters of the abelian group $A = G\times\hat{G}$. Now the only closed ideals in $L^1(A)$, A abelian, which are invariant under multiplication with characters $\alpha \in \hat{A}$ are 0 and $L^1(A)$ itself. This proves that $L^1(G\times\hat{G}; P)$ and $\Gamma(G)$ are simple algebras. As an easy consequence from [1] we get:

PROPOSITION 2. *If G is abelian then $\Gamma(G)$ is a simple and symmetric involutive Banach algebra.*

Proof. Let H be the central extension of the circle group T by $G\times\hat{G}$, defined by the 2-cycle P. Then all elements in H have relatively compact classes of conjugated elements, i.e. H is a $[FC]^-$-group and hence $L^1(H)$ is symmetric [1]. But $L^1(G\times\hat{G};P)$ and so $\Gamma(G)$ is a factoralgebra of $L^1(H)$ and therefore $\Gamma(G)$ is symmetric.

III

We will now investigate the L^1-algebras of certain nilpotent simply connected Lie groups. Let A be a real finite-dimensional associative nilpotent algebra of class r: $A^r\neq 0$, $A^{r+1}=0$. For x and y in A we define the product

$$x\circ y = x+y+xy, \qquad (\circ)$$

so in A we have $(1+x)(1+y)=1+x\circ y$. Obviously with $(\circ)$ as a product A becomes a n-dimensional nilpotent Lie group of class r. We denote this group by $N(A)$ or, if A is fixed, just by N. For $z\in A$ the mapping $t\to e^{tz}-1$ defines a one parameter subgroup in N and it is easy to see that this can be used to identify the Lie algebra $\mathfrak{a}$ defined by A (i.e. $\mathfrak{a}=A$ as linear space with $[u,v]=uv-vu$) with the Lie algebra of N. For every $a\in A$ the transformation $L_a x = ax$ is nilpotent, hence $1+L_a$ is a unipotent transformation of the real vectorspace A. Because of $a\circ x = a+(1+L_a)x$ it follows that the Lebesgue measure on A and the Haar measure on N coincide, so, considering A as a real locally compact vectorspace, we have $L^1(A)=L^1(N)$ as Banach spaces.

From now on we only consider the special case $A=A_1\oplus A_2$ where the n-dimensional subspace A_1 generates A, A_2 is spanned by the products xy, $x,y\in A_1$, $xy=-yx$, and $A^3=0$, in other words $A=G_n/R^3$, where G_n is the 2^n-dimensional Grassmann algebra and R^3 the third power of the radical R of G_n. If $\{b_i\}_{i=1,\dots,n}$ is a basis of A_1, the products b_jb_k, $j<k$, form a basis of A_2. The corresponding Lie algebra $\mathfrak{a}$ is the free nilpotent Lie algebra of class 2 with generators $\{b_i\}$: $\mathfrak{a}$ is generated by the elements b_i with the only relations $[b_j,b_k]=b_jb_k$, $\mathfrak{a}^3=0$.

For the product in $N=N(A)$ we find the following formula:

$$(a+b+ab)\circ x\circ(a-b+ab) = x+2a+2bx.$$

We conclude that a closed subspace $\mathcal{J}$ in $L^1(N)$ is a twosided ideal if and only if it is translation invariant, hence an ideal in $L^1(A)$, and invariant under all transformations

$$T_b\colon x\to x+bx$$

with $b\in A$. Now let $\mathcal{J}$ be such a twosided proper ideal. Then $\mathcal{J}$ as an ideal in $L^1(A)$ has a nonempty hull H in the dual group $\hat{A}$, invariant under the adjoints of the transformations T_b.

Now we introduce a euclidean inner product $(x|y)$ on A such that the b_i and $b_{jk}=b_jb_k$, $j<k$, form an orthonormal basis. Using this product we identify $\hat{A}$ with A: An element $y\in A$ corresponds to the character $\chi_y\colon x\to e^{2\pi i(x|y)}$. Because $T_b=1+L_b$ the adjoint $T_b'=1+L_b'$ with the adjoint L_b' of L_b. It follows that the orbit of an element $z\in\hat{A}$ is the coset

$$\mathcal{O}(z)=\{z+L_b'z;\ b\in A\}$$

of the linear subspace $L_A'z$ in $\hat{A}$, especially it follows that all orbits are closed and the kernels

$$M_z = k(\mathcal{O}(z)) = \{f\in L^1(A);\ \hat{f}|\mathcal{O}(z)=0\}$$

are T_b-invariant ideals in $L^1(A)$ and hence twosided closed ideals in $L^1(N)$ It follows immediately that $\mathcal{J}\subset M_z$ if and only if $z\in H$; so we have:

(1) The ideals M_z, $z\in A$, are maximal twosided closed ideals in $L^1(N)$. Every twosided closed ideal $\mathcal{J}$ in $L^1(N)$ is contained in an ideal M_z.

Our next aim is to determine explicitely the quotient algebras $L^1(N)/M_z$. Let

$$E=L_A'z,\qquad F=E^{\perp}=\{y\in A;\ (E|y)=0\}.$$

Then $A=E\oplus F$ and $\hat{E}$ can be identified with $L_A'z=E$. For $f\in L^1(N)$ define

$$\varrho(f)(x)=\int_F f(x+y)\langle x+y,z\rangle\,dy,\qquad x\in E$$

where $\langle u,v\rangle$ is defined as $e^{-2\pi i(u|v)}$, $u,v\in A$. $\varrho\colon f\to\varrho(f)$ is a bounded linear map from $L^1(N)$, respectively $L^1(A)$ onto the Banach space $L^1(E,dx)$, dx Lebesgue measure on E. For the Fourier transforms we have for $u\in E=L_A'z$:

$$\varrho(f)^{\wedge}(u)=\int_E\varrho(f)(x)\langle x,u\rangle\,dx=\int_E\int_F f(x+y)\langle x+y,z\rangle\langle x,u\rangle\,dy\,dx$$
$$=\int_A f(t)\langle t,z+u\rangle\,dt=\hat{f}(z+u)$$

because $\langle y,u\rangle=1$ for $y\in F$. It follows that $\varrho(f)=0$ if and only if $\hat{f}|\mathcal{O}(z)=0$, hence

$$\ker\varrho = M_z$$

and $L^1(N)/M_z=L^1(E)$ as Banach spaces. Now define the factor system $Q^z=\{Q_{u,v}\}$ by

$$Q_{u,v}=\langle uv,z\rangle,\qquad u,v\in E.$$

It is easily checked that Q^z is indeed a continuous factor system on E with values in $\boldsymbol{T}$. So we can form the algebra

$$L^1(E, \boldsymbol{C}; Q^z).$$

We will prove now that ϱ is a full isomorphism between the two involutive Banach algebras.

First we remark that $xy+yx = 0$ for all x and y in A. By definition this is true for $x, y \in A_1$ and it is trivial if x or y is in A_2, because $AA_2 = A_2A = 0$. It follows that $\langle xu, z\rangle = \langle ux, z\rangle = 1$ for all $x \in A$, $u \in F$. Next we see that

$$(\varrho) \qquad \varrho(f)(x) = \int_F f(x\circ y) < x\circ y, z > dy$$

because $x\circ y = x+y+xy = x+(1+x)y$ and $dy = d(1+x)y$. Similarly $\varrho(f)(x) = \int_F f(y\circ x)\langle y\circ x, z\rangle dy$. Now evidently $A_2 \subset F$, hence $x+y = x\circ y$ (mod F) and $-x = x^{-1}$(mod F). Because the right hand side in (ϱ) depends only on the coset of x mod F it follows that for x and s in E we have

$$\varrho(f)(x+s) = \int_F f(x\circ s\circ u)\langle x\circ s\circ u, z\rangle du,$$

$$\varrho(f)(-s) = \int_F f(v\circ s^{-1})\langle v\circ s^{-1}, z\rangle dv.$$

Using the fact that $\int_N h(y)dy = \int_F\int_E h(v\circ s)\,ds\,dv$ the convolution $f*g$ in $L^1(N)$ can be written as

$$(f*g)(x\circ u) = \int_F\int_E f(x\circ u\circ v\circ s)\,g(s^{-1}\circ v^{-1})\,ds\,dv.$$

Hence

$$\varrho(f*g)(x) = \iiint f(x\circ u\circ v\circ s)\,g(s^{-1}\circ v^{-1})\langle x\circ u, z\rangle\,du\,dv\,ds$$
$$= \iiint f(x\circ u\circ s)\langle x\circ u\circ s, z\rangle g(s^{-1}\circ v)\langle s^{-1}\circ v, z\rangle\,\vartheta\,du\,dv\,ds$$

with $\vartheta = \langle x\circ u\circ v - x\circ u\circ s - s^{-1}\circ v, z\rangle$. Because u and v are in F and $\langle x\circ u, z\rangle = \langle x+u, z\rangle$ etc. it follows that

$$\vartheta = \langle x+u+v-x-u-s-xs-s^{-1}-v, z\rangle$$
$$= \langle -xs-s^2, z\rangle = \langle -(x+s)s, z\rangle.$$

It is easy to see that F is normal in N and that the inner automorphisms of N leave the Haar measure on F fixed. It follows that

$$\int f(x\circ u\circ s)\langle x\circ u\circ s, z\rangle du = \int f(x\circ s\circ u^s)\langle x\circ s\circ u^s, z\rangle du$$
$$= \varrho(f)(x+s) \qquad (\text{here } u^s = s^{-1}\circ u\circ s).$$

Consequently the last formula for $\varrho(f*g)(x)$ yields

$$\varrho(f*g)(x) = \int \varrho(f)(x+s)\,\varrho(g)(-s)\langle -(x+s)s, z\rangle ds = \varrho(f)*\varrho(g)(x).$$

It follows that ϱ is multiplicative.

Finally $\varrho(f^*) = \varrho(f)^*$: F is a subgroup, so with $y \in F$ also $y^{-1} \in F$ and, as is easily checked, $dy = dy^{-1}$. Furthermore $x\circ y^{-1} + y\circ x^{-1} = x-y+y^2-xy+y-x+x^2-yx = x^2+y^2-xy-yx$, which implies for $y \in F$:

$$\langle x\circ y^{-1}, z\rangle = \overline{\langle y\circ x^{-1}, z\rangle}\,\overline{\langle -x^2, z\rangle} = Q^{-1}_{x,x^{-1}}\overline{\langle y\circ x^{-1}, z\rangle}.$$

It follows that

$$\varrho(f^*)(x) = \int \overline{f((x\circ y)^{-1})}\langle x\circ y, z\rangle dy = \int \overline{f(y\circ x^{-1})}\langle x\circ y^{-1}, z\rangle dy$$
$$= Q^{-1}_{x,x^{-1}}\Big(\int f(y\circ x^{-1})\langle y\circ x^{-1}, z\rangle dy\Big)^- = \varrho(f)^*(x).$$

We will now explicitely determine the algebras

$$\Gamma_z(N) = L^1(E, \boldsymbol{C}; Q^z).$$

Of course $\Gamma_z(N)$ depends only on the bilinear form

$$B(x, y) = (xy\,|\,z).$$

This form B is symplectic and non degenerate: $B(x, y) = -B(y, x)$ because $xy = -yx$, and $B(x, y) = 0$ for all $x \in E$ and fixed $y \in E$ implies $(y\,|\,L'_x z) = 0$ for all $x \in E$, $(y\,|\,L'_u z) = -(u\,|\,L'_y z) = 0$ also for all $u \in F$, hence $y \in E \cap L'_A z = 0$. It follows that $\dim E = 2m$ and that we can introduce coordinates

$$x = \{x_1, \ldots, x_m, \hat{x}_1, \ldots, \hat{x}_m\} \in \boldsymbol{R}^{2m}, \quad y = \{y_j, \hat{y}_j\}$$

with $B(x, y) = \frac{1}{2}\sum_1^m (x_i\hat{y}_i - \hat{x}_i y_i)$. With $c(x) = \frac{1}{2}\sum_{i=1}^m x_i\hat{x}_i$ we get the following identity

$$B(x, y) = -\sum \hat{x}_i y_i + c(x+y) - c(x) - c(y).$$

The factor system Q^z was defined by $Q_{x,y} = e^{-2\pi i B(x,y)}$. So with $C_x = e^{2\pi i c(x)}$ we find

$$Q_{x,y} = e^{2\pi i\Sigma\hat{x}_j y_j}\cdot C_x\cdot C_y\cdot C^{-1}_{x+y}.$$

It follows that Q^z and the factor system $P = \{P_{x,y}\}$, $P_{x,y} = e^{2\pi i\Sigma\hat{x}_j y_j}$ are equivalent, i.e. belong to the same class in $H^2(\boldsymbol{R}^{2m}, \boldsymbol{T})$. But identifying $\boldsymbol{R}^m$ with its dual $\hat{\boldsymbol{R}}^m$ and $\boldsymbol{R}^{2m} = E$ with $\boldsymbol{R}^m \times \hat{\boldsymbol{R}}^m$ the factor system $\{P_{x,y}\}$ is exactly the one which we used in our investigation of $\Gamma(G)$ and $L^1(G\times\hat{G}; P)$ for abelian G in the first part of this paper. So from [9], p. 263, we get the result:

(3) For every $z \in \hat{A}$ the factoralgebra $\Gamma_z(N)$ is canonically isomorphic with the involutive simple and symmetric Banach algebra $\Gamma(\boldsymbol{R}^m)$.

Now we can easily prove our main result:

THEOREM 2. *Let G be a connected nilpotent Lie group of class two, $L^1(G)$ be the L^1-group algebra and* Prim(G) *the set of primitive ideals of $L^1(G)$, i.e. the set of kernels of nontrivial irreducible continuous unitary representations of $L^1(G)$. Then*

1. *Every proper closed twosided ideal of $L^1(G)$ is contained in an ideal M from* Prim(G).

2. *If $M \in$* Prim(G) *then there exists a positive integer m such that the quotient algebra $L^1(G)/M$ is isomorphic to the simple and symmetric involutive Banach algebra $\Gamma(\boldsymbol{R}^m)$, defined in part* II.

Proof. Assume that Theorem 2 is true for G. If N is a closed normal subgroup of G, then the L^1-algebra $L^1(G/N)$ of the factorgroup G/N as is well known (see e.g. [15], p. 164) is a quotient algebra of $L^1(G)$. From this one easily derives that Theorem 2 holds also for G/N. Therefore it suffices to consider only simply connected groups. But the Lie algebra of such a group is a homomorphic image of a Lie algebra $\mathfrak{a}$ as considered in the first part of this section of the paper. Consequently every connected nilpotent Lie group G of class two is a factor group of a group $N(A)$ for a suitable algebra A as defined above. Theorem 2 now follows from (1), (2) and Theorem 1. In another paper we shall prove that for the groups G considered in Theorem 2 the algebras $L^1(G)$ are indeed symmetric.

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