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On perturbations of deviations of periodic differential-difference equations in Banach spaces

by

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Abstract. Let X be a Banach space and Y be a linear subset of X equipped with such a topology that it is a Banach space.

We consider a differential difference equation

$$x'(t)+\sum_{i=0}^{m} A_i(t)x(t-h_i) = y(t) \tag{1}$$

where $h = (h_0, \ldots, h_m)$ is a system of reals.

In the paper it is shown that if equation (1) has a unique ω-periodic solution, then an equation which we get from equation (1) replacing h by h' also has a unique ω-periodic solution provided that h' differs from h little enough. Moreover, it is proved that the solution depends on h in a continuous way.

Let X be a Banach space and let Y be a linear subset of X (not necessarily closed) equipped with a norm not weaker than the norm in the space X. Let us assume that Y with this new norm is a Banach space.

Let $A_i(t)$, $i = 0, 1, \ldots, m$, be continuous linear operators mapping Y into X. We assume that all $A_i(t)$ are continuous in the norm topology and are periodic with a period ω (briefly ω-periodic) with respect to the real parameter t.

Let C_ω^X (respectively C_ω^Y) denote the space of all continuous ω-periodic functions $x(t)$ of the real argument t with values in the space X (respectiwely Y).

Let us consider a differential-difference equation

$$(A^h x)(t) \stackrel{\mathrm{df}}{=} x'(t)+\sum_{i=0}^{m} A_i(t)x(t-h_i) = y(t) \tag{1}$$

where $h = (h_0, \ldots, h_m)$ is a system of real numbers.

Let us look for a solution $x \in C_\omega^X$ of the equation (1).

When all $h_0, \ldots, h_m$ are commensurable with ω, there is a simple method of reducing this problem to the problem of solving a certain system of differential equations in the space C_ω^X. It is done by the method of algebraic operators (see [2] and [3]).

When h_i are not all commensurable with ω, the solutions can be found by approximations on the basis of a perturbation theorem for finite-dimensional systems (proved in [3] Ch. V, § 3).

The aim of this note is to show the perturbation theorem for the infinite-dimensional case.

THEOREM 1. *Let the equation* (1) *have a unique solution* $x_h^y \in C_\omega^Y$ *for each* $y \in C_\omega^X$. *Then there is an* $\varepsilon_0 > 0$ *such that the equation*

$$A^{h'}(x) = y, \tag{1'}$$

where $h' = (h'_0, \ldots, h'_m)$, *has a unique solution* $x_{h'}^y \in C_\omega^Y$ *provided that*

$$|h_i - h'_i| < \varepsilon \leqslant \varepsilon_0 \quad i = 0, 1, \ldots, m.$$

Moreover, if $\varepsilon \to 0$, *then* $x_{h'}^y \to x_h^y$ *in the space* C_ω^Y.

Proof. To begin with we shall replace the equations (1) and (1′) by equivalent integral equations. For this purpose we shall introduce an operator R defined on the space C_ω^X and C_ω^Y by means of the formula

$$(Rx)(t) = \int_0^t x(s)\,ds - \left(\frac{t}{\omega} + 1\right) \int_0^\omega x(s)\,ds.$$

It is easy to verify (see [1] p. 317, [3] Ch. V § 3) that R is an isomorphism mapping C_ω^X (respectively C_ω^Y) onto the space $C_\omega^{X,1}$ (respectively $C_\omega^{Y,1}$) of the continuous differentiable ω-periodic functions with values in the space X (respectively Y) with the standard norm

$$|||x(\cdot)||| = \sup_{0 \leqslant t \leqslant \omega} [\|x(t)\| + \|x'(t)\|],$$

where $\| \ \|$ denotes the norm in X (respectively in Y).

Applying the operator R to the equations (1) and (1′) we get the equivalent integral equations

$$T_h x = Ry, \tag{2}$$

$$T_{h'} x = Ry, \tag{2'}$$

where

$$(T_h x)(t) = (RA_x^h)(t)$$

$$= x(t) - x(0) + \sum_{i=0}^m \left[\int_0^t A_i(s) x(s - h_i)\,ds - \left(\frac{t}{\omega} + 1\right) \int_0^\omega A_i(s) x(s - h_i)\,ds\right].$$

Let us observe that for each h the operator T_h is a continuous linear operator mapping $C_\omega^{Y,1}$ onto $C_\omega^{X,1}$. The assumption of the uniqueness of the solution of the equation (1) implies the uniqueness of a solution of the equation (2). In other words, the operator T_h is an isomorphic mapping of $C_\omega^{Y,1}$ onto $C_\omega^{X,1}$.

We shall show that T_h depends on h in a continuous way in the norm topology. Indeed, we may assume $0 < t \leqslant \omega$. Then

$$\begin{aligned}\|(T_h x - T_{h'} x)(t)\| &= \Bigg\| \sum_{i=0}^m \Bigg[\int_0^t A_i(s)[x(s - h_i) - x(s - h'_i)]\,ds - \\ &\quad - \left(\frac{t}{\omega} + 1\right) \int_0^\omega A_i(s)[x(s - h_i) - x(s - h'_i)]\,ds\Bigg]\Bigg\| \\ &\leqslant \sum_{i=0}^m \Bigg(\Big\|\int_0^t A_i(s)[x(s - h_i) - x(s - h'_i)]\,ds\Big\| + \\ &\quad + 2\Big\|\int_0^\omega A_i(s)[x(s - h_i) - x(s - h'_i)\,ds\Big\|\Bigg).\end{aligned} \tag{3}$$

Let us now estimate the first integral

$$\begin{aligned}&\Big\|\int_0^t A_i(s) x(s - h_i)\,ds - \int_0^t A_i(s) x(s - h'_i)\,ds\Big\| \\ &= \Big\|\int_0^{h'_i - h_i} A_i(s) x(s - h_i)\,ds + \int_{h'_i - h}^t A_i(s) x(s - h_i)\,ds - \\ &\quad - \int_0^{t - (h'_i - h_i)} A_i(s) x(s - h'_i)\,ds - \int_{t - (h'_i - h_i)}^t A_i(s) x(s - h'_i)\,ds\Big\| \\ &\leqslant \Big\|\int_0^{h'_i - h_i} A_i(s)\,ds\Big\| \cdot \|x\|_{C_\omega^Y} + \Big\|\int_{t - (h'_i - h_i)}^t A_i(s)\,ds\Big\| \cdot \|x\|_{C_\omega^Y} + \\ &\quad + \Big\|\int_{h'_i - h_i}^t [A_i(s) - A_i(s + h'_i - h_i)]\,ds\Big\| \cdot \|x\|_{C_\omega^Y}.\end{aligned} \tag{4}$$

The continuity of $A_i(s)$ in the norm topology implies that the right side of the inequality (4) tends to 0 when $h'_i \to h_i$.

Let us observe that similar estimations hold for the second integral. Then, by the inequality (3), T_h depends on h in a continuous way in the norm topology.

Thus by the perturbation theorem for bounded operators (see for instance [4] Ch. C III) there is an $\varepsilon_0 > 0$ such that for all h' satisfying the inequalities

$$|h'_i - h_i| < \varepsilon \leqslant \varepsilon_0 \qquad (i = 0, 1, \ldots, m)$$

the operator $T_{h'}$ is also an isomorphic mapping $C_\omega^{Y,1}$ onto $C_\omega^{X,1}$. This means that the equation (2′) has a unique solution $x_{h'}^y \in C_\omega^{Y,1}$ for each $y \in C_\omega^{X,1}$. Moreover, the solution $x_{h'}^y$ tends to the solution x_h^y of the equation (2) in the norm of the space $C_\omega^{Y,1}$.

Since equations (2) and (2′) are equivalent to the equations (1) and (1′), the equation (1′) has a unique solution $x_{h'}^y \in C_\omega^Y$ and, moreover, $x_{h'}^y$ tends to the solution of the equation (1) x_h^y in the norm of the space C_ω^Y.

EXAMPLE. Let us consider the equations

$$(4') \qquad \frac{\partial}{\partial t} u(x,t) + \sum_{i=0}^{m} \sum_{j=0}^{k} a_{i,j}(x,t) \frac{\partial^j}{\partial x^j} u \Big|_{x,t-h_i} = y(x,t)$$

and

$$(4'') \qquad \frac{\partial}{\partial t} u(x,t) + \sum_{i=0}^{m} \sum_{j=0}^{k} a_{i,j}(x,t) \frac{\partial^j}{\partial x_j} \Big|_{u,t-h_i'} = y(x,t)$$

defined for $-\infty < t < +\infty$ and $a \leqslant x \leqslant b$, where the functions $a_{i,j}(x,t)$ are continuous and ω-periodic with respect to t. If for each continuous function $y(x,t)$ ω-periodic with respect to t such that

$$(5) \qquad y(a,t) = y(b,t) = 0$$

there is a unique solution u_h^y of the equation (4) such that $u_h^y(x,t)$ is ω-periodic with respect to t and, moreover,

$$(5') \qquad u_h^y(a,t) = u_h^y(b,t) = 0,$$

then there is an $\varepsilon > 0$ such that the equation (4′) has a unique solution $u_{h'}^y(x,t)$ ω-periodic with respect to t and satisfying the boundary condition (5) provided

$$|h_i - h_i'| < \varepsilon \leqslant \varepsilon_0 \quad (i = 0, 1, \ldots, m).$$

Moreover, $u_{h'}^y$ tends uniformly with the derivatives of the order $1, 2, \ldots, k$ with respect to x when $\varepsilon \to 0$.

We get the above result putting instead of Y the space of continuous functions $v(x,t)$, ω-periodic with respect to t, k-time differentiable with respect to x and satisfying the boundary condition (5).

For differential-difference equations of higher orders we get in a similar way

THEOREM 2. *Let us consider a differential-difference equation*

$$(6) \qquad \sum_{j=0}^{k} \sum_{i=0}^{m} A_{i,j}(t) x^{(j)}(t - h_i) = y(t)$$

with the initial conditions

$$(7) \qquad x^{(j)}(0) = x_j, \quad j = 0, 1, \ldots, k-2$$

(where, as usual, we write $x^{(j)}(t) = \dfrac{d^j x}{dt^j}$*). We assume that* $A_{i,j}(t)$ *are continuous linear operators mapping* Y *into* X*, continuous with respect to* t *in the norm topology and* ω*-periodic with respect to* t.

Let us consider simultaneously the differential-difference equation

$$(6') \qquad \sum_{j=0}^{k} \sum_{i=0}^{m} A_{i,j}(t) x^{(j)}(t - h_i') = y(t)$$

with the initial condition (7).

If there is a unique ω*-periodic solution* x_h *for all initial values* $(x_0, \ldots, x_{k-2})$ *and all functions* $y \in C_\omega^X$*, then there is an* $\varepsilon_0 > 0$ *such that for*

$$|h_i - h_i'| < \varepsilon \leqslant \varepsilon_0 \qquad i = 0, 1, \ldots, m,$$

there is a unique ω*-periodic solution* $x_{h'}$ *of the equation* (6) *with the initial conditions* (7). *Moreover,* $x_{h'} \to x_h$ *in the space* C_ω^Y *when* $\varepsilon \to 0$.

Proof. In the same way as in the proof of Theorem 1 we apply to both sides of equations (6) and (6′) the operator R.

The right sides of the resulting equations together with the initial conditions (7) can be interpreted as a continuous linear operator T_h (respectively $T_{h'}$) mapping the space $\underbrace{X \times X \times \ldots \times X}_{n-1 \text{ time}} \times C_\omega^{Y,1}$ into the space C^X.

By the same calculations as in the proof of Theorem 1 we conclude that $\|T_{h'} - T_h\| \to 0$ provided $h' \to h$. Further the proof is the same as the proof of Theorem 1.

References

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