[8] A. E. Taylor, *Banach spaces of functions analytic in the unit circle*. I, Studia Math., 11 (1950), pp. 145–170.
[9] — *Banach spaces of functions analytic in the unit circle*. II, ibidem 12 (1951), pp. 25–50.
[10] A. Zygmund, *Trigonometric Series*, vol. I, Cambridge 1959.

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Received July 1, 1972 (563)

STUDIA MATHEMATICA, T. XLVII. (1973)

Operators associated with representations of amenable groups singular integrals induced by ergodic flows, the rotation method and multipliers

by

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Abstract. Suppose G is an amenable locally compact group, $k \in L^1(G)$ has compact support, and R_u is a uniformly bounded representation of G acting on $L^p(\mathfrak{M})$. It is shown that the operator

$$\hat{k}(R) = \int_G k(u) R_{u^{-1}} du$$

has $L^p(\mathfrak{M})$-operator norm not exceeding the $L^p(G)$-operator norm of the convolution operator defined by k. From this one can obtain an extension of the rotation method for singular integrals on $\boldsymbol{R}^n$ to Lie groups. Moreover, results of Calderón, on commutator operators, de Leeuw and Fife, on multipliers, are generalized.

§ 1. Introduction. In their work on Singular Integrals, Calderón and Zygmund observed that properties of those Singular Integrals having odd kernels could be derived easily from properties of the Hilbert transform

$$\tilde{f}(s) = \lim_{\varepsilon\to 0+} \frac{1}{\pi} \int_{\varepsilon\leqslant|t|} \frac{f(s-t)}{t} dt = \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s-t)}{t} dt.$$

The approach they used, called by them the *method of rotation*, can be described in the following way. An *odd kernel* has the form $k(y) = \Omega(y)/|y|^n$, where $y = (y_1, y_2, \ldots, y_n) \in \boldsymbol{R}^n$, $|y| = \left(\sum_{j=1}^{n} y_j^2\right)^{1/2}$ and Ω is an odd function which is homogeneous of degree 0 and whose restriction to the surface of the unit sphere $\Sigma_{n-1} = \{y \in \boldsymbol{R}^n\colon |y| = 1\}$ is integrable. Let us fix a point y' of Σ_{n-1}; we then consider the one-parameter group $\{U_{y'}^t\}$, $-\infty < t < \infty$, of transformations of $\boldsymbol{R}^n$ defined by

$$U_{y'}^t x = x + ty' \tag{1.1}$$

for each $x \in \boldsymbol{R}^n$. For each $y' \in \Sigma_{n-1}$ we can then form the "truncated" Hilbert transform of the function $g^x_{y'}(t) = f(U^t_{y'}x)$

$$(H^\delta_\varepsilon g^x_{y'})(s) = \int_{\varepsilon \leqslant |t| \leqslant \delta} \frac{g^x_{y'}(s-t)}{t} dt$$

and evaluate it at $s = 0$. The two principal features of the method of rotations are:

(1) the observation that the "truncated" singular integral with kernel k is a mean of these truncated Hilbert transforms, evaluated at 0, taken over Σ_{n-1}; that is,

$$(1.2) \qquad 2 \int_{\varepsilon \leqslant |y| \leqslant \delta} f(x-y) k(y) dy = \int_{\Sigma_{n-1}} (H^\delta_\varepsilon g^x_{y'}(0) \Omega(y') dy',$$

where dy' is the element of Lebesgue measure on Σ_{n-1}.

(2) Each of the operators $(H^*_{y'}f)(x) = \sup_{\varepsilon,\delta>0} |(H^\delta_\varepsilon g^x_{y'})(0)|$ satisfies the inequality

$$(1.3) \qquad \Big(\int_{\boldsymbol{R}^n} |(H^*_{y'}f)(x)|^p dx\Big)^{1/p} \leqslant A_p \|f\|_p$$

for $1 < p < \infty$, where the constant A_p is independent of y' and $f \in L^p(\boldsymbol{R}^n)$.

An easy application of Minkowski's integral inequality then gives us the existence of the singular integral operator

$$(Kf)(x) = \lim_{\varepsilon,\delta^{-1} \to 0} \int_{\varepsilon \leqslant |y| \leqslant \delta} f(x-y) k(y) dy \equiv \text{P.V.} \int_{\boldsymbol{R}^n} f(x-y) k(y) dy$$

for almost every $x \in \boldsymbol{R}^n$ and its boundedness in $L^p(\boldsymbol{R}^n)$. For details of this see [5] and [12].

Cotlar [7] has shown that the Hilbert transform, as well as many of its properties, can be generalized and put in the framework of ergodic theory. These results have been considerably simplified and extended by Calderón [4]. Briefly, the latter obtains a generalization of inequality (1.3) in the sense that, instead of $\boldsymbol{R}^n$, the functions f are defined on a general σ-finite measure space $\mathfrak{M}$ on which acts a one parameter group $\{U^t\}$, $-\infty < t < \infty$, of measure preserving transformations. Moreover, instead of taking the supremum of the collection of truncated Hilbert transforms he considers the supremum of a more general family of operators on $L^p(\boldsymbol{R})$ which commute with translations of the line. On the other hand, only a *single* one-parameter group $\{U^t\}$ is involved; consequently, neither Cotlar nor Calderón obtain "generalized" singular integral operators arising from taking means over an appropriate indexing family of one-parameter groups, as is done in (1.2).

In the case of a general Lie group G this can be done by considering operators having the form

$$(1.4) \qquad \int_{\Sigma_{n-1}} \Omega(y') \Big\{\text{P.V.} \int_{-\infty}^{\infty} f(u\exp(ty')) \frac{dt}{t}\Big\} dy'$$

where we have identified the Lie algebra of G with $\boldsymbol{R}^n$, $f \in L^p(G)$ and Ω is an odd function whose restriction to Σ_{n-1} is integrable (observe that (1.4) makes sense, but is 0, when Ω is even). Even though most examples we shall consider are essentially of this form, it turns out that it is more convenient to study operators having somewhat different forms which still allow us to apply these ideas. To illustrate this point let us examine a *Riesz transform* on $G = SU(2)$.

The only specific knowledge about $SU(2)$ we need is the form that Haar measure assumes for central functions. Any

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

in $SU(2)$ is conjugate to a matrix of the form

$$e(\lambda) = \begin{pmatrix} e^{-i\lambda} & 0 \\ 0 & e^{i\lambda} \end{pmatrix}$$

(the general element of a maximal torus). If $\tilde{f}$ is central and u is conjugate to $e(\lambda)$ then, by definition, $\tilde{f}(u) = \tilde{f}(e(\lambda))$; in this case,

$$(1.5) \qquad \int_{SU(2)} \tilde{f}(u) du = \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(e(\lambda)) \sin^2 \lambda d\lambda$$

(see [6], page 32). Since $\tilde{f}(w) = \int_{SU(2)} f(vwv^{-1}) dv$ is central whenever f is integrable, (1.5) gives us the general formula

$$(1.6) \qquad \int_{SU(2)} f(u) du = \frac{1}{\pi} \int_{SU(2)} \Big\{ \int_{-\pi}^{\pi} f(ve(\lambda)v^{-1}) \sin^2 \lambda d\lambda \Big\} dv.$$

A natural distance on $SU(2)$ is given by $d(u, v) = |||u-v|||$, where the last expression denotes the Hilbert–Schmidt norm of $u-v$. The operator in question is given by convolution with the kernel

$$K(u) = \frac{u_{12}}{|||u-e|||^4},$$

where $e = e(0)$ is the identity element [1]. The function $u \to |||u-e|||$ is obviously central and, therefore, it suffices to evaluate it when $u = e(\lambda)$ and we obtain $|||u-e||| = 2\sqrt{2}\left|\sin\frac{\lambda}{2}\right|$. A simple calculation now

[1] This operator differs from the Riesz transform studied in [6] by a convolution with an integrable kernel (see page 127).

gives us

$$K\big(ve(-\lambda)v^{-1}\big) = iv_{11}\overline{v_{21}}\sin\lambda/32\left|\sin\frac{\lambda}{2}\right|^4;$$

thus, using (1.6)

$$\text{P.V.}\int_{SU(2)} K(u)f(wu^{-1})\,du$$

$$= \frac{1}{\pi}\int_{SU(2)}\Big\{\text{P.V.}\int_{-\pi}^{\pi} K\big(ve(-\lambda)v^{-1}\big)f\big(wve(\lambda)v^{-1}\big)\sin^2\lambda\, d\lambda\Big\}\,dv$$

$$= \frac{i}{32\pi}\int_{SU(2)} v_{11}\overline{v_{21}}\left\{\int_{-\pi}^{\pi}\frac{\sin^3\lambda}{\left|\sin\frac{\lambda}{2}\right|^4} f\big(wve(\lambda)v^{-1}\big)\,d\lambda\right\}dv.$$

Letting $k(\lambda) = \frac{i}{4\pi}\left(\cot\frac{\lambda}{2} - \frac{\sin\lambda}{2}\right)$ and $U_v^\lambda w = wve(\lambda)v^{-1}$, the inner integral above takes the form

$$(1.7)\qquad (H_v f)(w) = \text{P.V.}\int_{-\pi}^{\pi} k(\lambda)f(U_v^\lambda w)\,d\lambda\ (^2).$$

It is an immediate consequence of the classical M. Riesz inequality for the conjugate function on the circle (convolution with $\cot\frac{\lambda}{2}$) that H_v is a bounded operator on $L^p\big(SU(2)\big)$, $1 < p < \infty$, with operator norm independent of v. In fact, since, for each v, the transformations of the one parameter group $\{U_v^\mu\}$ are measure preserving (on $SU(2)$ with Haar measure), we have

$$\int_{SU(2)} |(H_v f)(w)|^p\,dw = \int_{SU(2)} |(H_v f)(U_v^\mu w)|^p\,dw$$

$$= \frac{1}{2\pi}\int_{SU(2)}\Big\{\int_{-\pi}^{\pi}|(H_v f)(U_v^\mu w)|^p\,d\mu\Big\}\,dw$$

$$= \frac{1}{2\pi}\int_{SU(2)}\Big\{\int_{-\pi}^{\pi}\Big|\text{P.V.}\int_{-\pi}^{\pi}k(\lambda)f(U_v^{\lambda+\mu}w)\,d\lambda\Big|^p d\mu\Big\}\,dw$$

$$\leqslant \frac{1}{2\pi}\int_{SU(2)}\Big\{c_p\int_{-\pi}^{\pi}|f(U_v^\lambda w)|^p\,d\lambda\Big\}\,dw = \frac{c_p}{2\pi}\,\|f\|^p_{L^p(SU(2))}.$$

(2) In view of the identity $\text{P.V.}\int_{-\pi}^{\pi} f(x-t)\cot\frac{t}{2}\,dt = c\ \text{P.V.}\int_{-\pi}^{\pi} f(x-t)\frac{dt}{t}$ for periodic f, the integral in (1.7) is really of the same form as the inner integral in (1.4). In both cases one is averaging over a collection of one-parameter subgroups of $G = SU(2)$.

We now obtain the boundedness of the Riesz transform by an immediate application of Minkowski's integral inequality to

$$\Big(\int_{SU(2)}\Big|\text{P.V.}\int_{SU(2)} K(u)f(wu^{-1})\,du\Big|^p dw\Big)^{1/p}$$

$$= \Big(\int_{SU(2)}\Big|\int_{SU(2)} v_{11}\overline{v_{21}}(H_v f)(w)\,dv\Big|^p dw\Big)^{1/p}.$$

It is evident that this argument extends to general compact Lie groups G with a maximal torus T^m playing the role of $\{e(\lambda)\}$ and a singular integral replacing the conjugate function operator. In fact, the Riesz transform we have just considered is induced by a kernel which is a coefficient of matrix-valued functions having the form

$$K(u) = \frac{T(u)-I}{[p(u)]^{N+1}},$$

where T is a finite dimensional unitary representation of the compact group G, p is a central function "measuring the distance" from the identity (3) and N is the dimension of G. Writing

$$(K*f)(v) = T(v^{-1})\Big\{\text{P.V.}\int_G \frac{T(u)-T(v)}{[p(v^{-1}u)]^{N+1}}f(u)\,du\Big\}$$

and taking into account the fact that T is unitary leads us to consider operators of the following type

$$(1.8)\qquad \text{P.V.}\int_G \frac{A(u)-A(v)}{[p(v^{-1}u)]^{N+1}}f(u)\,du,$$

where A is a Lipschitz function on G; that is, $|A(u)-A(v)| \leqslant cp(v^{-1}u)$.

The method we described above is still applicable to such operators, even though they are not of convolution type, if the role played by the Hilbert transform is assumed by the commutator singular integral of Calderón [3] on the torus.

These techniques have a wider range of applications than the extension of singular integral theory. They also yield a refinement and generalization of theorems of de Leeuw and Fife on restrictions of multipliers to subgroups and certain measure spaces (see [8] and [9]) (see example (ii) in § 3). Moreover, we shall also indicate in § 3 how matrix multiplier theorems for compact Lie groups can be obtained by these means. The connections of these considerations with spectral theory and Stone's theorem will also be described in that section.

(3) The construction of such functions p is given explicitly by N. J. Weiss [12].

In the next section we state and prove the results that generalize the ergodic versions of the Hilbert transform and the Calderón commutator singular integral.

§ 2. Operators induced by representations. Suppose G is a locally compact group satisfying the following property: Given a compact subset C of G and $\varepsilon > 0$ then there exists an open neighborhood V of the identity e having finite measure such that

$$\frac{\mu(VC^{-1})}{\mu(V)} \leqslant 1+\varepsilon, \tag{2.1}$$

where μ is, say, left Haar measure (4). If G is compact this property is clearly valid; if G is abelian it is also true (see lemma 31.36 on page 234 of [11]). Let us suppose, further, that R is a representation of G acting on functions on a σ-finite measure space $\mathfrak{M}$ (5) satisfying, for some $p \in [1, \infty]$

$$\int\limits_{\mathfrak{M}} |(R_u f)(x)|^p\,dx \leqslant c^p \int\limits_{\mathfrak{M}} |f(x)|^p\,dx, \tag{2.2}$$

where c is independent of $f \in L^p(\mathfrak{M})$ $u \in G$. Observe that an application of (2.2) to $g = R_u f$, with u replaced by u^{-1}, gives us

$$\int\limits_{\mathfrak{M}} |f(x)|^p\,dx \leqslant c^p \int\limits_{\mathfrak{M}} |(R_u f)(x)|^p\,dx. \tag{2.3}$$

Let k be an integrable function on G with compact support C. Associated with k is the convolution operator $\varphi \to \int\limits_G k(u)\varphi(vu^{-1})\,du$ on L^p; let A denote the norm of this operator. The kernel k and the representation R induce an operator $\hat{k}(R) = K$ on functions defined on $\mathfrak{M}$:

$$(Kf)(x) = \int\limits_G k(u)(R_{u^{-1}}f)(x)\,d\mu(u) \ (6).$$

We shall show that K is a bounded operator on $L^p(\mathfrak{M})$ with operator norm not exceeding c^2A. First observe that, as a consequence of (2.3), we have

$$\int\limits_{\mathfrak{M}} |(Kf)(x)|^p\,dx \leqslant c^p \int\limits_{\mathfrak{M}} |(R_v Kf)(x)|^p\,dx$$

(4) These groups are called amenable (see [10]).

(5) That is, $R_u f$ is a measurable function whenever f is measurable on $\mathfrak{M}$ and $R_{uv}f = R_u(R_v f)$. We assume that R_e is the identity transformation and that $v \to R_v$ is continuous as a mapping of G into the bounded operators on $L^p(\mathfrak{M})$.

(6) In this case we tacitly assumed the μ-integrability of the integrand. We also would like to point out that the notation $\hat{k}(R)$ is consistent with the definition of the Fourier transform given in [11].

for all $v \in G$. Let us choose $\varepsilon > 0$ and V an open neighborhood of e, having finite measure, such that (2.1) holds (C being the support of k). Then integrating the left and right sides of this last inequality over V and dividing by $\mu(V)$ we obtain

$$\int\limits_{\mathfrak{M}} |(Kf)(x)|^p\,dx \leqslant \frac{c^p}{\mu(V)} \int\limits_V \Big\{ \int\limits_{\mathfrak{M}} |(R_v Kf)(x)|^p\,dx \Big\}\,dv.$$

Let $\chi_{VC^{-1}}$ be the characteristic function of VC^{-1}. If $u \in C$ and $v \in V$, obviously $\chi_{VC^{-1}}(vu^{-1}) = 1$. Using this as well as the fact that R is a representation we see that the last expression is equal to

$$\frac{c^p}{\mu(V)} \int\limits_{\mathfrak{M}} \Big\{ \int\limits_V \Big| \int\limits_G k(u)(R_{vu^{-1}}f)(x)\,\chi_{VC^{-1}}(vu^{-1})\,du \Big|^p dv \Big\}\,dx.$$

Since the norm of the convolution operator defined by k is A, this is less than or equal to

$$\frac{(Ac)^p}{\mu(V)} \int\limits_{\mathfrak{M}} \Big\{ \int\limits_G |(R_u f)(x)\,\chi_{VC^{-1}}(u)|^p\,du \Big\}\,dx.$$

An application of Fubini's theorem, (2.2) and (2.1) shows that this is smaller than

$$\frac{(c^2A)^p}{\mu(V)}\,\mu(VC^{-1}) \int\limits_{\mathfrak{M}} |f(x)|^p\,dx \leqslant (1+\varepsilon)\,c^{2p}A^p\,\|f\|_p^p.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small we obtain the desired result

$$\|Kf\|_p \leqslant c^2 A\,\|f\|_p. \tag{2.4}$$

This argument is an adaptation of a proof of Calderón [4] to our situation. As was mentioned in the introduction, he assumed that a one-parameter group of measure preserving transformations acted on $\mathfrak{M}$ instead of a representation R satisfying (2.2). On the other hand, he considered sequences of operators, and the maximal operator they defined, instead of a single operator, as we have done. The analog of this in our situation would be the following: Suppose $\{k_n\}$ is a sequence of integrable functions on G having bounded support. Let

$$(T\varphi)(v) = \sup_n \Big| \int\limits_G k_n(u)\,\varphi(vu^{-1})\,du \Big|$$

and suppose

$$\|T\varphi\|_{L^p(G)} \leqslant A\,\|\varphi\|_{L^p(G)}.\ (7) \tag{2.5}$$

(7) Calderón also considers weak-type (p, p) operators. The reader can easily verify that the extension we carry out for strong type is also valid for weak-type operators.

If $(K_nf)(x) = \int\limits_G k_n(u)(R_{u^{-1}}f)(x)\,d\mu(u)$ let

$$(T^{\#}f)(x) = \sup_n |(K_nf)(x)|.$$

Provided we make an assumption that is stronger than (2.3) we can then easily carry out the full argument of Calderón in order to obtain the L^p-boundedness of the maximal operator $T^{\#}$:

$$\text{(2.6)}\qquad \Big(\int\limits_{\mathfrak{M}} |(T^{\#}f)(x)|^p\,dx\Big)^{1/p} \leqslant Ac^2\Big(\int\limits_{\mathfrak{M}} |f(x)|^p\,dx\Big)^{1/p}.$$

This additional assumption, which is certainly true if the representation R consists of transformations on the measure space $\mathfrak{M}$, is

$$\text{(2.3')}\qquad \int\limits_{\mathfrak{M}} [\sup_n |g_n(x)|]^p\,dx \leqslant c^p \int\limits_{\mathfrak{M}} [\sup_n |(R_ug_n)(x)|]^p\,dx$$

(see example (i) in the next section). We leave the details of the proof of this extension to the reader.

In the introduction we mentioned the extension of the commutator singular integrals of Calderón. In order to obtain such an extension we first prove an analog to inequality (2.4). We again assume that G is a locally compact group satisfying (2.1). On the other hand, we make the more restrictive hypothesis on the representation R that it consist of measure-preserving transformations of the space $\mathfrak{M}$. The transformation we shall study will have the form

$$(Kf)(x) = \int\limits_G k(x, R_ux, u)f(R_ux)\,du,$$

where $k(x, y, u)$ is a measurable function on $\mathfrak{M}\times\mathfrak{M}\times G$ which is 0 if u does not belong to a compact set $C \subset G$. Moreover, we assume that for each $x\in\mathfrak{M}$ the kernel

$$k_x(v, u) = k(R_vx, R_{u^{-1}}R_vx, u) = k(R_vx, R_{u^{-1}v}x, u)$$

satisfies

$$\Big(\int\limits_G\Big|\int\limits_G k_x(v, u)g(u^{-1}v)\,du\Big|^p dv\Big)^{1/p} \leqslant A\Big(\int\limits_G |g(u)|^p\,du\Big)^{1/p}\ (^8),$$

where A is independent of $x\in\mathfrak{M}$ and $g\in L^p(G)$. We shall show that K is a bounded operator with norm not exceeding A:

$$\text{(2.7)}\qquad \Big(\int\limits_{\mathfrak{M}} |(Kf)(x)|^p\,dx\Big)^{1/p} \leqslant A\Big(\int\limits_{\mathfrak{M}} |f(x)|^p\,dx\Big)^{1/p}.$$

(8) If $k(x, y, u)$ depends only on u, so does $k_x(v, u)$, and we have, eassentially, the situation we considered in the first part of this section. The only difference is that we changed the convolution in order to conform with the usual notation for transformations.

The argument we use for establishing this inequality is practically the same as the one used to show the validity of (2.4). We include it there for the sake of completeness: Since R_v is measure-preserving

$$\int\limits_{\mathfrak{M}} |(Kf)(x)|^p\,dx = \int\limits_{\mathfrak{M}} |(Kf)(R_vx)|^p\,dx$$

for all $v\in G$. Let V and $\varepsilon > 0$ be as before and $\chi_{C^{-1}V}$ the characteristic function of $C^{-1}V$ (we are assuming (2.1) with $C^{-1}V$ instead of VC^{-1}) then $\|Kf\|_p^p$ is equal to

$$\frac{1}{\mu(V)}\int\limits_{\mathfrak{M}}\Big\{\int\limits_V\Big|\int\limits_G k_x(v, u)f(R_{u^{-1}v}x)\,\chi_{C^{-1}V}(u^{-1}v)\,du\Big|^p dv\Big\}\,dx$$

$$\leqslant \frac{A^p}{\mu(V)}\int\limits_{\mathfrak{M}}\Big\{\int\limits_G |f(R_ux)\,\chi_{C^{-1}V}(u)|^p\,du\Big\}\,dx = A^p\,\frac{\mu(C^{-1}V)}{\mu(V)}\int\limits_{\mathfrak{M}} |f(x)|^p\,dx.$$

Inequality (2.7) now follows from an application of (2.1).

As was the case above we can obtain a result analogous to (2.6) for suprema of operators of this type; however, because the representation consists of measure preserving transformations, we do not need the analog of assumption (2.3') (for the same reason we did not need analogs of (2.2) or (2.3) in the argument that was just given).

§ 3. Applications.

(i) Isometric representations on $L^p(\mathfrak{M})$. Suppose G is a locally compact group satisfying (2.1) and S is a representation of G consisting of transformations of a σ-finite measure space $\mathfrak{M}$ for which we have a "change of variables" formula

$$\text{(3.1)}\qquad \int\limits_{\mathfrak{M}} f(S_{u^{-1}}x)\,\Delta_u(x)\,dx = \int\limits_{\mathfrak{M}} f(x)\,dx$$

where

$$\text{(3.2)}\qquad \Delta_{vu}(x) = \Delta_v(x)\,\Delta_u(S_{v^{-1}}x)$$

for all $u, v\in G$ and $x\in\mathfrak{M}$. (For example, if G is a subgroup of $GL(n)$ satisfying (2.1), $\mathfrak{M} = \mathbf{R}^n$ and $S_u = u$ then $\Delta_u(x) = |1/\det u|$. Given $p\geqslant 1$ let $\alpha = (1/p)+i\gamma$, where γ is any real number. Then

$$(R_u^\alpha f)(x) = f(S_{u-1}x)[\Delta_u(x)]^\alpha$$

defines a representation R^α of G acting on $L^p(\mathfrak{M})$ satisfying

$$\int\limits_{\mathfrak{M}} |(R_u^\alpha f)(x)|^p\,dx = \int\limits_{\mathfrak{M}} |f(x)|^p\,dx$$

(thus, (2.2), (2.3) and (2.3') are satisfied). The results of § 2 are then applicable.

Such representations arise naturally when $\mathfrak{M}$ is the boundary of Symmetric Spaces (see the articles in [2] dealing with such boundaries).

(ii) Multiplier operators defined by actions of locally compact Abelian groups. Let G be a locally compact abelian group and S a representation of G consisting of measure preserving transformations on a σ-finite measure space $\mathfrak{M}$. Then, as was seen in the first application, S defines a representation R of G acting on $L^p(\mathfrak{M})$, $1 \leqslant p \leqslant \infty$, by letting $(R_u f)(x) = f(S_{-u}x)$ for $u \in G$ and $f \in L^p(\mathfrak{M})$. We shall show how a bounded "Fourier multiplier" operator on $L^p(G)$ induces, by means of R, a bounded operator on $L^p(\mathfrak{M})$. Special cases of this situation have been considered by de Leeuw in the case of certain homogeneous spaces [8], and by Fife [9] when a one-parameter group acts on $\mathfrak{M}$.

We already observed in § 2 that condition (2.1) is valid for all abelian locally compact groups. In order to avoid certain technical difficulties we shall further restrict ourselves to σ-compact groups. If G is such a group then it is known (see [11], Vol. I, page 255) that there exists a sequence of open sets $\{H_n\}$ having compact closure such that

$$(3.3)\quad \text{(a)}\ H_n \subset H_{n+1};\qquad \text{(b)}\ \bigcup_{n=1}^{\infty} H_n = G;\qquad \text{(c)}\ \lim_{n\to\infty} \frac{\mu([u+H_n]\cap H_n)}{\mu(H_n)} = 1.$$

Such a sequence of sets permits us to introduce a notion of "Cesàro Summability" on $\hat{G}$. This, in turn, can be used to describe those Fourier multiplier operators that can be approximated by convolution operators with kernels that belong to $L^1(G)$ and have compact support. Let us begin by showing how a sequence satisfying (3.3) can be used to introduce such a summability process. From now on, we let λ_E denote the characteristic function of E (the letter χ being preserved for characters).

LEMMA 3.4. *Suppose $\{H_n\}$ is a sequence of open subsets of G satisfying (3.3) and $\varphi_n(u) = (\lambda_{H_n} * \lambda_{-H_n})(u)/\mu(H_n)$, for $u \in G$ and $n = 1, 2, 3, \ldots$ Then the sequence $\{\hat{\varphi}_n\}$ of Fourier transforms satisfies*

$$(1)\qquad \hat{\varphi}_n(\chi) = \frac{|\hat{\lambda}_{H_n}(\chi)|^2}{\mu(H_n)} \geqslant 0;$$

$$(2)\qquad \int_{\hat{G}} \hat{\varphi}_n(\chi)\,d\chi = 1;$$

(3) *if C is a compact neighborhood of the identity of $\hat{G}$ then* $\lim_{n\to\infty} \int_{\chi\notin C} \hat{\varphi}_n(\chi)\,d\chi = 0.$

This lemma is well-known; however, for the sake of completeness, we shall indicate how it can be derived easily from (3.3). Property (1) is obvious. Using (1), the second property is then an immediate consequence of Plancherel's theorem. In order to show (3) we first choose

$\varphi \in L^1(G)$ such that for all $u \in G$, $\varphi(u) > 0$, $\varphi(u) = \varphi(-u)$ and, moreover, $\int_G \varphi(u)\,d\mu(u) = 1$.

It then follows that $|\hat{\varphi}(\chi)| < 1$ whenever χ is not the identity character 1 (since, for $\chi \in \hat{G}$ fixed, $\{\chi(u): u \in G\}$ is a subgroup of the unit circle; if it is not the trivial subgroup then there exists at least one u for which $\mathrm{Re}(\chi(u) < 0)$. Thus, we can find $\varepsilon > 0$ such that

$$|\hat{\varphi}(\chi)| < 1-\varepsilon \quad \text{for all } \chi \notin C$$

(if $\hat{G}$ is not compact, $\hat{\varphi}(\chi) \to 0$ as χ "tends to infinity"). Consequently,

$$\int_G \varphi(u)\frac{\mu([u+H_n]\cap H_n)}{\mu(H_n)}\,d\mu(u) = \int_{\hat{G}} \hat{\varphi}(\chi)\hat{\varphi}_n(\chi)\,d\chi \leqslant \Big|\int_{\chi\in C}\Big| + \Big|\int_{\chi\notin C}\Big|$$
$$\leqslant \int_{\chi\in C}\hat{\varphi}_n(\chi)\,d\chi + (1-\varepsilon)\int_{\chi\notin C}\hat{\varphi}_n(\chi)\,d\chi$$
$$= \int_{\hat{G}}\hat{\varphi}_n(\chi)\,d\chi - \varepsilon\int_{\chi\notin C}\hat{\varphi}_n(\chi)\,d\chi.$$

This shows

$$0 \leqslant \varepsilon\int_{\chi\notin C}\hat{\varphi}_n(\chi)\,d\chi \leqslant 1 - \int_G \varphi(x)\frac{\mu([u+H_n]\cap H_n)}{\mu(H_n)}\,d\mu(u).$$

Since, by (3.3), the last integral tends to 1 as $n\to\infty$ this gives us property (3).

We shall say that a bounded measurable function m on $\hat{G}$ is normalized (*with respect to* $\{\hat{\varphi}_n\}$) if

$$\lim_{n\to\infty}(\hat{\varphi}_n * m)(\chi) = m(\chi)$$

for all $\chi \in \hat{G}$. It is an immediate consequence of Lemma 3.4 that a bounded continuous function on $\hat{G}$ is normalized.

If m is a bounded measurable function on $\hat{G}$ then $f \to (m\hat{f})^{\check{}}$ is a bounded operator on $L^2(G)$. If, for some $p \in [1, \infty]$, this operator is bounded in the L^p-norm we say that m is a *Fourier multiplier* for $L^p(G)$. We let $N_p(m)$ denote the norm of this operator.

LEMMA 3.5. *Suppose m is normalized and is a Fourier multiplier for $L^p(G)$ then there exists a uniformly bounded sequence $\{m_n\}$ of functions satisfying*

$$(1)\qquad m(\chi) = \lim_{n\to\infty} m_n(\chi) \quad \textit{for all } \chi \in \hat{G};$$

(2) *if $h \in L^2(G)$ has compact support then the function $(m_n\hat{h})$ has compact support and is in $L^1(G)$;*

$$(3)\qquad N_p(m_n) \leqslant N_p(m).$$

Proof. Define, for $\chi \in \hat{G}$,

$$m_n(\chi) = (\hat{\varphi}_n * m)(\chi).$$

Since m is normalized, (1) clearly holds. Let h be in $L^2(G)$ having support in a compact set 0. An easy computation shows that the L^2-function

$$k_n = (m_n \hat{h})^{\check{}} \tag{3.6}$$

has support in $(H_n - H_n) - 0$, and, thus, belongs to $L^1(G)$. This establishes property (2). To show (3) we first observe that for each $\chi_0 \in \hat{G}$ the function $(\tau_{\chi_0} m)(\chi) = m(\chi\chi_0^{-1})$ is a Fourier multiplier for $L^p(G)$ with norm $N_p(m)$. Next an application of Minkowski's integral inequality and parts (1) and (2) of (3.4) give us

$$\Big(\int_G \Big|\int_{\hat{G}} \Big[\int_{\hat{G}} m(\chi\chi_0^{-1})\hat{\varphi}_n(\chi_0)\,d\chi_0\Big]\hat{f}(\chi)\overline{\langle\chi, u\rangle}\,d\chi\Big|^p du\Big)^{1/p}$$
$$\leqslant \int_{\hat{G}} \hat{\varphi}_n(\chi_0)\,\|[(\tau_{\chi_0} m)\hat{f}]^{\check{}}\|_p\,d\chi_0 \leqslant N_p(m)\,\|f\|_p.$$

Thus, the operator $f \to ([m*\hat{\varphi}_n]\hat{f})^{\check{}}$ is bounded with norm not exceeding $N_p(m)$.

Let us now apply the generalized theorem of Stone to the representation R defined by S as was done at the beginning of (ii). Thus, we obtain a spectral measure E on the character group $\hat{G}$ such that

$$R_u = \int_{\hat{G}} \langle\chi, u\rangle\,dE(\chi)$$

for all $u \in G$. Hence, with k_n defined by (3.6) and $f, g \in L^2(\mathfrak{M})$

$$\int_G k_n(u)(R_{-u}f, g)\,d\mu(u) = \int_{\hat{G}} \Big\{\int_G k_n(u)\langle\chi, u\rangle\,d\mu(u)\Big\}\,d(E(\chi)f, g)$$
$$= \int_{\hat{G}} \hat{h}(\chi)\,m_n(\chi)\,d(E(\chi)f, g).$$

But, by the dominated convergence theorem of Lebesgue and lemma (3.5), the last expression converges to

$$\int_{\hat{G}} \hat{h}(\chi)\,m(\chi)\,d(E(\chi)f, g).$$

Since the Fourier multiplier operator defined by m_n, followed by convolution with h, is the operator $f \to (\hat{k}_n f)^{\check{}}$, it follows from (3.5), part (3) that $N_p(\hat{k}_n) \leqslant \|h\|_1 N_p(m_n) \leqslant \|h\|_1 N_p(m)$. Thus, applying (2.4) to the integrable kernel with compact support k_n and then passing to the limit we see that the operator

$$\int_{\hat{G}} \hat{h}(\chi)\,m(\chi)\,dE(\chi)$$

has norm not exceeding $N_p(m)$ for all $h \in L^2(G)$ having compact support and L^1-norm equal to 1. It follows that the same must be true for the operator $\int_{\hat{G}} m(\chi)\,dE(\chi)$. We have, therefore, obtained the following result:

THEOREM 3.7. *Suppose m is a normalized function of $\hat{G}$ such that the operator*

$$f \to (m\hat{f})^{\check{}}$$

is bounded on $L^p(G)$ with operator norm $N_p(m)$ for some $p \in [1, \infty]$. Then the operator

$$\int_{\hat{G}} m(\chi)\,dE(\chi)^{(9)}$$

is a bounded operator on $L^p(\mathfrak{M})$ with operator norm not exceeding $N_p(m)$.

It $\hat{H}$ is a closed subgroup of $\hat{G}$ and $H = \{u \in G\colon \langle\chi, u\rangle = 1$ for all $\chi \in \hat{H}\}$ then H is a closed subgroup of G. Let $\mathfrak{M} = G/H$ and S the representation defined by the action of G on the cosets in G/H. Then Theorem 3.7 asserts, in this case:

COROLLARY 3.8. *Suppose m is a normalized function on $\hat{G}$ which is a Fourier multiplier for $L^p(G)$ with operator norm $N_p(m)$ for some $p \in [1, \infty]$. Then the restriction of m to $\hat{H}$ is an L^p multiplier for $L^p(G/H)$ with operator norm not exceeding $N_p(m)$.*

This is a version of the original theorem of de Leeuw. Theorem 3.7 generalizes a theorem of Fife. If $G = \boldsymbol{R}^n$ and $\hat{H} = Z^n = \{j = (j_1, \ldots, j_n)\colon j_l$ integral$\}$ then H can be naturally identified with Z^n and G/H with the torus T^n. Corollary 3.8 then tells us that a normalized function m on $\boldsymbol{R}^n$ which is a Fourier multiplier for $L^p(\boldsymbol{R}^n)$ with operator norm $N_p(m)$ has the property that $\{m(j)\}$ (the restriction of m *to* Z^n) is a Fourier multiplier for $L^p(T^n)$ with operator norm less than or equal to $N_p(m)$.

We remind the reader that this method also yields similar results for suprema of operators (see (2.6)) and can be used to obtain weak-type inequalities. For example, for the Calderón-Zygmund singular integrals described in § 1 we obtain the fact that the operator $K^{\#}$ defined for periodic functions f on $\boldsymbol{R}^n$ by

$$(K^{\#}f)(x) = \sup_{\varepsilon,\delta>0}\Big|\int_{\varepsilon<|y|<\delta} \frac{\Omega(y)}{|y|^n}\,f(x-y)\,dy\Big|,$$

is bounded on $L^p(T^n)$, $1 < p < \infty$.

(iii) Generalized Calderón commutator operators. Let τ be an ergodic flow on the σ-finite measure space $\mathfrak{M}$ (that is, τ is a one-param-

(9) This operator can, formally, be written in the form $\int_G \check{m}(u) R_{-u}\,du$.

eter group of measure preserving transformations). A function A on $\mathfrak{M}$ is said to be *Lipschitz relative to* τ if

$$|A(\tau_s x)-A(x)| \leqslant (\text{const})\,|s|$$

for all $x\in\mathfrak{M}$ and $s\in\boldsymbol{R}$, where the constant is independent of $x\in\mathfrak{M}$. In the sequel we shall assume, for simplicity, that this constant is 1. If $0<\varepsilon<\delta$ let

$$k_\varepsilon^\delta(x,y,s) = \begin{cases} \dfrac{A(x)-A(y)}{s^2} & \text{if } \varepsilon<|s|<\delta,\\ 0 & \text{otherwise}. \end{cases}$$

We claim that this kernel k satisfies the conditions we need in order to obtain inequality (2.7) when $G=\boldsymbol{R}$. In fact this is an immediate consequence of the following result of Calderón:

If a is a Lipschitz function on the line then

$$\Big(\int_{-\infty}^{\infty}\Big|\int_{\varepsilon<|t|<\delta}\frac{a(s)-a(s-t)}{t^2}\varphi(s-t)\,dt\Big|^p ds\Big)^{1/p} \leqslant c_p\|\varphi\|_p$$

for $1<p<\infty$, where C_p depends only on p and a's Lipschitz constant.

Hence, we obtain the result:

THEOREM 3.9. *The operator K_ε^δ defined by*

$$(K_\varepsilon^\delta f)(x) = \int_{\varepsilon<|s|<\delta}\frac{A(x)-A(\tau_s x)}{s^2}f(\tau_s x)\,ds$$

is a bounded operator on $L^p(\mathfrak{M})$, $1<p<\infty$, with operator norm independent of ε and δ.

This theorem can be extended to an analogous result involving an action S of $\boldsymbol{R}^n$ on $\mathfrak{M}$ (instead of the one-parameter group we have just considered). Suppose, then, that A is a function on $\mathfrak{M}$. We say that A is *Lipschitz relative to* S if

$$|A(S_u x)-A(x)| \leqslant (\text{const})\,|u|$$

for all $x\in\mathfrak{M}$ and $u\in\boldsymbol{R}^n$. Let h be an even function homogeneous of degree $-(n+1)$ which is integrable in $|u|\geqslant 1$.

COROLLARY 3.10. *The operators C_ε^δ, $0<\varepsilon<\delta$, defined by*

$$(C_\varepsilon^\delta f)(x) = \int_{\varepsilon<|u|<\delta} h(u)[A(x)-A(S_u x)]f(S_u x)\,du$$

have uniformly bounded norms as transformations of $L^p(\mathfrak{M})$, $1<p<\infty$.

To reduce this to (3.9) we observe that

$$(C_\varepsilon^\delta f)(x) = \tfrac{1}{2}\int_{\Sigma_{n-1}} h(u')\Big\{\int_{\varepsilon<|t|<\delta}\frac{A(x)-A(S_{tu'}x)}{t^2}f(S_{tu'}x)\,dt\Big\}du'.$$

Now an application of (3.9) and Minkowski's integral inequality gives (3.10) in complete analogy with the method described in § 1 for singular integrals.

Calderón, himself, extended his commutator operators to n-dimensions [3]. In fact, he also includes odd functions h as well (satisfying more restrictive conditions). A corresponding more general version of corollary 3.10 is obtainable directly from his results and inequality (2.7).

We again stress the special case when $\mathfrak{M}=T^n$ and $G=\boldsymbol{R}^n$. In this case the Lipschitz function A is periodic. With h as above, we have the uniform boundedness of the $L^p(T^n)$-operator norms of the transformations

$$(C_\varepsilon^\delta f)(x) = \int_{\varepsilon<|u|<\delta} h(u)[A(x)-A(x-u)]f(x-u)\,du, \tag{3.11}$$

where f is periodic.

(iv) Singular integrals on compact Lie groups. Let G be an N-dimensional compact connected Lie group and T^n a fixed maximal torus. We shall identify T^n with the cube $Q=\{\theta=(\theta_1,\theta_2,\ldots,\theta_n)\in\boldsymbol{R}^n: -\pi\leqslant\theta_j<\pi,\ j=1,\ldots,n\}$ via the map $\theta\mapsto t(\theta)\in T^n$. Let $m(u)=\det(Ad_u-I)$, where Ad denotes the adjoint representation of G on its Lie algebra. The function m is central; thus, it is completely determined by its values on T^n which are known to be

$$m(t(\theta)) = \prod_{j=1}^{q}4\sin^2(a_j(\theta)) \tag{3.12}$$

for $\theta\in Q$, where $2q=N-n$ and a_j is a non-trivial linear functional on $\boldsymbol{R}^n$ (see [1], Chapter VI). Moreover, m can be used to evaluate the Haar integral of all central functions: if $f\in L^1(G)$ is central then

$$\int_G f(u)\,d\mu(u) = \frac{1}{|w|}\int_{T^n} f(t)m(t)\,dt,$$

where $|w|$ denotes the order of the Weyl group. This formula allows us to obtain the following extension of (1.6):

$$\int_G f(u)\,du = \frac{1}{|w|}\int_G\Big\{\int_{T^n} f(vtv^{-1})m(t)\,dt\Big\}dv. \tag{3.13}$$

We choose a central function p such that

$$p(t(\theta)) = \Big(\sum_{j=1}^{n}\theta_j^2\Big)^{1/2} = |\theta|$$

for θ in a neighborhood of the origin and $p(t) > \delta > 0$ for t outside this neighborhood (see [13]).

THEOREM 3.14. *Let T be a finite dimensional unitary representation of G, then the operator*

$$(K_T^0 f)(v) = \text{P.V.} \int_G \frac{T(u)-T(u^{-1})}{[p(u)]^{N+1}} f(vu^{-1})\,du$$

is a bounded operator on $L^p(G)$, $1 < p < \infty$.

Proof. Using (3.13) we have

$$(K_T^0 f)(v) = \frac{1}{|w|}\int_G T(u)\Big\{\text{P.V.} \int_{T^n} \frac{T(t)-T(t^{-1})}{[p(t)]^{N+1}} f(vutu^{-1})\,m(t)\,dt\Big\}\, T(u^{-1})\,du.$$

It follows immediately from (3.12) that

$$H(t) = \frac{T(t)-T(t^{-1})}{[p(t)]^{n+1}}\,\frac{m(t)}{[p(t)]^{N-n}} = \frac{\Omega(\theta)}{|\theta|^n} + O(|\theta|^{1-n}),$$

where Ω is an odd matrix-valued function that is homogeneous of degree 0 and bounded. This shows that the kernel $H(t)$ in the inner integral defines a bounded operator on $L^p(T^n)$, $1 < p < \infty$ (see (3.8) and the last paragraph of (ii)).

Let R^u, $u \in G$, be the representation of T^n acting on functions on G defined by $(R_t^u f)(v) = f(vutu^{-1})$. Applying (2.3) we see that the operators defined by the expression in the curly brackets are bounded on $L^p(G)$ independently of $u \in G$. Now, an application of Minkowski's integral inequality (using matrix operator norms and taking into account that $T(u)$ is unitary) yields the theorem.

THEOREM 3.15. *Suppose A is a function on G satisfying the Lipschitz condition $|A(u)-A(v)| \leqslant cp(u^{-1}v)$ then*

$$(Cf)(u) = \text{P.V.} \int_G \frac{A(u)-A(uv)}{[p(v)]^{N+1}} f(uv)\,dv$$

is a bounded operator on $L^p(G)$, $1 < p < \infty$.

Proof. We proceed in a manner completely analogous to the last proof. Using (3.13) we have

$$(Cf)(u) = \int_G \Big\{\text{P.V.} \int_{T^n} \frac{A(u)-A(vutu^{-1})}{|t|^{N+1}}\, m(t) f(vutu^{-1})\,dt\Big\}\,du.$$

It follows from Corollary (3.10) that the expression in the curly brackets defines a bounded operator on $L^p(G)$ with norm independent of u, where $h(t) = \dfrac{m(t)}{[p(t)]^{N+1}} + O([p(t)]^{-n+1})$ is an even homogeneous function of degree $-(n+1)$ (Instead of $\mathbf{R}^n$ the group T^n is acting on $\mathfrak{M} = G$; (3.11) assures us that this can be done). As before, the theorem now follows from Minkowski's integral inequality.

COROLLARY 3.16. *Let T be a finite dimensional unitary representation of G, then the operator*

$$(K_T f)(v) = \text{P.V.} \int_G \frac{T(u^{-1})-J}{[p(u)]^{N+1}} f(vu^{-1})\,du$$

is a bounded operator on $L^p(G)$, $1 < p < \infty$.

This is immediately reduced to (3.15) by multiplying this operator on the left by the unitary operator $T(v)$ (we can use, say, the Hilbert–Schmidt norm of T to reduce this to scalar-valued functions).

Moreover, we can obtain precise estimates of the operator norms of the matrix entries of K_T (with respect to a basis that diagonalizes $T(u)$ for $u \in T^n$). These estimates involve the Lipschitz norms

$$l_T = \sup_{u \in G} \frac{\|T(u)-I\|}{p(u)},$$

where $\|\ \|$ denotes the operator norm of $T(u)-I$. Let $(t_{k,j})$ be the matrix entries with respect to such a basis. We first observe that for $w \in G$ and $t \in T^m$ we have (since T is unitary and $T(t)$ diagonal for $t \in T^m$)

(3.17) $$t_{kj}(wtw^{-1}) - \delta_{kj} = \sum_{l=1}^{d_T} t_{kl}(w)\overline{t_{jl}(w)}\,(t_{ll}(t)-1),$$

where d_T is the dimension of the representation space. Thus,

$$\text{P.V.} \int_G \frac{t_{kj}(u^{-1})-\delta_{kj}}{[p(u)]^{N+1}} f(vu^{-1})\,du$$
$$= \int_G \Big\{\text{P.V.} \int_{T^n} \frac{t_{kj}(wt^{-1})-\delta_{kj}}{[p(t)]^{N+1}}\, m(t) f(vwt^{-1}w^{-1})\,dt\Big\}\,dw$$
$$= \sum_{l=1}^{d_T} \int_G t_{kl}(w)\overline{t_{jl}(w)}\Big\{\text{P.V.} \int_{T^n} (t_{ll}(t^{-1})-1)\frac{m(t)}{[p(t)]^{N+1}} f(vwt^{-1}w^{-1})\,dt\Big\}\,dw.$$

We now claim that the kernel in the inner integral defines a bounded convolution operator on $L^p(T^m)$.

To see this we use the fact that multiplication by the character t_{ll} of the torus does not change absolute values; thus convolution by this kernel has the same absolute value as

$$\int_{T^n} \frac{t_{ll}(st^{-1})-t_{ll}(s)}{|t|^{n+1}}\, h(t)\varphi(t^{-1}s)\,dt,$$

where h is the (asymptotically) even homogeneous function of degree 0 given by $h(t) = m(t)/[p(t)]^{N-m}$. By (3.11) the operator norm of this transformation does not exceed $\sup_{s,t\in T^n} \frac{|t_{ll}(s)-t_{ll}(t)|}{|st^{-1}|} A_p \leqslant l_T A_p$, where A_p depends only on p and n. (10) As seen often (by now) $l_T A_p$ dominates the $L^p(G)$-norm of the operator defined by the last curly brackets. Since $T(w)$ is unitary we have

$$\sum_{l=1}^{d_T} \int_G |t_{kl}(w)\overline{t_{jl}(w)}|\,dw \leqslant \int_G \Big(\sum_l |t_{kl}(w)|^2\Big)^{1/2}\Big(\sum_l |\overline{t_{jl}(w)}|^2\Big)^{1/2} dw = 1.$$

This, and Minkowski's integral inequality give us

$$(3.18)\quad \left(\int_G \left|\text{P.V.} \int_G \frac{t_{kj}(u^{-1})-\delta_{kj}}{[p(u)]^{N+1}} f(vu^{-1})\,du\right|^p dv\right)^{1/p} \leqslant l_T A_p \|f\|_{L^p(G)}$$

for $1 < p < \infty$.

This inequality yields the following result.

THEOREM 3.19. *Suppose $\{T_\lambda\}$ is a complete system of irreducible representations of G, d_λ the dimension of the space on which T_λ acts and $l_\lambda = l_{T_\lambda}$. If $\sum_\lambda d_\lambda l_\lambda \sum_{k,j=1}^{d_\lambda} |a^\lambda_{kj}| < \infty$ then the function (with $\hat{a}(\lambda) = (a^\lambda_{jk})$)*

$$a(u) = \sum d_\lambda tr\big(\hat{a}(\lambda)T^\lambda(u)\big)$$

defines a bounded operator on $L^p(G)$ by letting

$$(K_a f)(v) = \text{P.V.} \int_G \frac{a(u)-a(e)}{[p(u)]^{N+1}} f(vu^{-1})\,du.$$

To conclude we observe that in the case $G = SU(2)$ one can make a choice of a natural matricial Fourier transform so that multiplication by $[p(u)]^2$ yields a difference operator on the Fourier coefficients (see [6]). Theorem 3.19 can be reworded as a matrix multiplier theorem. Such choices can be made for other compact Lie groups.

References

[1] J. Frank Adams, *Lectures on Lie Groups*, New York 1969.
[2] W. Boothby and Guido Weiss, (editors) *Symmetric Spaces*, New York 1972.
[3] A. P. Calderón, *Commutators of singular integral operators*, Proc. Nat. Acad. Sci., U.S.A. 53 (1965), pp. 1092–1099.

(10) By decomposing the kernel into its even and odd parts, the odd part, being a singular integral, is of type discussed at the end of (ii). The even part is an integrable kernel with norm $\leqslant l_T$. We could have, therefore, avoided using the commutator operator.

[4] — *Ergodic theory and translation invariant operators*, Proc. Nat. Acad. Sci., U.S.A. 59 (1968), pp. 349–353.
[5] — and A. Zygmund, *On singular integrals*, Amer. J. Math, 78 (1956), pp. 289–309.
[6] R. R. Coifman and Guido Weiss, *Analyse Harmonique non Commutative sur Certains Espaces Homogènes*, Lecture notes in Mathematics, Vol. 242, Berlin 1971.
[7] M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Revista Mat. Cuyana 1. (1955), pp. 105–167.
[8] Karel de Leeuw, *On L^p Multipliers*, Am. J. Math, 91 (1965), pp. 364–379.
[9] D. Fife, *Spectral decompositions of ergodic flows on L^p*, Bull A.M.S. 7, Vol. 76 (1970), pp. 138–141.
[10] Frederick P. Greenleaf, *Invariant means on topological groups and their applications*, Canada 1969.
[11] Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis II*, Berlin 1970.
[12] E. M. Stein and Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton 1971.
[13] Norman J. Weiss, *L^p Estimates for Bi-invariant Operators on Compact Lie Groups*, preprint.

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Received August 9, 1972 (572)