STUDIA MATHEMATICA, T. XLVII. (1973)

On linear functionals in Hardy–Orlicz spaces. III

by

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Abstract. The paper contains theorems on the representation of linear functionals on spaces H^1 and H^∞, and their dual spaces.

This paper is a continuation of papers [5] and [6]. We adopt the notation and continue the section numbering of papers I and II. We cite the results of papers I, II and III, writing the number of the section and the number of the result in the section; within the same section the section number is omitted.

V. THE CASE OF SPACES H^1 AND H^∞

1.1. We shall first consider the representation of linear functionals and the question of reflexivity of Hardy space H^1. For H^1 we shall use the usual norm

$$\|F\|_1 = \sup\Big\{\int_0^{2\pi} |F(re^{it})|\,dt:\ 0 \leqslant r < 1\Big\} = \int_0^{2\pi} |F(e^{it})|\,dt, \qquad (F \in H^1).$$

Here we have $H^1 = H^{*\varphi} = H^{\circ\varphi}$ and $(H^1)^\# = (H^1)_0^\# = (H_m^1)^\# = (H_m^1)_0^\#$, since $\varphi(u) = u$ satisfies conditions $(\varDelta_2)$ and (∇_2). We equip $(H^1)^\#$ with a usual norm

$$\|\xi\|_1^\# = \sup\{|\xi(F)|:\ F \in H^1,\ \|F\|_1 \leqslant 1\}, \qquad (\xi \in (H^1)^\#).$$

From III.3.1, III.3.2, III.6.1 and II.8.1 we infer that the space $[(H^1)^\#, \|\cdot\|_1^\#]$ is isometric isomorphic with the space $[(H^1)', \|\cdot\|_1']$, where

$$\|G\|_1' = \sup\{2\pi|(F*G)(z)|:\ F \in H^1,\ \|F\|_1 \leqslant 1,\ z \in D\}, \qquad (G \in (H^1)').$$

1.2. *For every function $G \in (H^1)'$ there exists a function $g \in L^\infty$ such that G is its Cauchy integral and $\|G\|_1' = \|g\|_\infty^\#$* ([9]).

Proof, analogous to that of IV.1.2 and based on the integral representation of linear functionals from $(L^1)^\#$ by functions from L^∞, is omitted.

1.3. *For every function $g \in L^\infty$ its Cauchy integral G belongs to $(H^1)'$. Besides there holds the inequality $\|G\|_1' \leqslant \|g\|_\infty^*$* ([9]).

Proof, analogous to that of IV.1.1, is omitted.

1.4. We designate by L_+^∞ the class of all functions $f \in L^\infty$ for which

$$\int_0^{2\pi} f(t) e^{-int} dt = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

L_+^∞ is a closed linear subspace of $[L^\infty, \|\cdot\|_\infty^*]$. As in IV.1.4 we denote by $\tilde{L}^\infty$ the quotient space L^∞ / L_+^∞. The norm in $\tilde{L}^\infty$ is defined by

$$\|g^\sim\|_\infty^\sim = \inf\{\|f+g\|_\infty^*: f \in L_+^\infty\}, \quad (g \in L^\infty).$$

The space $[(H^1)', \|\cdot\|_1']$ *is isometric isomorphic to* $[\tilde{L}^\infty, \|\cdot\|_\infty^\sim]$. *This isomorphism establishes the operation of Cauchy integral.*

This theorem is a consequence of 1.2, 1.3 and the fact that the Cauchy integral of a function $g \in L^\infty$ equals 0 if and only if $g \in L_+^\infty$.

1.5. *The space* H^∞ *of analytic and bounded functions in* D *is contained in* $(H^1)'$; *besides* $\|F\|_1' \leqslant \|F\|_\infty$ *for every* $F \in H$ ([9]).

This theorem follows from 1.3 in the analogical way to that in which IV.1.5 follows from IV.1.3.

2.1. By C we shall designate a space of these functions f continuous on $[0, 2\pi]$ for which $f(0) = f(2\pi)$. For C we use the usual norm

$$\|f\|^* = \sup\{|f(t)|: 0 \leqslant t \leqslant 2\pi\}, \quad (f \in C).$$

Clearly, $[C, \|\cdot\|^*]$ is a closed linear subspace of $[L^\infty, \|\cdot\|_\infty^*]$.

We shall demonstrate that

For every function $g \in C$ *its Cauchy integral* G *belongs to* $(H_{vw}^1)'$.

Proof. Let G be the Cauchy integral of $g \in C$. It is known that the sequence of functions $\{T_r g\}$, where

$$(T_r g)(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\tau)+r^2} g(\tau) d\tau, \quad 0 \leqslant r < 1,$$

converges uniformly to g as $r \to 1-$, i.e. $\|T_r g - g\|^* \to 0$ as $r \to 1-$. The function $T_r G - G$ is the Cauchy integral of $T_r g - g$, and so, by 1.3, we have $\|T_r G - G\|_1' \leqslant \|T_r g - g\|^*$. This means that $\|T_r G - G\|_1' \to 0$ as $r \to 1-$, and hence $G \in (H_{vw}^1)'$.

2.2. *The space* K *of functions analytic in* D *and continuous in* $\bar{D}$ *is contained in* $(H_{vw}^1)'$.

This theorem follows immediately from 1.5, III.1.7 and 2.1.

2.3. We shall designate by $\mathscr{V}$ a class of all functions λ of finite variation on $[0, 2\pi]$, equal to 0 for $t = 0$ and right-continuous on $[0, 2\pi]$. Obviously, $\mathscr{V}$ is a Banach space relative to the norm

$$\|\lambda\|_{\mathscr{V}}^* = \operatorname{var}\{\lambda(t): 0 \leqslant t \leqslant 2\pi\}, \quad (\lambda \in \mathscr{V}).$$

It is well known that:

For every functional $\eta \in (C)^\#$ there is a unique function $\lambda \in \mathscr{V}$ such that

$$\eta(g) = \int_0^{2\pi} g(t) d\lambda(t) \quad \text{for every } g \in C;$$

moreover $\|\eta\|^\# = \|\lambda\|_{\mathscr{V}}^*$.

2.4. Let us write $C_+ = C \cap L_+^\infty$ and, as in IV.3.2, $\tilde{C}$ will designate the quotient space C/C_+. We equip C with the norm given by

$$\|g^\simeq\|^\simeq = \inf\{\|f+g\|^*: f \in C_+\}, \quad (g \in C).$$

The space $\tilde{C}$ is complete relative to this norm since it is the quotient of the complete space $[C, \|\cdot\|^*]$ by its closed linear subspace C_+.

We shall show that

The space $[(\tilde{C})^\#, \|\cdot\|^\#]$ *is isometric isomorphic to the space* $[H^1, \|\cdot\|_1]$. *More precisely, for every functional* $\eta^\simeq \in (\tilde{C})^\#$ *there is a unique function* $F \in H^1$ *such that*

$$\eta^\simeq(g^\simeq) = \int_0^{2\pi} F(e^{-it}) g(t) dt \quad \text{for } g \in C,$$

and, conversely, each functional $\eta^\simeq$ *represented by this formula for a function* $F \in H^1$ *belongs to* $(\tilde{C})^\#$, *and* $\|\eta^\simeq\|^\# = \|F\|_1$.

Proof. Let $\eta^\simeq \in (\tilde{C})^\#$. We notice first that the functional $\eta(g) = \eta^\simeq(g^\simeq)$ for $g \in C$ belongs to $(C)^\#$ and its norm equals $\|\eta\|^\# = \|\eta^\simeq\|^\#$. On account of 2.3, there is a unique function $\lambda \in \mathscr{V}$ such that

$$\eta(g) = \int_0^{2\pi} g(t) d\lambda(t) \quad \text{for } g \in C;$$

moreover $\|\eta\|^\# = \|\lambda\|_{\mathscr{V}}^*$. Since the functions $e^{-ik\cdot}$ for $k = 1, 2, \ldots$ all belong to C_+, it follows then that

$$0 = \eta(e^{-ik\cdot}) = \int_0^{2\pi} e^{-ikt} d\lambda(t) \quad \text{for } k = 1, 2, \ldots$$

Since for $0 \leqslant r < 1$ we have

$$\frac{1}{1-re^{-it}} + \sum_{k=1}^{\infty} r^k e^{ikt} = \frac{1-r^2}{1-2r\cos t + r^2},$$

we see that an analytic function F defined by the Cauchy–Stieltjes integral

$$F(z) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-ze^{-it}} d\lambda(2\pi - t), \quad (z \in D),$$

is representable in the form of the Poisson–Stieltjes integral

$$F(re^{it}) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\tau)+r^2}\, d\lambda(2\pi-\tau), \qquad (0 \leqslant r < 1).$$

Thus we get for $0 \leqslant r < 1$

$$\int_0^{2\pi} |F(re^{it})|\,dt \leqslant \int_0^{2\pi} |d\lambda(2\pi-\tau)| = \|\lambda\|^*_{\mathscr{V}}.$$

This means that $F \in H^1$. Now let

$$\lambda_1(t) = \int_0^t F(e^{i\tau})\,d\tau, \qquad (0 \leqslant t \leqslant 2\pi).$$

Clearly

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-ze^{-it}}\, d\big(\lambda_1(t)+\lambda(2\pi-t)\big) \qquad \text{for } z \in D.$$

From this we get

$$\int_0^{2\pi} e^{-ikt} d\big(\lambda_1(t)+\lambda(2\pi-t)\big) = 0 \qquad \text{for } k = 0, 1, 2, \ldots$$

Besides, we have also

$$\int_0^{2\pi} e^{ikt} d\big(\lambda_1(t)+\lambda(2\pi-t)\big) = \int_0^{2\pi} e^{-ikt} d\lambda(t) = 0 \qquad \text{for } k = 1, 2, \ldots$$

Hence

$$\lambda(2\pi) = \lambda_1(2\pi) = 0 \quad \text{and} \quad \int_0^{2\pi} \big(\lambda_1(t)+\lambda(2\pi-t)\big)e^{ikt}\,dt = 0$$

for $k = \pm 1, \pm 2, \ldots$ From this and the right-continuity of λ we deduce that $\lambda_1(t)+\lambda(2\pi-t) = 0$ for all $t \in [0, 2\pi]$. This implies that λ is absolutely continuous and

$$\frac{d\lambda}{dt}(t) = F(e^{-it}) \qquad \text{for almost all } t \in [0, 2\pi].$$

Further, we infer that $\|\lambda\|^*_{\mathscr{V}} = \|F\|_1$. Thus for every functional $\eta^{\simeq} \in (\tilde{C})^{\#}$ there is a unique function $F \in H^1$ such that

$$\eta^{\simeq}(g^{\simeq}) = \int_0^{2\pi} F(e^{-it})g(t)\,dt \qquad \text{for every } g \in C;$$

moreover $\|\eta^{\simeq}\|^{\#} = \|F\|_1$.

Let $F \in H^1$. Let us consider the functional

$$\eta(g) = \int_0^{2\pi} F(e^{-it})g(t)\,dt \qquad \text{for } g \in C.$$

This functional belongs to $(C)^{\#}$, since

$$|\eta(g)| = \Big|\int_0^{2\pi} F(e^{-it})g(t)\,dt\Big| \leqslant \|F(e^{-i\cdot})\|^*_1\|g\|^* = \|F\|_1\|g\|^*$$

for every $g \in C$. The functional represented, for $f \in L^1$, by an integral $\int_0^{2\pi} f(t)g(t)\,dt$, where g is a fixed function from C, is an element of $(L^1)^{\#}$. Hence for any $g \in C_+$ we have

$$\eta(g) = \lim_{r\to 1-}\int_0^{2\pi} F(re^{-it})g(t)\,dt = \lim_{r\to 1-}\sum_{n=0}^{\infty}\gamma_n(F)r^n\int_0^{2\pi} e^{-int}g(t)\,dt = 0.$$

This implies that the functional $\eta^{\simeq}(g^{\simeq}) = \eta(g)$ for $g \in C$ is well defined and belongs to $(\tilde{C})^{\#}$.

From the preceding proof we conclude that

Every function $\lambda \in \mathscr{V}$ such that

$$\int_0^{2\pi} e^{-int}d\lambda(t) = 0 \qquad \text{for } n = 1, 2, \ldots$$

is absolutely continuous on $[0, 2\pi]$ (cf. [10], Chap. VII (8.2)).

2.5. *The space* $[\tilde{C}, \|\cdot\|^{\simeq}]$ *is isometric isomorphic to the space* $[(H^1_{vw})', \|\cdot\|'_1]$. *This isomorphism establishes the operation of Cauchy integral.*

Proof, analogous to that of IV.3.3 and based on 2.4, is omitted.

2.6. Several corollaries follow from 2.5. For instance, we have the following

Every function $G \in (H^1_{vw})'$ *is the Cauchy integral of some function* $g \in C$.

This implies further that the space $\tilde{C}$ is isomorphic to the space $\{g^{\sim} \in \tilde{L}^{\infty}: g \in C\}$; this isomorphism is the operation of correspondence between classes $g^{\simeq} \in \tilde{C}$ and $g^{\sim} \in \tilde{L}^{\infty}$. Besides, in view of 1.4 and 2.5, we have, for $g \in C$, $\|g^{\sim}\|^{\sim}_{\infty} = \|g^{\simeq}\|^{\simeq}$.

3.1. *For every function* $F \in H^1$ *the functional defined by*

$$(+) \qquad \eta^{\circ}(G) = \lim_{r\to 1-} 2\pi(F*G)(r) = 2\pi(F*G)(1) \qquad \text{for } G \in (H^1_{vw})'$$

belongs to $\big((H^1_{vw})'\big)^{\#}$; *besides* $\|\eta^{\circ}\|^{\#}_1 = \|F\|_1$. *Conversely, for every functional* $\eta^{\circ} \in \big((H^1_{vw})'\big)^{\#}$ *there is a unique function* F *analytic in* D *and such that* $(+)$ *holds; this function belongs to* H^1 *and is defined by*

$$(++) \qquad F(z) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\eta^{\circ}(U_n)z^n, \qquad (z \in D),$$

where $U_n(z) = z^n$, $(z \in D)$, *for* $n = 0, 1, 2, \ldots$

This theorem follows from 2.4; the proof, quite analogous to that of IV.4.1, is omitted.

3.2. *For every function $F \in H^1$ the functional η defined by $(+)$ for $F \in (H^1)'$ belongs to $((H^1)'_{vw})^{\#}$; besides $\|\eta\|_1^{\#} = \|F\|_1$. Conversely, for every functional $\eta \in ((H^1)'_{vw})^{\#}$ there exists a unique function F analytic in D and such that $(+)$ holds for $F \in (H^1)'$; this function belongs to H^1 and is represented by $(++)$.*

Proof. Let $F \in H^1$ and η be defined by $(+)$ for $F \in (H^1)'$. Then we have

$$|\eta(G)| = 2\pi |(F*G)(1)| \leqslant \|F\|_1 \|G\|_1' \quad \text{for every } G \in (H^1)'.$$

This implies that $\eta \in ((H^1)')^{\#}$ and $\|\eta\|_1^{\#} \leqslant \|F\|_1$. Applying the theorem on attaining a norm by functionals, we see that there is a functional $\xi \in (H^1)^{\#}$ such that $\|\xi\|_1^{\#} = 1$ and $\|F\|_1 = \xi(F)$. Since $(H^1)^{\#} = (H^1_m)^{\#}$, this yields, in view of III.3.4, the existence of a function $G \in (H^1)'$ such that $\|G\|_1' = 1$ and $\|F\|_1 = 2\pi(F*G)(1) = \eta(G)$. Hence $\|F\|_1 \leqslant \|\eta\|_1^{\#}$ and so $\|\eta\|_1^{\#} = \|F\|_1$. For a function $F \in H^1$ we have, by I.3.6, $\|T_rF - F\|_1 \to 0$ as $r \to 1-$. Let us notice that $\|T_r^{\#}\eta - \eta\|_1^{\#} = \|T_rF - F\|_1$. This implies that $\|T_r^{\#}\eta - \eta\|_1^{\#} \to 0$ as $r \to 1-$ and this entails $\eta \in ((H^1)'_{vw})^{\#}$.

Conversely, let $\eta \in ((H^1)'_{vw})^{\#}$. Then the restriction η° of η to $(H^1_{vw})'$ belongs to $((H^1_{vw})')^{\#}$. By 3.1 there is a unique function F analytic in D and such that $(+)$ holds for $G \in (H^1_{vw})'$; this function belongs to H^1 and is represented by $(++)$. Let us consider the functional $\eta_1(G) = 2\pi(F*G)(1)$ for $G \in (H^1)'$. In view of the reasoning of the first part of this proof this functional belongs to $((H^1)'_{vw})^{\#}$. For $G \in (H^1_{vw})'$ we have $\eta_1(G) = \eta^\circ(G) = \eta(G)$. For arbitrary $G \in (H^1)'$ and $0 \leqslant r < 1$ we have $T_rG \in K \subset (H^1_{vw})'$ and the sequence $\{T_rG\}$ converges very weakly to G as $r \to 1-$. Hence for arbitrary $G \in (H)'$ it is true that

$$\eta_1(G) = \lim_{r\to 1_-} \eta_1(T_rG) = \lim_{r\to 1_-} \eta(T_rG) = \eta(G).$$

This accomplishes the proof.

3.3. *Every functional $\eta \in ((H^1)')^{\#}$ may be uniquely represented in the form*

$$\eta = \eta_1 + \eta_2, \quad \text{where } \eta_1 \in ((H^1)'_{vw})^{\#} \text{ and } \eta_2 \in ((H^1)'^{\sim})^{\#}.$$

$((H^1)'^{\sim})^{\#}$ denotes a space of functionals $\eta \in ((H^1)')^{\#}$ such that $\eta(G) = 0$ for every $G \in (H'_{vw})'$.

Proof, analogous to that of IV.4.5 and based on 3.1 and 3.2, is omitted.

3.4. By $(H^1)''$ we denote the class of all functions F analytic in D for which

$$\|F\|_1'' = \sup\{2\pi|(F*G)(z)| : G \in (H^1)', \|G\|_1' \leqslant 1, z \in D\} < \infty.$$

As in IV.4.7, we get here

$$[(H^1)'', \|\cdot\|_1''] = [H^1, \|\cdot\|_1].$$

3.5. $(H^1)' \neq (H^1_{vw})'$.

Proof. Let us define a sequence of real functions

$$f_n(t) = \begin{cases} 2^{n-1} & \text{if } 2\pi - t \in E_n, \\ \dfrac{1}{2^{n+2}\pi} & \text{for other } t \text{ from } [0, 2\pi), \end{cases}$$

where $E_n = (2^{-n}, 2^{-n+1})$, $n = 1, 2, \ldots$ Next we define a sequence of analytic functions

$$F_n(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log f_n(t)\,dt\right), \quad (z \in D).$$

It is known ([10] Chap. VII (7.33)) that these functions $F_n \in N'$ and are such that $|F_n(e^{it})| = f_n(t)$ for almost all $t \in [0, 2\pi)$. This together with

$$\int_0^{2\pi} f_n(t)\,dt \leqslant 2^{n-1}\cdot 2^{-n} + (2^{n+2}\pi)^{-1}\cdot 2\pi \leqslant 1$$

implies, in view of I.3.3, that $F_n \in H^1$ and $\|F_n\|_1 \leqslant 1$ for $n = 1, 2, \ldots$ Notice that $|F_n(e^{it})| = f_n(t) \to 0$ as $n \to \infty$ for almost every $t \in [0, 2\pi)$. This, by virtue of the Ostrowski theorem, means that $\{F_n\}$ converges very weakly to 0. Now we define the function

$$g(t) = \begin{cases} \operatorname{sgn} F_n(e^{-it}) & \text{for } t \in E_n,\ n = 1, 2, \ldots \\ 0 & \text{for other } t \text{ from } [0, 2\pi). \end{cases}$$

Clearly, $g \in L^\infty$. Hence the Cauchy integral G of g belongs to $(H^1)'$. Besides, from III.1.6 and from the norm continuity of a functional represented by the integral $\int_0^{2\pi} f(t)g(t)\,dt$, on L^1 we get

$$2\pi(F_n*G)(1) = \lim_{r\to 1-}\int_0^{2\pi} F_n(re^{-it})g(t)\,dt = \int_0^{2\pi} F_n(e^{-it})g(t)\,dt$$

for $n = 1, 2, \ldots$ We verify that

$$\int_{E_n} F_n(e^{-it})g(t)\,dt = \int_{E_n} |F_n(e^{-it})|\,dt = 2^{n-1}\cdot 2^{-n} = 1/2$$

and

$$\left|\int_{[0,2\pi)\setminus E_n} F_n(e^{-it})g(t)\,dt\right| \leqslant \frac{1}{2^{n+2}\pi}\cdot 2\pi = \frac{1}{2^{n+1}}.$$

Hence we get

$$2\pi|(F_n*G)(1)| \geqslant \frac{1}{2}-\frac{1}{2^{n+1}} \geqslant \frac{1}{4} \quad \text{for } n = 1, 2, \ldots$$

This implies that $G \notin (H^1_{vw})'$.

3.6. *The space $((H^1)'^{\sim})^{\#}$ is not a trivial one, i.e. there exist non-trivial functionals $\eta \in ((H^1)')^{\#}$ such that $\eta(G) = 0$ for every $G \in (H^1_{vw})'$. For these functionals there is no function F analytic in D and such that $(+)$ holds.*

The proof, analogous to that of IV.5.2, is omitted.

3.7. *The space H^1 is not reflexive.*

This is a direct consequence of 3.6.

4.1. We shall now deal with the representation of linear functionals and the reflexivity of H^∞ and its subspace K. The space H^∞ is not of the Hardy–Orlicz type, however, there is a close similarity between these types of spaces; it is our objective to investigate this similarity. For H^∞ we shall use the usual norm

$$\|F\|_\infty = \sup\{|F(z)|: z \in D\} = \|F(e^{i\cdot})\|^*_\infty, \qquad (F \in H^\infty).$$

Two types of convergence will be distinguished in H^∞: the norm convergence and the very weak convergence, which will be defined as in the case of the Hardy–Orlicz space $H^{*\varphi}$ (see I.4.1). Let us note here, that the results of I.4.3, I.4.4 and I.4.5 remain true also for H^∞.

In analogy to the notation used in the case of Hardy–Orlicz spaces, $(H^\infty)^{\#}$ will denote the space of norm continuous linear functionals and $(H^\infty_{vw})^{\#}$ — the space of very weakly continuous linear functionals on H^∞. The space $(H^\infty)^{\#}$ is equipped with the norm

$$\|\xi\|^{\#}_\infty = \sup\{|\xi(F)|: F \in H^\infty, \|F\|_\infty \leqslant 1\}, \qquad (\xi \in (H^\infty)^{\#}).$$

Similarly, $(K)^{\#}$ denotes a space of norm continuous linear functionals and $(K_{vw})^{\#}$ a space of very weakly continuous linear functionals on K. The space $(K)^{\#}$ is equipped with the norm

$$\|\xi^\circ\|^{\#} = \sup\{|\xi^\circ(F)|: F \in K, \|F\|_\infty \leqslant 1\}, \qquad (\xi^\circ \in (K)^{\#}).$$

From Hahn–Banach theorem we deduce that for every functionar $\xi^\circ \in (K)^{\#}$ there exists a functional $\xi \in (H^\infty)^{\#}$ such that $\xi(F) = \xi^\circ(F)$ fol $F \in K$ and $\|\xi\|^{\#}_\infty = \|\xi^\circ\|^{\#}$.

Lastly, $(\tilde{H}^\infty)^{\#}$ will denote the space of functionals $\xi \in (H^\infty)^{\#}$ such that $\xi(F) = 0$ for all $F \in K$.

It is clear that $(H^\infty)^{\#}$ is complete with respect to the norm $\|\cdot\|^{\#}_\infty$ and that $(\tilde{H}^\infty)^{\#}$ is its closed linear subspace. And also the space $(K)^{\#}$ is complete with respect to the norm $\|\cdot\|^{\#}$.

4.2. *For every functional $\xi^\circ \in (K_{vw})^{\#}$ there is a unique functional $\xi \in (H^\infty_{vw})^{\#}$ such that $\xi(F) = \xi^\circ(F)$ for $F \in K$; this functional is defined by*

$$\xi(F) = \lim_{r\to 1-} \xi^\circ(T_r F) \quad \textit{for } F \in H^\infty.$$

Besides $\|\xi\|^{\#}_\infty = \|\xi^\circ\|^{\#}$ (see II.6.5).

Proof. Let $\xi^\circ \in (K_{vw})^{\#}$. Further, let $F \in H^\infty$ and $\{r_{1n}\}$, $\{r_{2n}\}$ be two sequences such that $0 \leqslant r_{1n}, r_{2n} < 1$, $r_{1n} \to 1$, $r_{2n} \to 1$ as $n \to \infty$. Observing that then the sequence $\{T_{r_{1n}}F - T_{r_{2n}}F\}$ of elements of K is very weakly convergent to 0, we get

$$\lim_{n\to\infty} \big(\xi^\circ T(r_{1n}F) - \xi^\circ(T_{r_{2n}}F)\big) = 0.$$

The above combined with

$$\sup\{|\xi^\circ(T_r F)|: 0 \leqslant r < 1\} \leqslant \sup\{\|\xi^\circ\|^{\#}\|T_r F\|_\infty: 0 \leqslant r < 1\} = \|\xi^\circ\|^{\#}\|F\|_\infty$$

yields the existence of the limit $\lim_{r\to 1-} \xi^\circ(T_r F) = \xi(F)$ and, moreover, $|\xi(F)| \leqslant \|\xi^\circ\|^{\#}\|F\|_\infty$. For ξ the relation $\xi(F) = \xi^\circ(F)$ obviously holds for all $F \in K$. Besides $\xi \in (H^\infty)^{\#}$ and $\|\xi\|^{\#}_\infty = \|\xi^\circ\|^{\#}$. We shall now demonstrate that $\xi \in (H^\infty_{vw})^{\#}$. Let $\{F_n\} \subset H^\infty$ be any sequence very weakly converging to 0. We find a sequence $\{r_n\}$ such that $0 \leqslant r_n < 1$, $r_n \to 1$ as $n \to \infty$ and

$$|\xi(F_n) - \xi^\circ(T_{r_n}F_n)| \leqslant 1/n \quad \text{for } n = 1, 2, \ldots$$

Elements of the sequence $\{T_{r_n}F_n\}$ belong to K and this sequence is also very weakly convergent to 0. Thus $\xi(F_n) \to 0$ as $n \to \infty$. Hence $\xi \in (H^\infty_{vw})'$. The uniqueness of such a functional ξ is obvious.

4.3. *The functional $\xi \in (H^\infty)^{\#}$ belongs to $(H^\infty_{vw})^{\#}$ if and only if*

$$\lim_{r\to 1-} \|T^{\#}_r\xi - \xi\|^{\#}_\infty = 0.$$

Proof. Let $\xi \in (H^\infty)^{\#}$ and $\|T^{\#}_r\xi - \xi\|^{\#}_\infty \to 0$ as $r \to 1-$. Further, let $\{F_n\} \subset H^\infty$ be a sequence very weakly converging to 0 and such that $\sup_n \|F_n\|_\infty < M$. For a fixed $\varepsilon > 0$ we take $0 < r < 1$ such that $\|T^{\#}_r\xi - \xi\|^{\#}_\infty \leqslant \varepsilon M^{-1}$. Hence we get

$$|\xi(F_n)| \leqslant |T^{\#}_r\xi(F_n)| + |\xi(F_n) - T^{\#}_r\xi(F_n)| \leqslant |\xi(T_rF_n)| + \varepsilon$$

and next

$$\limsup_{n\to\infty} |\xi(F_n)| \leqslant \varepsilon.$$

This implies that $\xi(F_n) \to 0$ as $n \to \infty$ and hence $\xi \in (H^\infty_{vw})^{\#}$.

Assume now that $\|T^{\#}_r\xi - \xi\|^{\#}_\infty \nrightarrow 0$ as $r \to 1-$. Then there are a number $\varepsilon > 0$, a sequence $\{r_n\}$ and a sequence $\{F_n\} \subset H^\infty$ such that $0 \leqslant r_n < 1$, $r_n \to 1$ $(n \to \infty)$, $\|F_n\|_\infty \leqslant 1$ and $|T^{\#}_{r_n}\xi(F_n) - \xi(F_n)| \geqslant \varepsilon$ for $n = 1, 2, \ldots$

Since the unit ball in $[H^\infty, \|\cdot\|_\infty]$ is sequentially very weakly compact, there is a very weakly convergent subsequence $\{F_{n_k}\}$ of $\{F_n\}$. The sequence $\{T_{r_{n_k}} F_{n_k} - F_{n_k}\}$ then converges very weakly to 0. For this sequence we have the following estimation

$$|\xi(T_{r_{n_k}} F_{n_k} - F_{n_k})| \geqslant \varepsilon \quad \text{for } k = 1, 2, \ldots$$

Hence $\xi \notin (H^\infty_{vw})^{\#}$.

4.4. $(H^\infty_{vw})^{\#}$ *is a closed linear subspace of* $[(H^\infty)^{\#}, \|\cdot\|^{\#}_\infty]$.

Proof. Let $\{\xi_n\} \subset (H^\infty_{vw})^{\#}$ be a sequence convergent in norm $\|\cdot\|^{\#}_\infty$ to $\xi \in (H^\infty)^{\#}$. Since for $\xi \in (H^\infty)^{\#}$ and $0 \leqslant r < 1$ we have $\|T^{\#}_r \xi\|^{\#}_\infty \leqslant \|\xi\|^{\#}_\infty$ we get the following estimation

$$\begin{aligned}\|T^{\#}_r \xi - \xi\|^{\#}_\infty &\leqslant \|T^{\#}_r \xi - T^{\#}_r \xi_n\|^{\#}_\infty + \|T^{\#}_r \xi_n - \xi_n\|^{\#}_\infty + \|\xi_n - \xi\|^{\#}_\infty \\ &\leqslant 2\|\xi_n - \xi\|^{\#}_\infty + \|T^{\#}_r \xi_n - \xi_n\|^{\#}_\infty.\end{aligned}$$

This, in view of our assumption together with 4.3, implies that $\xi \in (H^\infty_{vw})^{\#}$.

4.5. *Let z be a fixed point in the circle D. The functionals*

$$\gamma_{0,z}(F) = F(z) \quad \text{and} \quad \gamma_{n,z}(F) = \frac{1}{n!} F^{(n)}(z) \quad \text{for } n = 1, 2, \ldots$$

are easily seen to belong to $(H^\infty_{vw})^{\#}$. For these functionals the following relations hold

$$|\gamma_{n,z}(F)| \leqslant \frac{1}{(1-|z|)^n} \|F\|_\infty \quad \text{for } n = 0, 1, 2, \ldots$$

Proof. The above inequality is obvious for $n = 0$. For $n \geqslant 1$ let us take the circumference C_r of a radius $0 < r < 1-|z|$ and centre in 0. By Cauchy integral formulae we get

$$\begin{aligned}|\gamma_{n,z}(F)| &= \left|\frac{1}{2\pi i} \int_{C_r} \frac{F(\zeta+z)}{\zeta^{n+1}} d\zeta\right| \leqslant \frac{1}{2\pi} \cdot \frac{1}{r^{n+1}} \cdot 2\pi r \sup\{|F(\zeta+z)|:\ \zeta \in C_r\} \\ &\leqslant \frac{1}{r^n} \|F\|_\infty.\end{aligned}$$

Passing to the limit as $r \to 1-|z|$ we get the desired inequality.

This result allows us to apply Theorems II.1.4, II.1.5 and II.1.6 to the space H^∞.

5.1. We denote by $(H^\infty)'$ the space of all functions G analytic in D for which

$$\|G\|'_\infty = \sup\{2\pi|(F*G)(z)|:\ F \in H^\infty, \|F\|_\infty \leqslant 1, z \in D\} < \infty.$$

We shall show the following

For any function G analytic in D

$$\|G\|'_\infty = \sup\{2\pi|(F*G)(z)|:\ F \in K, \|F\|_\infty \leqslant 1, z \in D\}.$$

Proof. Since $K \subset H^\infty$, we have the following inequality

$$\sup\{2\pi|(F*G)(z)|:\ F \in K, \|F\|_\infty \leqslant 1, z \in D\} \leqslant \|G\|'_\infty.$$

Let now $F \in H^\infty$ be any function such that $\|F\|_\infty \leqslant 1$ and z any number from D. $T_r F \in K$ for every $0 \leqslant r < 1$, and so we have

$$|(F*G)(rz)| \leqslant \sup\{|(F*G)(z)|:\ F \in K, \|F\|_\infty \leqslant 1, z \in D\} \quad \text{for } 0 \leqslant r < 1.$$

Passing to the limit as $r \to 1-$, we get

$$|(F*G)(z)| \leqslant \sup\{|(F*G)(z)|:\ F \in K, \|F\|_\infty \leqslant 1, z \in D\}.$$

Hence

$$\|G\|_\infty \leqslant \sup\{2\pi|(F*G)(z)|:\ F \in K, \|F\|_\infty \leqslant 1, z \in D\}.$$

5.2. *For every $G \in (H^\infty)'$ and every $F * H^\infty$ the function $F \in G$ is bounded in D. Moreover, for every $z \in D$ the inequality*

$$2\pi|(F*G)(z)| \leqslant \|F\|_\infty \|G\|'_\infty$$

is satisfied.

The easy proof of this theorem is omitted.

Let us notice the function $I(z) = (1-z)^{-1}$, $(z \in D)$, belongs to $(H^\infty)'$. Indeed, for this function we have $(F*I)(z) = F(z)$ for every $F \in H^\infty$ and $z \in D$. It follows that $\|I\|'_\infty = 2\pi$. This means that functions $G \in (H^\infty)'$ and $F \in H^\infty$ are not always such that their convolution could be completed so as to form a continuous function in $\bar{D}$.

5.3. *For every $G \in (H^\infty)'$ and every $F \in K$ the function $F*G$ has the radial limits everywhere on the circumference $\{z:\ |z| = 1\}$ and its completion with these limits is a continuous function in the circle $\bar{D}$.*

Proof. Let $G \in (H^\infty)'$ and $F \in K$. Since $F \in K$, we have $\|S_h F - F\|_\infty \to 0$ as $h \to 0$. Thus for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|S_{h_1} F - S_{h_2} F\|_\infty = \|S_{h_1-h_2} F - F\|_\infty \leqslant \varepsilon$ for $|h_1 - h_2| \leqslant \delta$. Hence we get, for $|h_1 - h_2| \leqslant \delta$ and $0 \leqslant r < 1$,

$$\begin{aligned}2\pi|(F*G)(re^{ih_1}) - (F*G)(re^{ih_2})| &= 2\pi\left|((S_{h_1}F - S_{h_2}F)*G)(r)\right| \\ &\leqslant \|S_{h_1}F - S_{h_2}F\|_\infty \|G\|'_\infty \leqslant \varepsilon\|G\|'_\infty.\end{aligned}$$

Hence the functions $f_r(t) = (F*G)(re^{it})$, $0 \leqslant r < 1$, are uniformly continuous with respect to t. From 5.2 it follows also that these functions are commonly bounded. By Fatou's and Arzela's theorems we now conclude that the sequence $\{f_r\}$ is uniformly convergent as $r \to 1-$. Thus the theorem is proved.

5.4. *For any function G analytic in D, $\|T_rG\|'_\infty$ is a non-decreasing function for $0 \leqslant r < 1$ and*

$$\|G\|'_\infty = \lim_{r\to 1-} \|T_rG\|'_\infty.$$

Thus, a function G analytic in D belongs to $(H^\infty)'$ if and only if

$$\sup\{\|T_rG\|'_\infty\colon\ 0 \leqslant r < 1\} < \infty.$$

Proof. We take $0 \leqslant r_1 < r_2 < 1$ and $F \in H^\infty$. Let us notice that

$$\begin{aligned}\sup\{|(F*T_{r_1}G)(z)|\colon\ z\in D\} &= \sup\{|(F*G)(r_1z)|\colon\ z\in D\}\\ &\leqslant \sup\{|(F*G)(r_2z)|\colon\ z\in D\}\\ &= \sup\{|(F*T_{r_2}G)(z)|\colon\ z\in D\}.\end{aligned}$$

It follows that $\|T_{r_1}G\|'_\infty \leqslant \|T_{r_2}G\|'_\infty$. Hence $\|T_rG\|'_\infty$ is a non-decreasing function for $0 \leqslant r < 1$, and so we get further

$$\begin{aligned}\lim_{r\to 1-} \|T_rG\|'_\infty &= \sup\{\|T_rG\|'_\infty\colon\ 0 \leqslant r < 1\}\\ &= \sup\{2\pi|(F*G)(rz)|\colon\ F\in H^\infty,\ \|F\|_\infty \leqslant 1,\ z\in D,\ 0\leqslant r<1\}\\ &= \sup\{2\pi|(F*G)(z)|\colon\ F\in H^\infty,\ \|F\|_\infty \leqslant 1,\ z\in D\} = \|G\|'_\infty.\end{aligned}$$

6.1. *A functional defined by*

$$(+)\qquad \xi^\circ(F) = 2\pi(F*G)(1) = \lim_{r\to 1-} 2\pi(F*G)(r) \quad \textit{for } F\in K,$$

where $G \in (H^\infty)'$, belongs to $(K)^\#$. Moreover $\|\xi^\circ\|^\# = \|G\|'_\infty$. Conversely, for every functional $\xi^\circ \in (K)^\#$ there is a unique function G analytic in D and that $(+)$ holds; this function belongs to $(H^\infty)'$ and is represented by the formula

$$(++)\qquad G(z) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \xi^\circ(U_n)z^n, \quad (z\in D),$$

where $U_n(z) = z^n$, $(z \in D)$, $n = 0, 1, 2, \ldots$

Proof. Let $G \in (H^\infty)'$ and ξ° be defined as in $(+)$. From 5.3 it follows that ξ° is well defined on K, and from III.1.1 it follows that ξ° is linear. Taking into account 5.1 and the fact that $\|S_hF\|_\infty = \|F\|_\infty$ for every $F\in H^\infty$ and every real h, we get here

$$\begin{aligned}\|\xi^\circ\|^\# &= \sup\{2\pi|(F*G)(1)|\colon\ F\in K,\ \|F\|_\infty\leqslant 1\}\\ &= \sup\{2\pi|(F*G)(z)|\colon\ F\in H^\infty,\ \|F\|_\infty \leqslant 1,\ z\in D\} = \|G\|'_\infty.\end{aligned}$$

Conversely, let $\xi^\circ \in (K)^\#$ and G be a function defined by $(++)$. We have $|\xi^\circ(U_n)| \leqslant \|\xi^\circ\|^\#$ for $n = 0, 1, 2, \ldots$ since $U_n\in K$ and $\|U_n\|_\infty = 1$ for

$n = 0, 1, 2, \ldots$ This implies that G is an analytic function in D. As in the proof of III.3.2, we check that G satisfies $(+)$ and that G is a unique analytic function in D satisfying $(+)$. We claim that G belongs to $(H^\infty)'$. Indeed, for arbitrary $F\in H^\infty$ and $z = re^{it}\in D$ we have

$$\begin{aligned}2\pi|(F*G)(z)| &= 2\pi|(T_rS_tF*G)(1)| = |\xi^\circ(T_rS_tF)|\\ &\leqslant \|\xi^\circ\|^\#\|T_rS_tF\|_\infty \leqslant \|\xi^\circ\|^\#\|F\|_\infty\end{aligned}$$

and hence $\|G\|'_\infty \leqslant \|\xi^\circ\|^\#$.

6.2. *If G is a function analytic in D and such that*

$$\lim_{r\to 1-}(F*G)(r) = (F*G)(1)$$

exists for every $F\in H^\infty$, then $G\in(H^\infty)'$. Moreover, the functional $\xi(F) = 2\pi(F*G)(1)$ for $F\in H^\infty$ belongs to $(H^\infty)^\#$ and $\|\xi\|^\#_\infty = \|G\|'_\infty$.

Proof. Let G be a function analytic in D and such that the convolution $F*G$ has a radial limit $(F*G)(1)$ for every $F\in H^\infty$. As in the proof of III.3.5, we notice that, for $0\leqslant r<1$, $2\pi(F*G)(r) = \xi(T_rF) = T^\#_r\xi(F)$ is satisfied for every $F\in H^\infty$. By 4.5 we get

$$|T^\#_r\xi(F)| = 2\pi\Big|\sum_{n=0}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big| \leqslant 2\pi\|F\|_\infty\sum_{n=0}^{\infty}|\gamma_n(G)|r^n.$$

The series on the right-hand side of the above inequality is convergent for $0\leqslant r<1$, since G is analytic in D. This means that $T^\#_r\xi\in(H^\infty)^\#$ for $0\leqslant r<1$. The functional ξ is a pointwise limit of a sequence of functionals $\{T^\#_r\xi\}$ as $r\to 1-$. Thus, by virtue of the Banach theorem, $\xi\in(H^\infty)^\#$. Let ξ° be the restriction of ξ to the domain K. Obviously, $\xi^\circ\in(K)^\#$. Applying 6.1 to the functional ξ°, we get $G\in(H^\infty)'$ and $\|\xi^\circ\|^\# = \|G\|'_\infty$. Finally we obtain for any $F\in H^\infty$

$$\begin{aligned}|\xi(F)| &= \lim_{r\to 1-}|\xi^\circ(T_rF)| \leqslant \lim_{r\to 1-}(\|\xi^\circ\|^\#\|T_rF\|_\infty)\\ &= \|\xi^\circ\|^\#\|F\|_\infty \leqslant \|\xi\|^\#_\infty\|F\|_\infty\end{aligned}$$

and

$$\|\xi\|^\#_\infty = \|\xi^\circ\|^\# = \|G\|'_\infty.$$

6.3. *For every functional $\xi\in(H^\infty)^\#$ there is at most one function G analytic in D and such that*

$$(+)\qquad \xi(F) = 2\pi(F*G)(1) = \lim_{r\to 1-}2\pi(F*G)(r) \quad \textit{for } F\in H^\infty.$$

If such a function G exists, it belongs to $(H^\infty)'$ and is represented by $(++)$.

Proof. Let us assume that for a functional $\xi\in(H^\infty)^\#$ there exists a function G analytic in D and satisfying $(+)$. Then, by 6.2, this function belongs to $(H^\infty)'$. G is the only function satisfying $(+)$, since its coefficients are determined uniquely

$$\gamma_n(G) = (U_n * G)(1) = \frac{1}{2\pi}\xi(U_n) \quad \text{for } n = 0, 1, 2, \ldots$$

6.4. *For every functional $\xi \in (H^\infty_{vw})^\#$ there exists a unique function G analytic in D satisfying $(+)$ for $F \in H^\infty$. This function belongs to $(H^\infty)'$, is represented by $(++)$ and is such that $\|\xi\|^\#_\infty = \|G\|'_\infty$.*

Proof. In view of 6.2 and 6.3 it suffices to show here that for $\xi \in (H^\infty_{vw})^\#$ there is a function G analytic in D and satisfying $(+)$ for $F \in H^\infty$. Let ξ° be the restriction of ξ to K. Obviously, $\xi^\circ \in (K_{vw})^\# \subset (K)^\#$. By 6.1 there is a function G analytic in D and such that $\xi^\circ(F) = 2\pi(F*G)(1)$ for $F \in K$. Since it is $T_rF \in K$ for $F \in H^\infty$ and $0 \leqslant r < 1$, we infer by 4.2 that the following limit exists and the following equalities hold

$$\xi(F) = \lim_{r\to 1-} \xi^\circ(T_rF) = \lim_{r\to 1-} 2\pi(F*G)(r) = 2\pi(F*G)(1) \quad \text{for } F \in H^\infty.$$

6.5. In the sequel $(H^\infty_{vw})'$ will denote a class of all functions $G \in (H^\infty)'$ for which the functional ξ defined by $(+)$ for $F \in H^\infty$ belongs to $(H^\infty_{vw})^\#$.

From 6.4, 4.4, 4.2 and 6.1 it easily follows that $(H^\infty_{vw})'$ is a closed linear subspace of $[(H^\infty)', \|\cdot\|'_\infty]$.

We shall show that

If G is a function analytic in D, then $T_rG \in (H^\infty_{vw})'$ for $0 \leqslant r < 1$.

Proof. For a fixed r, $0 \leqslant r < 1$, we define a functional

$$\xi(F) = 2\pi(F*G)(r) = 2\pi(F*T_rG)(1) \quad \text{for } F \in H^\infty.$$

Let $\{F_m\} \subset H^\infty$ be a sequence very weakly converging to 0. Then $\sup_m \|F_m\|_\infty = R < \infty$ and $\gamma_n(F_m) \to 0$ as $m \to \infty$ for $n = 0, 1, 2, \ldots$ In view of 4.5 we have for every m and k

$$\Big|\sum_{n=k}^{\infty} \gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant R\sum_{n=k}^{\infty}|\gamma_n(G)|r^n.$$

Since the series on the right-hand side of above inequality is convergent, then for every $\varepsilon > 0$ there exists a k such that

$$2\pi\Big|\sum_{n=k}^{\infty}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \frac{\varepsilon}{2} \quad \text{for } m = 1, 2, \ldots$$

Now, the fact that $\gamma_n(F_m) \to 0$ as $m \to \infty$ for $n = 0, 1, 2, \ldots$ implies that for an already fixed $\varepsilon > 0$ there exists an m_0 such that

$$2\pi\Big|\sum_{n=0}^{k-1}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \frac{\varepsilon}{2} \quad \text{for } m \geqslant m_0.$$

Hence we get for $m \geqslant m_0$

$$|\xi(F_m)| = 2\pi|(F_m*G)(r)| = 2\pi\Big|\sum_{n=0}^{\infty}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \varepsilon.$$

This proves that $\xi \in (H^\infty_{vw})^\#$. Hence $T_rG \in (H^\infty_{vw})'$.

6.6. *A function $G \in (H^\infty)'$ belongs to $(H^\infty_{vw})'$ if and only if*

$$\lim_{r\to 1-}\|T_rG - G\|'_\infty = 0.$$

Proof. Let $G \in (H^\infty_{vw})'$. Let us consider a functional ξ corresponding to the function G. For every $F \in H^\infty$ and $0 \leqslant r < 1$ we have

$$(T^\#_r\xi - \xi)(F) = 2\pi\big(F*(T_rG - G)\big)(1).$$

This, in view of 6.4 and 4.2, implies that

$$\|T_rG - G\|'_\infty = \|T^\#_r\xi - \xi\|^\#_\infty \to 0 \quad \text{as } r \to 1-.$$

On the other hand, if for $G \in (H^\infty)'$ is $\|T_rG - G\|'_\infty \to 0$ as $r \to 1-$, then by 6.5 we have $G \in (H^\infty_{vw})'$.

6.7. *The space $[(H^\infty_{vw})', \|\cdot\|'_\infty]$ is separable. Polynomials with rational coefficients form a dense set in this space.*

Proof. Let $G \in (H^\infty_{vw})'$ and ε be any positive number. By virtue of 6.6 there exists an r, $0 \leqslant r < 1$, such that $\|T_rG - G\|'_\infty \leqslant \frac{\varepsilon}{2}$. Since the series $\sum_{n=0}^{\infty}|\gamma_n(G)|r^n$ is convergent, it follows that there exists a k such that

$$2\pi\Big|\sum_{n=k}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big| \leqslant 2\pi\sum_{n=k}^{\infty}|\gamma_n(G)|r^n \leqslant \frac{\varepsilon}{4}$$

for $F \in H^\infty$ such that $\|F\|_\infty \leqslant 1$. Let us now take rational numbers a_n such that

$$|\gamma_n(G)r^n - a_n| \leqslant (8\pi k)^{-1}\varepsilon \quad \text{for } n = 0, 1, \ldots, k-1,$$

and we construct a polynomial $Q(z) = \sum_{n=0}^{k-1} a_n z^n$.

For $F \in H^\infty$ such that $\|F\|_\infty \leqslant 1$ we now have

$$2\pi\big|\big(F*(T_rG - Q)\big)(1)\big| \leqslant 2\pi\sum_{n=0}^{k-1}|\gamma_n(F)|\,|\gamma_n(G)r^n - a_n| + 2\pi\Big|\sum_{n=k}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big| \leqslant \frac{\varepsilon}{2}.$$

By virtue of 5.1 we get $\|T_rG - Q\|'_\infty \leqslant \varepsilon/2$. This yields

$$\|G - Q\|'_\infty \leqslant \|G - T_rG\|'_\infty + \|T_rG - Q\|'_\infty \leqslant \varepsilon.$$

6.8. *For non-trivial functionals $\xi\in(H^\infty)^\#$ such that $\xi(F)=0$ for every $F\in K$ there exists no function G analytic in D satisfying $(+)$ for $F\in H^\infty$.*

Proof. Let ξ be a non-trivial functional from $(\tilde{H}^\infty)^\#$. Let us suppose that there is a function G analytic in D satisfying $(+)$ for $F\in H^\infty$. Functions $U_n(z)=z^n$, $z\in D$, $n=0,1,2,\ldots$ all belong to K. Hence

$$\gamma_n(G)=(U_n*G)(1)=\frac{1}{2\pi}\xi(U_n)=0 \quad \text{for } n=0,1,2,\ldots$$

Thus we get $G(z)=0$ for all $z\in D$ and then $\xi(F)=2\pi(F*G)(1)=0$ for every $F\in H^\infty$, which contradicts our assumption on ξ.

The functionals referred to in 6.8 exist, since the functional

$$p(F)=\inf\{\|F-G\|_\infty\colon G\in K\}, \quad (F\in H^\infty),$$

is a non-trivial homogeneous pseudonorm on H^∞ such that $p(F)\leqslant\|F\|_\infty$ for every $F\in H^\infty$ and $p(F)=0$ if and only if $F\in K$. This enables us, in view of the Hahn–Banach theorem, to construct such functionals.

7.1. *For every function $G\in(H^\infty)'$ there is a function $\lambda\in\mathscr{V}$ such that G is a Cauchy–Stieltjes integral of λ, i.e.*

$$G(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{1-ze^{-it}}d\lambda(t), \quad (z\in D),$$

and $\|G\|'_\infty=\|\lambda\|^*_{\mathscr{V}}=\operatorname{var}\{\lambda(t)\colon 0\leqslant t\leqslant 2\pi\}$ (cf. [9]).

Proof. Let ξ° be a functional from $(K)^\#$ corresponding to $G\in(H^\infty)'$ according to 6.1. Since the space $[K,\|\cdot\|_\infty]$ is, via functions $F(e^{-i\cdot})$, $F\in K$, isometric isomorphic to a linear subspace of $[C,\|\cdot\|^*]$, there is, by virtue of the Hahn–Banach theorem, a functional $l\in(C)^\#$ such that $\xi^{\circ}(F)=l(F(e^{-i\cdot}))$ for $F\in K$ and $\|l\|^\#=\|\xi^{\circ}\|^\#$. In view of 2.3 for l there is a function $\lambda\in\mathscr{V}$ such that

$$l(f)=\int_0^{2\pi}f(t)\,d\lambda(t) \quad \text{for } f\in C \quad\text{and}\quad \|l\|^\#=\|\lambda\|^*_{\mathscr{V}}.$$

Thus we get

$$\xi^{\circ}(F)=\int_0^{2\pi}F(e^{-it})\,d\lambda(t) \quad \text{for } F\in K \text{ and } \|\xi^{\circ}\|^\#=\|\lambda\|^*_{\mathscr{V}}.$$

Let G_1 be the Cauchy–Stieltjes integral of λ. Then for $0\leqslant r<1$ and $F\in K$ we get

$$2\pi(F*G_1)(r)=\sum_{n=0}^\infty\gamma_n(F)r^n\int_0^{2\pi}e^{-int}d\lambda(t)$$
$$=\int_0^{2\pi}F(re^{-it})\,d\lambda(t)=\xi^{\circ}(T_rF),$$

and so

$$2\pi(F*G_1)(1)=\xi^{\circ}(F)=2\pi(F*G)(1) \quad \text{for every } F\in K.$$

By 6.1 we have $G=G_1$. Hence G is the Cauchy–Stieltjes integral of λ and $\|G\|'_\infty=\|\lambda\|^*_{\mathscr{V}}$.

7.2. *For every function $\lambda\in\mathscr{V}$ its Cauchy–Stieltjes integral G belongs to $(H^\infty)'$ and $\|G\|'_\infty\leqslant\|\lambda\|^*_{\mathscr{V}}$* (cf. [9]).

Proof. Let G be the Cauchy–Stieltjes integral of $\lambda\in\mathscr{V}$. Then for $F\in H^\infty$ and $z\in D$ we have

$$2\pi(F*G)(z)=\sum_{n=0}^\infty\gamma_n(F)z^n\int_0^{2\pi}e^{-int}d\lambda(t)$$
$$=\int_0^{2\pi}F(ze^{-it})\,d\lambda(t).$$

Thus we get for $F\in H^\infty$ and $z\in D$

$$2\pi|(F*G)(z)|\leqslant\sup\{|F(ze^{-it})|\colon 0\leqslant t\leqslant 2\pi\}\|\lambda\|^*_{\mathscr{V}}\leqslant\|F\|_\infty\|\lambda\|^*_{\mathscr{V}}.$$

Hence $\|G\|'_\infty\leqslant\|\lambda\|^*_{\mathscr{V}}$ and so 7.2 is proved.

7.3. We denote by $\mathscr{V}_+$ the class of all functions $\lambda\in\mathscr{V}$ for which

$$\int_0^{2\pi}e^{-int}d\lambda(t)=0 \quad \text{for } n=0,1,2,\ldots$$

It is easy to see that $\mathscr{V}_+$ is a closed linear subspace of $[\mathscr{V},\|\cdot\|^*_{\mathscr{V}}]$. For $\lambda_1,\lambda_2\in\mathscr{V}$ we say that $\lambda_1\sim\lambda_2$ if and only if $\lambda_1-\lambda_2\in\mathscr{V}_+$. This relation is an equivalence relation on $\mathscr{V}$. The quotient space $\mathscr{V}/\!\sim\,=\mathscr{V}/\mathscr{V}_+$ will be denoted by $\tilde{\mathscr{V}}$. As usual, $\lambda^\sim$ designates the equivalence class determined by λ. We equip the space $\tilde{\mathscr{V}}$ with the norm

$$\|\lambda^\sim\|^\sim_{\mathscr{V}}=\inf\{\|\lambda+\lambda_1\|^*_{\mathscr{V}}\colon \lambda_1\in\mathscr{V}_+\}, \quad (\lambda\in\mathscr{V}).$$

The space $[(H^\infty)',\|\cdot\|'_\infty]$ is isometric isomorphic to the space $[\tilde{\mathscr{V}};\|\cdot\|^\sim_{\mathscr{V}}]$. This isomorphism establishes the operator of Cauchy–Stieltjes integration.

This is an immediate conclusion from 7.1 and 7.2, and also from the fact that the Cauchy–Stieltjes integral G of $\lambda\in\mathscr{V}$ equals 0 if and only if $\lambda\in\mathscr{V}_+$.

7.4. *For every function $g\in L^1$ its Cauchy integral G belongs to $(H^\infty_{vw})'$ and $\|G\|'_\infty\leqslant\|g\|^*_1$.*

Proof. Let G be the Cauchy integral of a function $g\in L^1$. By III.1.6 we have for $F\in H^\infty$ and $z\in D$

$$2\pi(F*G)(z)=\int_0^{2\pi}F(ze^{-it})g(t)\,dt,$$

and so

$$2\pi|(F*G)(z)| \leqslant \|F\|_\infty \|g\|_1^*.$$

Hence $\|G\|'_\infty \leqslant \|g\|_1^*$ and $G \in (H^\infty)'$. Let us now observe that for $0 \leqslant r < 1$ the function $T_r G$ is the Cauchy integral of $T_r g$. In view of what is already proved we have for $0 \leqslant r < 1$

$$\|T_r G - G\|'_\infty \leqslant \|T_r g - g\|_1^*.$$

It is known ([2] p. 33) that for $g \in L^1$ $\|T_r g - g\|_1^* \to 0$ as $r \to 1-$. Thus $\|T_r G - G\|'_\infty \to 0$ as $r \to 1-$ and, by virtue of 6.6, $G \in (H^\infty_{vw})'$.

7.5. We denote by L^1_+ the class of all functions $g \in L^1$ for which

$$\int_0^{2\pi} e^{-int} g(t)\,dt = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

$\tilde{L}^1$ designates the quotient space L^1/L^1_+. On the space $\tilde{L}^1$ we define the norm by

$$\|g^{\simeq}\|_1^{\simeq} = \inf\{\|g+f\|_1^* : f \in L^1_+\}, \qquad (g \in L^1).$$

Since L^1 is complete with respect to the norm $\|\cdot\|_1^*$ and L^1_+ is its closed linear subspace, the space $\tilde{L}^1$ is complete with respect to this norm.

We shall show that:

The space $[(\tilde{L}^1)^\#, \|\cdot\|_1^\#]$ is isometric isomorphic to the space $[H^\infty, \|\cdot\|_\infty]$. More precisely, for every functional $\eta^{\simeq} \in (\tilde{L}^1)^\#$ here is a unique function $F \in H^\infty$ such that

$$\eta^{\simeq}(g^{\simeq}) = \int_0^{2\pi} F(e^{-it}) g(t)\,dt \quad \text{for } g \in L^1,$$

and, conversely, every functional represented by this formula with a function $F \in H^\infty$ belongs to $(\tilde{L}^1)^\#$. Besides $\|\eta^{\simeq}\|_1^\# = \|F\|_\infty$.

Proof. Let $\eta^{\simeq} \in (\tilde{L}^1)^\#$. The functional $\eta(g) = \eta^{\simeq}(g^{\simeq})$ for $g \in L^1$ clearly belongs to $(L^1)^\#$ and $\|\eta\|_1^\# = \|\eta^{\simeq}\|_1^\#$. It is known that for η there exists a unique (up to the set of zero measure) function $f \in L^\infty$ such that

$$\eta(g) = \int_0^{2\pi} f(t) g(t)\,dt \quad \text{for } g \in L^1.$$

Moreover, then $\|\eta\|_1^\# = \|f\|_\infty^*$. Since the functions $e^{-ik\cdot}$ belong to L^1_+ for $k = 1, 2, \ldots$, it follows that

$$0 = \eta(e^{-ik\cdot}) = \int_0^{2\pi} f(t) e^{-ikt}\,dt \quad \text{for } k = 1, 2, \ldots$$

By virtue of III.1.7 we now conclude that the Cauchy integral F of $f(2\pi - t)$ is also the Poisson integral of the same function. This implies

that $F \in H^\infty$ and $\|F\|_\infty = \|f\|_\infty^*$. Hence for $\eta^{\simeq} \in (\tilde{L}^1)^\#$ there is a unique function $F \in H^\infty$ such that

$$\eta^{\simeq}(g^{\simeq}) = \int_0^{2\pi} F(e^{i(2\pi - t)}) g(t)\,dt = \int_0^{2\pi} F(e^{-it}) g(t)\,dt$$

for $g \in L^1$. Then also $\|F\|_\infty = \|\eta^{\simeq}\|_1^\#$.

Conversely, let $F \in H^\infty$. Let us consider the functional

$$\eta(g) = \int_0^{2\pi} F(e^{-it}) g(t)\,dt \quad \text{for } g \in L^1.$$

This functional belongs to $(L^1)^\#$ since $F(e^{-i\cdot}) \in L^\infty$. Let $g \in L^1_+$. By 7.4, 6.5 and III.1.6 we get

$$\eta(g) = \lim_{r \to 1-} \int_0^{2\pi} F(re^{-it}) g(t)\,dt$$

$$= \lim_{r = 1-} \sum_{n=0}^{\infty} \gamma_n(F) r^n \int_0^{2\pi} e^{-int} g(t)\,dt = 0.$$

This means that $\eta^{\simeq}(g^{\simeq}) = \eta(g)$ is well defined for $g^{\simeq} \in \tilde{L}^1$ and belongs to $(\tilde{L}^1)^\#$.

7.6. *The space $[\tilde{L}^1, \|\cdot\|_1^{\simeq}]$ is isometric isomorphic to the space $[(H^\infty_{vw})', \|\cdot\|'_\infty]$. This isomorphism is established by the Cauchy integral operation.*

The proof of this theorem, as quite analogous to that of IV.3.3, will be omitted.

7.7. From 7.6 follow a number of corollaries. For instance

Every function $G \in (H^\infty_{vw})'$ is the Cauchy integral of some function $g \in L^1$.

Let us denote by $\mathscr{AC}$ a space of functions λ defined, absolutely continuous on $[0, 2\pi]$ and satisfying $\lambda(0) = 0$. Clearly $\mathscr{AC} \subset \mathscr{V}$. By the remark made in 2.4 it follows that $\mathscr{V}_+ \subset \mathscr{AC}$. The correspondence

$$\lambda(t) = \int_0^t g(\tau)\,d\tau, \quad 0 \leqslant t \leqslant 2\pi, \; g \in L^1,$$

is easily seen to be an isomorphism of L^1 onto $\mathscr{AC}$ and also L^1_+ onto $\mathscr{V}_+$. We then have also $\|\lambda\|_{\mathscr{V}}^* = \|g\|_1^*$. Hence the space $[\tilde{L}^1, \|\cdot\|_1^{\simeq}]$ and $[\mathscr{AC}/\mathscr{V}_+, \|\cdot\|_{\tilde{\mathscr{V}}}^{\simeq}]$ are isometric isomorphic to each other.

7.8. *$H^1 \subset (H^\infty_{vw})'$ and $H^1 \neq (H^\infty_{vw})'$; besides $\|F\|_\infty \leqslant \|F\|_1$ for every $F \in H^1$.*

Proof. Let $F \in H^1$. Then $F(e^{i\cdot}) \in L^1$. By 7.4 the Cauchy integral F of $F(e^{i\cdot})$ belongs to $(H^\infty_{vw})'$ and $\|F\|_\infty \leqslant \|F(e^{i\cdot})\|_1^* = \|F\|_1$. Let us now suppose that $H^1 = (H^\infty_{vw})'$. We take an arbitrary function $g \in L^1$. By virtue of 7.4 its Cauchy integral G belongs to $(H^\infty_{vw})'$. Since $(H^\infty_{vw})' = H^1$, we have, $G \in H^1$. Hence $G(e^{i\cdot}) \in L^1$ and so

$$\hat{g}(t) = i\left(\frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx + g(t) - 2G(e^{it})\right), \quad (0 \leqslant t < 2\pi),$$

is a member of L^1. This implies that $g \to \hat{g}$ maps L^1 into itself, which, as is well known, is not true (cf. [3]). Hence $H^1 \neq (H^\infty_{vw})'$.

8.1. In the space $(H^\infty)'$ we may consider, besides norm convergence, also the very weak convergence, which will be defined analogously to the previous cases.

In the same manner as in III.5.1, III.5.2, III.5.3 we obtain the following theorems:

A sequence $\{G_n\} \subset (H^\infty)'$ converges very weakly to $G \in (H^\infty)'$ if and only if the sequence of functionals $\{\xi_n^\circ\} \subset (K)^\#$ corresponding to the sequence $\{G_n\}$ by 6.1 pointwise converges to $\xi^\circ \in (K)^\#$ corresponding to G.

A sequence $\{G_n\} \subset (H^\infty)'$ is very weakly convergent if and only if $\sup \|G_n\|'_\infty < \infty$ and the sequence $\{\gamma_k(G_n)\}$ is convergent for $k = 0, 1, 2, \ldots$

A sequence $\{G_n\} \subset (H^\infty)'$ is very weakly convergent if and only if $\sup_n \|G_n\|'_\infty < \infty$ and the sequence $\{G_n(z)\}$ is convergent on the set of points $z \in D$ having a cluster point in D.

The unit ball $\{G \in (H^\infty)': \|G\|'_\infty \leqslant 1\}$ is sequentially very weakly compact.

8.2. *For every $F \in H^\infty$ the functional defined by*

$$(+) \qquad \eta^\circ(G) = \lim_{r\to 1-} 2\pi(F*G)(r) = 2\pi(F*G)(1) \qquad \text{for } G \in (H^\infty_{vw})'$$

belongs to $((H^\infty_{vw})')^\#$; besides $\|\eta^\circ\|^\#_\infty = \|F\|_\infty$. Conversely, for every functional $\eta^\circ \in ((H^\infty_{vw})')^\#$ there exists a unique function F analytic in D for which $(+)$ holds; this function belongs to H^∞ and is represented by

$$(++) \qquad F(z) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \eta^\circ(U_n) z^n, \quad (z \in D).$$

The proof of this theorem, based on 7.5 and analogous to that of IV.4.1, will be omitted.

8.3. *For every $F \in K$ the functional η defined by $(+)$ for $G \in (H^\infty)'$ belongs to $((H^\infty)'_{vw})^\#$; besides $\|\eta\|^\#_\infty = \|F\|_\infty$. Conversely, for every functional $\eta \in ((H^\infty)'_{vw})^\#$ there is a unique function F analytic in D satisfying $(+)$ for $G \in (H^\infty)'$; this function belongs to K and is represented by $(++)$.*

The proof of this theorem greatly resembles that of IV.4.6 and will be neglected.

8.4. $(H^\infty)' \neq (H^\infty_{vw})'$.

This follows directly from the fact that $(H^\infty)'$ is isomorphic to $\tilde{\mathscr{V}} = \mathscr{V}/\mathscr{V}_+$ and $(H^\infty_{vw})'$ to $\mathscr{AC}/\mathscr{V}_+$ and then $\mathscr{V}_+ \subset \mathscr{AC} \subset \mathscr{V}$ and $\mathscr{AC} \neq \mathscr{V}$.

8.5. *For non-trivial functionals $\eta \in ((H^\infty)')^\#$ such that $\eta(G) = 0$ for every $G \in (H^\infty_{vw})'$ there exists no function F analytic in D and satisfying $(+)$ for $G \in (H^\infty)'$.*

The proof, based on the fact that functions $U_n(z) = z^n$, $z \in D$, $n = 0, 1, 2, \ldots$ all belong to $(H^\infty_{vw})'$, and similar to that of 6.8, is omitted.

Functionals referred to in this theorem exist since

$$p(G) = \inf\{\|G - F\|'_\infty : F \in (H^\infty_{vw})'\}, \qquad (G \in (H^\infty)'),$$

is a non-trivial homogeneous pseudonorm on $(H^\infty)'$ such that $p(G) \leqslant \|G\|'_\infty$ for $G \in (H^\infty)'$ and $p(G) = 0$ if and only if $G \in (H^\infty_{vw})'$. This enables us, in view of the Hahn–Banach theorem, to construct such functionals.

8.6. $(H^\infty)''$ denotes the class of all functions F analytic in D for which

$$\|F\|''_\infty = \sup\{2\pi|(F*G)(z)| : G \in (H^\infty)', \|G\|'_\infty \leqslant 1, z \in D\} < \infty.$$

As in IV.4.7 we obtain that

$$[(H^\infty)'', \|\cdot\|''_\infty] = [H^\infty, \|\cdot\|_\infty] \ ([8]).$$

As a consequence of our considerations we get

8.7. *The spaces H^∞ and K are not reflexive.*

8.8. The results given in 1.4, 2.5 and 7.6 can be presented in another form if we consider the operator: $g(t) \to h(e^{it}) = e^{-it} g(2\pi - t)$ for g defined on $(0, 2\pi)$. This operator maps isometrically the space L^∞ onto itself and, respectively, for C and L^1. For $g \in L^1$ and $n = 0, 1, 2, \ldots$ the equality $\int_0^{2\pi} g(t) e^{-int}\,dt = 0$ holds if and only if $\int_0^{2\pi} g(2\pi - t) e^{-it} e^{i(n+1)t}\,dt = 0$. From this we deduce that this operator maps the space L^∞_+ onto H^∞, C_+ onto K and L^1_+ onto H^1. Thus we have

The space $(H^1)'$ is isometric isomorphic to the quotient space L^∞/H^∞, and so is the space $(H^1_{vw})'$ to C/K and the space $(H^\infty_{vw})'$ to L^1/H^1. This isomorphism establishes the operator

$$G(z) = \frac{1}{2\pi i}\int_C \frac{h(\zeta)}{1 - z\zeta}\,d\zeta, \qquad (z \in D),$$

where C is the boundary of D with the positive orientation. Moreover, then for $F \in H^1$ and $h(e^{i\cdot}) \in L^\infty$ or for $F \in H^\infty$ and $h(e^{i\cdot}) \in L^1$ we have

$$\lim_{r\to 1-} (F*G)(r) = \frac{1}{2\pi_i}\int_C F(\zeta) h(\zeta)\,d\zeta, \qquad (cf.\ \text{IV.5.6}).$$

8.9. The result given in 7.3 can also be presented in form analogous to that of 8.8. We denote by h^1 the space of all functions h harmonic in D for which

$$\|h\|_1 = \sup\Big\{\int_0^{2\pi} |h(re^{it})|\,dt:\ 0 \leqslant r < 1\Big\} < \infty.$$

The space L^1 can be identified with the closed linear subspace of h^1. We prove that

The space $[h^1, \|\cdot\|_1]$ *is isometric isomorphic to the space* $[\mathscr{V}, \|\cdot\|^*_{\mathscr{V}}]$. *This isomorphism establishes the operator*

$$h(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t+\theta)+r^2}\, e^{i\theta}\, d\lambda(\theta),$$

($\lambda \in \mathscr{V}$), $0 \leqslant r < 1$). *Moreover, this operator maps the space* $\mathscr{V}_+$ *onto* H^1.

Proof. Let h be the function constructed as above for $\lambda \in \mathscr{V}$. Then by the easy estimation we obtain $\|h\|_1 \leqslant \|\lambda\|^*_{\mathscr{V}}$. Conversely, let $h \in h^1$. We observe that

$$h(r\varrho e^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t+\theta)+r^2}\, e^{i\theta}\, d\lambda_\varrho(\theta), \qquad 0 \leqslant r,\ \varrho < 1,$$

where

$$\lambda_\varrho(\theta) = \int_0^{\theta} e^{-i\tau} h(\varrho e^{-i\tau})\, d\tau.$$

We see that $\lambda_\varrho(0) = 0$ and $\operatorname{var}\{\lambda_\varrho(\theta):\ 0 \leqslant \theta \leqslant 2\pi\} \leqslant \|h\|_1$ for every $0 \leqslant \varrho < 1$. By the well known Helly theorems we obtain a function λ_1 such that

$$h(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t+\theta)+r^2}\, e^{i\theta}\, d\lambda_1(\theta), \qquad (0 \leqslant r < 1),$$

and $\operatorname{var}\{\lambda_1(\theta):\ 0 \leqslant \theta \leqslant 2\pi\} \leqslant \|h\|_1$. Since for any fixed t and r ($0 \leqslant r < 1$) the first function under this Riemann–Stieltjes integral belongs to C, we may replace in this integral the function λ_1 by a function $\lambda \in \mathscr{V}$ such that $\|\lambda\|^*_{\mathscr{V}} \leqslant \operatorname{var}\{\lambda_1(\theta):\ 0 \leqslant \theta \leqslant 2\pi\} \leqslant \|h\|_1$. For $\lambda \in \mathscr{V}$ let

$$\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t+\theta)+r^2}\, e^{i\theta}\, d\lambda(\theta) = 0 \quad \text{for all } re^{it} \in D.$$

Then we have

$$\int_0^{2\pi} \cos n\theta\, e^{i\theta}\, d\lambda(\theta) = 0 \quad \text{and} \quad \int_0^{2\pi} \sin n\theta\, e^{i\theta}\, d\lambda(\theta) = 0$$

for $n = 0, 1, 2, \ldots$, and next

$$\int_0^{2\pi} e^{in\theta}\, d\lambda(\theta) = 0 \quad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

From this, as in 2.4, we obtain $\lambda = 0$. Thus we have a complete proof of the first part of the theorem.

Now let $\lambda \in \mathscr{V}_+$. Then λ is absolutely continuous on $[0, 2\pi]$. Hence the harmonic function h corresponding to λ is the Poisson integral of $e^{-i\theta}\frac{d\lambda}{dt}(2\pi-\theta)$, and $\frac{d\lambda}{dt} \in L^1_+$. From this we infer, as in 8.8, that $h \in H^1$. Conversely, let $h \in H^1$. Then we have $e^{-it}h(e^{-it}) \in L^1_+$ and $\lambda(\theta) = \int_0^{\theta} e^{-it}h(e^{-it}dt.)$ Hence $\lambda \in \mathscr{V}_+$.

From this theorem it follows immediately that

The space $(H^\infty)'$ *is isometric isomorphic to the quotient space* h^1/H^1. *This isomorphism establishes the operator*

$$G(z) = \lim_{r\to 1-} \frac{1}{2\pi i}\int_C \frac{h(r\zeta)}{1-z\zeta}\, d\zeta, \qquad (z \in D,\ h \in h^1),$$

where C *is the boundary of* D *with the positive orientation. Moreover, then for* $F \in K$ *and* $h \in h^1$ *we have*

$$\lim_{r\to 1-} (F * G)(r) = \lim_{r\to 1-} \frac{1}{2\pi i}\int_C F(\zeta)\, h(r\zeta)\, d\zeta.$$

I wish to express my warmest thanks to Professor W. Orlicz for his helpful criticism, valuable advice and inspiration.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.
[2] K. Hoffman, *Banach Spaces of Analytic Functions*, N. J. 1962.
[3] R. Leśniewicz, *On conjugate Functions and Fourier series in Orlicz spaces*, Bull. Acad. Polon. Sci., 14 (1966), pp. 627–636.
[4] — *On Hardy-Orlicz spaces*. I, Comm. Math., 15 (1971), pp. 3–56.
[5] — *On linear functionals in Hardy-Orlicz spaces*. I, Studia Math., 46 (1973), pp. 53–77.
[6] — *On linear functionals in Hardy-Orlicz spaces*. II, ibidem 46 (1973), pp. 259–295.
[7] A. Ostrowski, *Über die Bedeutung der Jensenschen Formel für einige Fragen der komplexen Funktionentheorie*, Acta Lit. Ac., Sc. Szeged 1 (1923), p. 80.

[8] A. E. Taylor, *Banach spaces of functions analytic in the unit circle*. I, Studia Math., 11 (1950), pp. 145–170.
[9] — *Banach spaces of functions analytic in the unit circle*. II, ibidem 12 (1951), pp. 25–50.
[10] A. Zygmund, *Trigonometric Series*, vol. I, Cambridge 1959.

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Received July 1, 1972 (563)

STUDIA MATHEMATICA, T. XLVII. (1973)

Operators associated with representations of amenable groups singular integrals induced by ergodic flows, the rotation method and multipliers

by

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Abstract. Suppose G is an amenable locally compact group, $k \in L^1(G)$ has compact support, and R_u is a uniformly bounded representation of G acting on $L^p(\mathfrak{M})$. It is shown that the operator

$$\hat{k}(R) = \int_G k(u) R_{u^{-1}} du$$

has $L^p(\mathfrak{M})$-operator norm not exceeding the $L^p(G)$-operator norm of the convolution operator defined by k. From this one can obtain an extension of the rotation method for singular integrals on $\boldsymbol{R}^n$ to Lie groups. Moreover, results of Calderón, on commutator operators, de Leeuw and Fife, on multipliers, are generalized.

§ 1. Introduction. In their work on Singular Integrals, Calderón and Zygmund observed that properties of those Singular Integrals having odd kernels could be derived easily from properties of the Hilbert transform

$$\tilde{f}(s) = \lim_{\varepsilon \to 0+} \frac{1}{\pi} \int_{\varepsilon \leqslant |t|} \frac{f(s-t)}{t} dt = \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(s-t)}{t} dt.$$

The approach they used, called by them the *method of rotation*, can be described in the following way. An *odd kernel* has the form $k(y) = \Omega(y)/|y|^n$, where $y = (y_1, y_2, \ldots, y_n) \in \boldsymbol{R}^n$, $|y| = \left(\sum_{j=1}^{n} y_j^2\right)^{1/2}$ and Ω is an odd function which is homogeneous of degree 0 and whose restriction to the surface of the unit sphere $\Sigma_{n-1} = \{y \in \boldsymbol{R}^n: |y| = 1\}$ is integrable. Let us fix a point y' of Σ_{n-1}; we then consider the one-parameter group $\{U_{y'}^t\}$, $-\infty < t < \infty$, of transformations of $\boldsymbol{R}^n$ defined by

$$U_{y'}^t x = x + ty' \tag{1.1}$$