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Integral representation of additive transformations on L_p spaces

by

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Abstract. Consider a complete positive measure space, a transformation F on an L_p space $(1 \leqslant p \leqslant \infty)$ of vector-valued functions, and a vector-valued measure m defined on certain measurable sets. We obtain sets of conditions which guarantee the existence of a suitable connecting map, so that F has an integral representation n terms of m-integrals of a certain type.

1. Introduction. Let v be a complete positive measure on a σ-algebra S of subsets of a nonempty set T, and let X, Y, Z denote Banach spaces. (All vector spaces in this paper are real.) Consider a Z-valued transformation F on an L_p space $(1 \leqslant p \leqslant \infty)$ of Y-valued functions on T, and an X-valued measure m on the ring $\{A \in S | v(A) < \infty\}$. Our aim is to study necessary and sufficient conditions, under which there exists a suitable continuous mapping φ of Y into the space $L(X, Z)$ of continuous linear operators, such that

$$F(h) = \int \varphi h\, dm \qquad \big(h \in L_p(Y)\big).$$

The transformations we consider are additive, i.e. $F(h+h') = F(h) + F(h')$, whenever h, h' are in L_p and $v\big(\{t \in T |\ h(t) \neq 0, h'(t) \neq 0\}\big) = 0$. Representations of additive transformations have been studied by several authors in recent times (see the references). Our results complement (and in some cases extend) the theorems of Dinculeanu ([3], pp. 145–158, 259–261) on the integral representation of linear mappings with respect to vector measures, the theorems of Martin and Mizel [9], Mizel and Sundaresan [11] on the representation of additive functionals with respect to a positive measure, and some of the theorems of Bartle and Joichi [1] concerning nonlinear operators on function spaces. We use a new technique and introduce certain new conditions mainly because the transformation is no longer linear and the measure no longer real valued.

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2. Notation. We use the theory of vector integration presented in Dinculeanu [3]. Let S_σ denote the collection of all sets in S which are σ-finite with respect to v, and $H(Y)$ the vector space of all Y-valued v-integrable simple functions on T. Let $1 \leqslant P < \infty$ and $1/P+1/Q = 1$. The Q-semi-variation v_Q of the measure m with respect to v and the space $L(X, Z)$ is defined by

$$v_Q(A) = \sup\Big|\int g\,dm\Big| \quad (A \subset T),$$

where the supremum is taken over all $g \in H\big(L(X,Z)\big)$ such that $|g|_P \leqslant 1$ and g vanishes outside A. ($|\cdot|$ denotes norm in general and $|\cdot|_P$ denotes the L_P-norm with respect to v.) If v_Q is finite on S_σ, then for a function G in $L_P\big(L(X,Z)\big)$ the integral $\int G\,dm$ is uniquely defined as $\lim\int g_n dm$, where $\{g_n\}$ is any sequence of functions in $H\big(L(X,Z)\big)$, Cauchy in L_P and converging to G v-a.e. (See [3], pp. 246–255.) Finally, we define a function $J \equiv J_{m,F}$ which plays an important role in the next section:

$$J(u, y) = \sup\Big|\sum r_i[F(u\chi_{A_i}) - F(y\chi_{A_i})]\Big| \quad (u, y \in Y),$$

where the supremum is taken over all functions $\sum r_i\chi_{A_i}$ in $H\big((-\infty, \infty)\big)$ with $\big|\sum r_i m(A_i)\big| \leqslant 1$ (i ranging over a finite set).

3. The principal results. In this section we state two representation theorems and prove one of them. We assume for convenience that the linear subspace X_m generated by the range of m is dense in X.

THEOREM 1. *Let $1 \leqslant p < \infty$, $1 \leqslant P < \infty$, and $1/P+1/Q = 1$. Let there be a set $T_0 \in S_\sigma$ such that $v(T_0) = \infty$ and T_0 contains no atoms of v, and let v_Q be finite on S_σ. Then for a transformation F: $L_p(Y) \to Z$, there exists a continuous mapping φ: $Y \to L(X, Z)$, such that for every function h in $L_p(Y)$ the composition φh is in $L_P\big(L(X,Z)\big)$ and*

$$F(h) = \int \varphi h\,dm,$$

if and only if

(i) *F is additive,*

(ii) *F is continuous with respect to L_p-convergence,*

(iii) $\lim J(u, y) = 0$ *whenever y approaches a fixed u, and*

(iv) *the set $\{J(0, y)|y|^{-p/P} | 0 \neq y \in Y\}$ is bounded.*

Proof. The 'if' part:
Let the conditions (i) through (iv) hold. We have

$$\Big|\sum r_i F(y\chi_{A_i})\Big| \leqslant J(0, y)\Big|\sum r_i m(A_i)\Big| \quad \Big(y \in Y, \sum r_i m(A_i) \in X_m\Big),$$

where $J(0, y)$ is finite by (iv). Hence for a fixed $y \in Y$, we can define a continuous linear mapping φ_y of X_m into Z by setting

$$\varphi_y\Big(\sum r_i m(A_i)\Big) = \sum r_i F(y\chi_{A_i}).$$

Consider the mapping φ: $Y \to L(X, Z)$, where $\varphi(y)$ is the unique extension of φ_y to X, for all $y \in Y$. By virtue of (iii) and (iv), φ is continuous and satisfies

$$|\varphi(y)| \leqslant K|y|^{p/P} \quad (y \in Y),$$

for some constant $K > 0$. Since F is additive, $\varphi(0) = 0$, and $F(y\chi_A) = \varphi(y)m(A)$, we have

$$F(f) = \int \varphi f\,dm \quad \big(f \in H(Y)\big).$$

Now, for any function $h \in L_p(Y)$, there exists a sequence $\{f_n\}$ in $H(Y)$ that converges to h in L_p and v-a.e. It follows from the properties of φ and the Vitali convergence theorem ([6], Th. 15, p. 150) that φh is in $L_P\big(L(X,Z)\big)$, and that

$$\lim|\varphi h - \varphi f_n|_P = 0 \quad \text{as } n \to \infty.$$

Since

$$\Big|\int\varphi h\,dm - \int\varphi f_n dm\Big| \leqslant |\varphi h - \varphi f_n|_P\, v_Q(A),$$

where $A \in S_\sigma$ is such that $h, f_1, f_2, \ldots$ all vanish outside A, we get

$$\lim\int\varphi f_n dm = \int\varphi h\,dm.$$

Hence

$$F(h) = \lim F(f_n) = \lim\int\varphi f_n dm = \int\varphi h\,dm.$$

The 'only if' part: Consider a continuous mapping φ: $Y \to L(X, Z)$ such that φh is in $L_P\big(L(X,Z)\big)$ whenever h is in $L_p(Y)$. φ must satisfy

$$|\varphi(y)| \leqslant K|y|^{p/P} \quad (y \in Y),$$

for some constant $K > 0$. For, otherwise there exists a sequence $\{y_n\}$ in Y such that

$$|\varphi(y_n)| > n|y_n|^{p/P} > 0 \quad (n = 1, 2, \ldots).$$

Since v is nonatomic and σ-finite on T_0 and $v(T_0) = \infty$, we can always choose a sequence $\{A_n\}$ of pairwise disjoint sets in S with

$$v(A_n) = n^{-2}|y_n|^{-p} \quad (n = 1, 2, \ldots).$$

Then the function $h = \sum y_n\chi_{A_n}$ is in $L_p(Y)$, while the composition $\varphi h = \sum\varphi(y_n)\chi_{A_n}$ is not in $L_P\big(L(X,Z)\big)$ (the sums being taken from $n = 1$ to ∞).

The transformation $F: L_p(Y) \to Z$, defined by $F(h) = \int \varphi h\, dm$, is additive, since $\varphi(0) = 0$, and $v_Q(A) = 0$ whenever $v(A) = 0$. The continuity of F can be established by an argument similar to the one given in an earlier paragraph, if we remember that every subsequence of every sequence $\{h_n\}$ converging to h in L_p, itself has a subsequence converging to h v-a.e. and in L_p. (iii) and (iv) are easily obtained from the properties of φ, and we conclude the proof of this theorem.

Remarks. The partial nonatomicity of v is not needed for the proof of the 'if' part. And in the presence of the conditions (i)-(iv), F uniquely determines the mapping φ.

THEOREM 2. *Let $1 \leqslant p \leqslant \infty$, $1 \leqslant P < \infty$, and $1/P + 1/Q = 1$. Let $v(T)$ and $v_Q(T)$ be finite, and let there be a set $T_0 \in S$ such that $v(T_0)$ is positive and T_0 contains no atoms of v. Then for a transformation $F: L_p(Y) \to Z$, there exists a continuous mapping $\varphi: Y \to L(X, Z)$ with $\varphi(0) = 0$, such that for every h in $L_p(Y)$ the composition φh is in $L_P(L(X, Z))$ and*

$$F(h) = \int \varphi h\, dm,$$

if and only if

(i) *F is additive,*

(ii) *F is continuous with respect to L_p-convergence, if $p < \infty$, and is continuous relative to bounded a.e. convergence, if $p = \infty$,*

(iii) *$\lim J(u, y) = 0$ whenever y approaches a fixed u,*

(iv) *$J(0, \cdot)$ is bounded on bounded subsets of Y, and*

(v) *$J(0, y) \leqslant K|y|^{p/P}$, whenever $|y| \geqslant K$, for some constant $K > 0$, if $p < \infty$.*

References

[1] R. G. Bartle and J. R. Joichi, *The preservation of convergence of measurable functions under composition*, Proc. Amer. Math. Soc. 12 (1961), pp. 122–126.

[2] R. V. Chacon and N. Friedman, *Additive functionals*, Arch. Rational Mech. Anal. 18 (1965), pp. 230–240.

[3] N. Dinculeanu, *Vector measures*, Int. Series Monog. Pure App. Math. 95, Berlin, 1967.

[4] L. Drewnowski and W. Orlicz, *On orthogonally additive functionals*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), pp. 883–888.

[5] — *On representation of orthogonally additive functionals*, ibid. 17 (1969), pp. 167–173.

[6] N. Dunford and J. T. Schwartz, *Linear operators* I, Pure App. Math. 7, New York, 1957.

[7] N. Friedman and M. Katz, *A representation theorem for additive functionals*, Arch. Rational Mech. Anal. 21 (1966), pp. 49–57.

[8] — *Additive functionals on L_p spaces*, Canad. J. Math. 18 (1966), pp. 1264–1271.

[9] A. D. Martin and V. J. Mizel, *A representation theorem for certain nonlinear functionals*, Arch. Rational Mech. Anal. 15 (1964), pp. 353–367.

[10] V. J. Mizel, *Characterization of nonlinear transformations possessing kernels*, Canad. J. Math. 22 (1970), pp. 449–471.

[11] V. J. Mizel and K. Sundaresan, *Representation of additive and biadditive functionals*, Arch. Rational Mech. Anal. 30 (1968), pp. 102–126.

[12] — *Representation of vector valued nonlinear functions*, Carnegie-Mellon U. Math. Dept. Report, 69–30 (1969).

[13] — *Additive functionals on spaces with non-absolutely-continuous norm*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 385–389.

[14] K. Sundaresan, *Additive functionals on Orlicz spaces*, Studia Math. 32 (1969), pp. 61–68.

[15] W. A. Woyczyński, *Additive functionals on Orlicz spaces*, Colloq. Math. 19 (1968), pp. 319–326.

[16] — *Additive operators*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 17 (1969), pp. 447–451.

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