A local factorization of analytic functions and its applications

by

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Abstract. The following notion is introduced: The family \((p_1): E \rightarrow E_{\infty}^\infty\) of linear operators is called a basic system for \(E\) iff the inverse images of neighbourhoods of zero in \(E\) form the base of neighbourhoods of zero in \(E\) and the family \(I\) is ordered in a suitable manner.

The local factorization, by some projection \(p_1\), of analytic mappings \(R \supseteq U \rightarrow E\), where \(U\) is a domain in \(E\) and \(E\) is a normed linear space, is proved and the following consequences are obtained:

- If \(E\) and \(E_1\) are complex and every \(E\) has the Baire property, then every \(g\)-analytic mapping of a domain \(U \subseteq E\), continuous at some point of \(U\), is continuous on the whole set \(U\).

The polynomial approximation property is studied in the case of locally convex spaces \(E\) and \(E_1\). Some results are also obtained in the case of real \(E\) and \(E_1\).

INTRODUCTION

A. Hirschowitz has proved in [4] that every analytic complex-valued function on an open subset of the Cartesian product of a family of linear topological spaces can be locally factorized by the projection on a finite number of coordinates. (See also [3] for the case of \(C^\infty\) and [9] for the case of \(C^\infty\).) An analogous fact was proved by L. Nachbin [7] for the case of a locally convex space \(E\) such that the canonical projections \(E \rightarrow E_1\) are open. For the continuous seminorm \(q\) on \(E_1\), \(E_1\) denotes the space \(E/(q^{-1}(0))\) with the norm induced by \(q\).

The aim of this paper is to generalize this fact and to apply it to the proof of some theorems about analytic function on linear topological spaces. To obtain these results we introduce the notion of a basic system. A topological vector space endowed with a basic system generalizes both the Cartesian product of linear topological spaces and the locally convex space with its system of seminorms and it is also a special case of the projective limit in the sense of [10].

In part I of this paper we prove some fundamental facts concerning this notion and apply them to the proof of two theorems about the conti-
nity of $G$-analytic and weakly analytic functions. Theorem 6.1.1 is a generalization of Theorem 6.1.2. The proofs of Propositions 1.1.1 and 1.2 are based on methods used by A. Hirschowitz in [3].

In part II the results of part I are applied to the study of analytic functions on locally convex spaces. There is proved a theorem about an extension of an analytic function over an open set in the completion of a locally convex space. This theorem is known in the case of a normed space. Next we generalize the Oka–Weil theorem about a polynomial approximation on polynomially convex compact sets on the large class of locally convex complete spaces containing all nuclear complete spaces.

Some results of part I are proved for the real case in part III. Theorem 3.2 is a generalization of the real analytic function extension theorem due to A. Hirschowitz [3].

All the notions and facts concerning analytic functions which are not defined here may be found in [10]. The notions concerning the functional analysis may be found in [10]. All spaces are assumed to be Hausdorff. We now recall some standard notations: "neighbourhood of zero" stands for "open, balanced neighbourhood of zero"; "subspace" stands for "vector subspace". The abbreviations t.v.s. and l.c.s. stand for "topological vector space" and "locally convex space", respectively. The completion of $E$ is always denoted by $\overline{E}$ and the complexification of $E$ by $\overline{E}^c$.

In the case where $F$ is not complete we shall call a function $f$ with values in $F$ "an analytic function" iff it is analytic as a function with values in $\overline{F}$.

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**I. BASIC SYSTEM AND ANALYTIC FUNCTIONS ON TOPOLOGICAL VECTOR SPACES OVER C**

**Definition 1.1.** Let $E$ be a t.v.s. and let $\{(E_i, p_i)\}_{i \in I}$ be a family of t.v.s.'s $E_i$ and linear continuous mappings $p_i$ of $E$ onto $E_i$. We say that $\{(E_i, p_i)\}_{i \in I}$ is a basic system for $E$ iff the following conditions are satisfied:

1. $I$ is a directed set.
2. The sets $p_i^{-1}(V_i)$, where $i \in I$ and $V_i$ is a neighbourhood of zero in $E_i$, form the base of neighbourhoods of zero in $E$.
3. If $i \geq i'$, then for every neighbourhood of zero $V_{i'} \subset E_{i'}$ there exists a neighbourhood of zero $V_i \subset E_i$ such that $p_i^{-1}(V_i) \subset p_{i'}^{-1}(V_{i'})$.

**Example 1.1.** Let $E$ be an l.c.s. and let $E$ be a family of seminorms on $E$, corresponding to some fixed base of neighbourhoods of zero in $E$.

We shall name such a family "a basic system of seminorms". Denoting by $E_i$ the space $E/p_i^{-1}(0)$ with the norm $q_i$ and by $p_i$ the canonical projection $p_i: E \to E_i$ we obtain the basic system $\langle (E_i, p_i) \rangle_{i \in \mathbb{N}}$ for $E$.

**Example 1.2.** Let $E = \bigoplus_{i \in A} E_i$ be a Cartesian product of t.v.s.'s $E_i$.

Let $I$ be the set of finite subsets of $A$ ordered by inclusion. Denote by $E_i$ the product $E_{i_1} \times \cdots \times E_{i_n}$ for $i = \{i_1, \ldots, i_n\}$ and by $p_i$ the projection of $E$ on $E_i$. The family $\langle (E_i, p_i) \rangle_{i \in I}$ is a basic system for $E$.

**Example 1.3.** For every t.v.s. $E$ there exists a trivial basic system, where $I = \{I\}$, $E_i = E$ and $p_i = \text{id}_E$.

**Remark 1.1.** Let $E$ be a t.v.s. with a basic system $\langle (E_i, p_i) \rangle_{i \in I}$. Let $F$ be a subspace of $E$. Then the family $\langle (p_F) : p_i | F \rangle_{i \in I}$ is a basic system for $F$.

**Remark 1.2.** Let $E$ be a t.v.s. with a basic system $\langle (E_i, p_i) \rangle_{i \in I}$.

The following statements hold:

1. If $i \geq i'$, then $\ker p_i \subset \ker p_{i'}$.
2. Let $U, V \in E$ be open sets in $E_i$. Then for every $i \geq i_0$ the set

$$V = p_i(p_{i_0}^{-1}(V_{i_0}))$$

is open in $E_i$.

**Proof.**

Ad 1. It follows from the condition 2 of Definition 1.1, that

$$\ker p_{i_0} = \bigcap_{U \in \mathcal{U}} p_{i_0}^{-1}(U) = \bigcap_{V \in \mathcal{V}} p_{i_0}^{-1}(V) = \ker p_i$$

where $\mathcal{U}$, $\mathcal{V}$ are the bases of neighbourhoods of zero in $E_{i_0}$ and $E_i$, respectively.

Ad 2. Since $\ker p_{i_0} \supset \ker p_i$, the mapping $h = p_{i_0} \circ p_{i_0}^{-1}$ is well-defined and linear. Condition 2 of Definition 1.1 implies that $h$ is continuous at zero, and so it is continuous. Hence $V = h^{-1}(V_{i_0})$ is open.

The following two results on factorization will be essential in the sequel.

**Proposition 1.1.** Let $E$ be a t.v.s. over $C$ with a basic system $\langle (E_i, p_i) \rangle_{i \in I}$. Let $f$ be an analytic function on an open set $U \subset E$ into a normed space $F$. Then for every $x \in U$ there exist $i \in I$ and a neighbourhood of zero $V_i \subset E_i$ such that $x + p_i^{-1}(V_i) \subset U$ and $f(x) + p_i^{-1}(V_i) = f(x) + p_i^{-1}(V_i)$, where $f_i$ is an analytic function on $p_i(x) + V_i$ into $F$.

**Proof.** Let $x \in U$, then there exist $i \in I$ and a neighbourhood $V_i \subset E_i$ such that $x + p_i^{-1}(V_i) \subset U$ and the set $f(x + p_i^{-1}(V_i))$ is bounded in $F$. Take $y = x + p_i^{-1}(V_i)$ and put $f_b(y) = f(y + m)$ for $m \in \ker p_i$. The function $f_b$ is analytic and bounded on $\ker p_i$, and by the Liouville theorem it is constant. This implies that the function $f_b(y) = f(y')$, where $y' + p_i^{-1}(y)$,
is well-defined for \( y \in p_1(x) + V_1 \). Now, let \( y_1, y_2 \in p_1(x) + V_1 \) and \( x, y \in p_1(x) + p_2^(-1)(y_2) \). The function \( f(t) = f_1(y_1 + t y_2) = f(x + t x_2) \), \( t \in [0, 1] \), is analytic in a neighbourhood of \( t = 0 \), and so \( f_t \) is \( G \)-analytic. Since \( f_t \) is bounded on \( p_2(x) + V_1 \), it is analytic on this set. We have \( f = f_0 + p_0 \) on \( x + p_2^(-1)(V_1) \).

**Proposition 1.2.** Let \( E \) be a t.s.s. over \( C \) with a basic system \( (E_0, p_0)_{loc} \).

Let \( f \) be a \( G \)-analytic function on an open connected set \( U \subseteq E \) into a normed space \( F \).

Assume that \( f \) is continuous at a point \( x_0 \in U \). We can then select some

1. \( x + V_1(x) + V_2(x) = U \),
2. For every \( x' \in x + V_1(x) \) and for every \( m \in V_2(x) \), \( m \perp ker p_1 \).

\[ f(x' + m) = f(x') \]

Proof. Since \( F \) is normed, there exists a neighbourhood \( U_0 \) of \( x_0 \) such that \( f(U_0) \) is bounded in \( F \). By Proposition 1.1, there exist \( i \in I \) and a neighbourhood \( V \) of \( x_0 \) for which \( f(x) = f(x + m) \) if \( x' \in V \) and \( m \perp ker p_1 \).

We denote by \( D \) the set of all \( x \in U \) for which there exist neighbourhoods \( V_1(x) \) and \( V_2(x) \) such that \( f \) is analytic on \( x + V_1(x) \) and \( V_2(x) \) satisfying conditions 1) and 2).

It is clear that \( D \) is open and non-empty. It is enough to show that \( D \) is closed in \( U \). Let \( x' \in x + V_1(x) \) be an accumulation point of \( D \). Then \( V_1(x') \) is a neighbourhood of zero in \( E \) such that \( x' + V_1(x') + V_2(x') \subseteq U \) and \( x' + V_1(x') \subseteq D \). Put \( V_2(x') = V_1(x') \cap V_2(x') \) and fix \( m \in V_2(x') \perp ker p_1 \). We define the function \( h \) on \( x + V_1(x') \) as follows: \( h(x) = f(x) - f(x + m) \). This function is \( G \)-analytic and vanishes on the set \( \{ x + V_1(x') \} \cap \{ x + V_2(x') \} \), and so it is zero on the whole set \( x + V_1(x') \). Hence \( f(x) = f(x + m) \) for \( x' + V_1(x') \) and \( m \in V_2(x') \perp ker p_1 \) and so \( x' \in D \).

**Theorem 1.1.** Let \( E \) be a t.s.s. over \( C \) with a basic system \( (E_0, p_0)_{loc} \).

Suppose that every \( E_i \) is a Baire space. Let \( f \) be a \( G \)-analytic function on an open and connected set \( U \subseteq E \) into an L.c.c. \( F \), continuous at a point \( x_0 \in U \). Then \( f \) is analytic on the whole set \( U \).

Proof. The mapping \( f \) into a l.c.c. \( F \) is continuous if and only if every \( x \in F(P) \), the composition \( p_2 f \) is continuous. We can hence assume without loss of generality, that \( F \) is normed. Let \( D \) denote the set of all \( x \in U \) such that \( f \) is continuous at \( x \). The set \( D \) is non-empty. Since \( f \) is \( G \)-analytic and \( D \) is normed, \( D \) is open. It is enough to show that \( D \) is closed in \( U \). Let \( x' \) be an accumulation point of \( D \), \( x' \in U \). Take \( i \in I \) and \( V_1(x') \) as in Proposition 1.2, and select \( t_n \to t \) and a neighbourhood \( V_1(x) \) of zero in \( E_0 \) such that \( p_1(x_0) = V_1(x') \). We choose \( x' + p_1(x_0) \cap D \) and take \( t_n > t \), and a neighbourhood \( V_2(x) \) of zero in \( E_0 \) such that \( x' + p_1(x_0) \cap D \cap \{ x' + p_1(x_0) \} \), and \( f \) is bounded on \( x' + p_1(x_0) \). From Remark 1.2 and Proposition 1.2 we obtain:

1. \( p_0(x') + p_1(x_0) \) is open in \( E_0 \).
2. \( f(x + m) = f(x) \) if \( x + x' + p_1(x_0) \) and \( m \in ker p_1 \).

Condition (2) is satisfied, since the function \( m \in ker p_1 \) is constant on \( V_1(x') \perp ker p_1 \) and \( x' + ker p_1 \subseteq U' \).

Hence \( f = f_0 + p_0 \) on \( x' + p_1(x_0) \). We have \( V_1(x') \perp ker p_1 \) and \( x' + ker p_1 \subseteq U' \).

The function \( f \) is bounded on \( p_0(x') + V_0 \). Hence \( f \) is continuous on this set. Since \( E_0 \) is a Baire space, this implies by Theorem 6.1.1 [1] that \( f \) is continuous on the whole set \( p_0(x') + p_1(x_0) \). Hence \( f \) is continuous on \( x' + p_1(x_0) \) as a composition of continuous mappings and \( x' + D \). So, \( D = U \). Every \( G \)-analytic and continuous function is analytic, and so \( f \) is analytic on \( U \).

**Example 1.** Every subspace of \( C^X \) satisfies assumptions of Theorem 1.1. Particularly, Theorem 1.1 holds for every l.c.c. with a weak topology.

**Example 2.** Let \( E = P E_0 \), where for every \( i \), \( E_i \) is a complete metric linear space. Let \( H \) be the subspace of \( E \) containing all those elements of \( E \) which have only finitely 
many coordinates different from zero. Theorem 1.1 holds for \( H \).

**Example 3.** Let \( X \) be a \( T_0 \)-topological space. Let \( C(X) \) denote the space of complex-valued continuous functions on \( X \), with seminorms \( \| f \| = sup |f(x)| \), where \( K \) is a compact set in \( X \). It follows from the Tietze–Urysohn theorem that \( C(X) \) is isomorphic to \( C(K) \). Since \( C(K) \) is a Banach space for every compact \( K \), Theorem 1.1 holds for \( C(X) \).

**Remark 1.3.** Theorem 1.1 remains true if the Baire property of \( E_i \) is replaced by the following property (B):

The t.s.s. \( E \) has the property (B) if, for every open, connected set \( U \subseteq E \) and every \( G \)-analytic function \( f \) defined on \( U \), the continuity of \( f \) on a non-empty open subset \( U_0 \subseteq U \) implies the continuity of \( f \) on the whole set \( U \). (Hence \( f \) is analytic on the whole set \( U \).

It is easy to check that the property (B) is equivalent to the following property: For every open, connected set \( U \subseteq E \) and each \( G \)-analytic function \( f \) defined on \( U \), if \( f \) is continuous at some point \( x_0 \in U \) then \( f \) is analytic on the whole set \( U \).

We can say that property (B) is an invariant of basic systems (If every \( E_i \) has the property (B) then \( E \) also has this property too.) For the application of this property see [5].
The examples 1 and 2 given above show that (B) is essentially weaker than the Baire property.

Remark 1.4. Theorem 1.1 can be false without the assumption that all $E_i$ have property (B). A suitable example, due to A. Hirschowitz, can be found in the preprint [1] published by J. H. E. E. 1970.

Theorem 1.2. Let $E$ be a t.c.s. over $C$ with a basic system $(\{E_i; p_i\})_{i\in I}$. Assume that every $E_i$ is metrizable and every $p_i$ is open. Let $f$ be a weakly analytic function on an open and connected set $U \subset E$ into a l.c.s. $F$, continuous at a point $x_0 \in U$. Then $f$ is analytic on all $U$.

Proof. We can assume, as above, without loss of generality, that $F$ is a normed space. The weak analyticity of $f$ implies that $f$ is $G$-analytic. Let $x \in U$. Take $\varepsilon > 0$ and the neighbourhood $V(x)$ of zero in $E_i$ such that $p_i^{-1}(V(x)) \subset V_i(x)$. We infer from Remark 1.2 and Proposition 1.2 that $f(x) = f(x + m)$ if $x \in x + p_i^{-1}(V(x))$ and $m \in A(E)$. Hence $f = f \circ p_i$ on $x + p_i^{-1}(V(x))$, where $f$ is a $G$-analytic function on $p_i^{-1}(V(x)) + V(x)$. For every $x \in F^*$ the composition $\omega f = \omega f \circ p_i$ is continuous on $x + p_i^{-1}(V(x))$. Since $p_i$ is open, this implies, that $\omega f$ is continuous on $p_i^{-1}(V(x)) + V(x)$ and therefore $f$ is weakly analytic. Since $E_i$ is metrizable, we infer by Theorem 6.3 in [1] that $f$ is analytic. This implies that $f$ is analytic on $x + p_i^{-1}(V(x))$. This ends the proof.

Corollary 1.1. Suppose that Theorem 1.2 holds for $E$. Let $U$ be an open and connected subset of $E$ and let $f$ be an analytic function on an open subset $V$ of $U$ into a l.c.s. $F$ such that for every $x \in F$ the function $\omega f$ can be extended to the analytic function $f(x)$ on $U$. If it follows from Theorem 1 of [5] that $f$ can be extended to a $G$-analytic function on $U$ into $F$. Then, by Theorem 1.2, this extension is analytic.

Example. Theorem 1.2 holds if $E = \prod_{\alpha \in A} E_{\alpha}$ where every $E_{\alpha}$ is a metrizable t.v.s.

For example, l.c.s. $E$ such that the canonical projections $E \rightarrow E_{\alpha}$, $\alpha \in A$ are open see Nachbin [7].

Finally, we give a proposition which is partially a converse of Theorem 1.1.

Proposition 1.3. Let $E$ be a t.c.s. with a basic system $(\{E_i; p_i\})_{i\in I}$. Suppose that $E$ has property (B) (see Remark 1.2) and every $p_i$ is open. Then every $E_i$ has the property (B).

Proof. Assume that there exists an $i \in I$ such that $E_i$ does not have property (B). Then we can find an open connected set $U_i \subset E_i$ a $G$-analytic function $f_i$ on $U_i$ into a normed space $F$, an open subset $V_i$ of $U_i$ and a point $x_0 \in U_i \setminus V_i$ such that $f_i$ is continuous on $V_i$ and not continuous at $x_0$. Take $U = p_i^{-1}(U_i)$, $V = p_i^{-1}(V_i)$, $f = f_i \circ p_i$ and choose $y_0 \in V_i$.

It is clear that $U$ is open and connected, $V$ is open and $f$ is $G$-analytic on $U$ and continuous on $V$. It remains to show that $f$ is not continuous at $y_0$. We shall prove that for every open neighbourhood $W$ of $y_0$, the function $f$ is not bounded on $W$. Since $p_i$ is open, $p_i(W)$ is an open neighbourhood of $y_0$. The function $f_i$ is not continuous at $x_0$, and so it is not bounded on $p_i(W)$. Hence $f = f_i \circ p_i$ is not bounded on $U$. This implies that $f$ is not continuous at $y_0$. We obtain a contradiction of property (B).

Corollary 1.2. A Cartesian product of t.c.s. $E_{\alpha}, E = \prod_{\alpha \in A} E_{\alpha}$ has property (B) if every Cartesian product of a finite number of spaces $E_{\alpha}$ has this property.

II. ANALYTIC FUNCTIONS ON LOCALLY CONVEX SPACES OVER $C$

Theorem 2.1. Let $V$ be an open set in a l.c.s. $E$ over $C$ and let $f$ map $U$ analytically into a Banach space $E$. Then there exist an open set $U' \supset U$ in the completion $\overline{E}$ of $E$ and an analytic extension $\tilde{f}: U' \rightarrow \overline{E}$ of $f$.

Proof. We can treat $E$ as a subspace of the cartesian product $G = \prod E_{\alpha}$, where $E_{\alpha}$ is the completion of $E_{\alpha}$ in $\Gamma(E)$ is a basic system of seminorms on $E$ (see Example 1.1). By the uniqueness of completion (see [10], p. 158) the completion $\overline{E}$ of $E$ is the closure of $E$ in $G$. Take $x \in U$. By Proposition 1.1 there exist $x_0 \in \Gamma(E)$ and $r > 0$ such that $x + p_i^{-1}(K(0, r)) \subset U$ and $f = f \circ p_i$, where $f$ is an analytic function on $p_i^{-1}(K(0, r))$. $K(0, r)$ denotes the ball with centre zero and radius $r$ in $E_{\alpha}$). Take the Taylor series of $f$ at $p_i(x)$, $f[p_i(x) + x] = \sum_{n=0}^{\infty} f_n(x)$. Let $\sum f_n(x)$ denote the symmetric $r$-linear mapping corresponding to $f_n$. We prove that $f_n$ maps Cauchy sequences in $E_{\alpha}$ onto Cauchy sequences in $F$. Let $\{a_n\}$ be a Cauchy sequence in $E_{\alpha}$. Clearly, there exists an $M > 0$ such that $\|q(a)_n\| < M$ for every $n$. We have

$$\|f_n(a_n) - f_n(a_0)\| = \|f_n(a_0, \ldots, a_n - a_0)\| = \|\sum_{n=0}^{\infty} f_n(x) - f_n(x_{n-1})\|$$

where $x_{n-1} = (a_0, \ldots, a_{n-1})$. Therefore $\{f_n(a_n)\}$ is a Cauchy sequence in $F$. Hence $f_n$ can be extended to the $r$-homogeneous polynomial $f_n$ on $\mathbb{R}^n$. The norm of $f_n$ is the same as the norm of $f_n$, and therefore if $r' < r$ is less than the radius of convergence.
gence of $\sum_{n=0}^{\infty} f_n$ in $E_n$, then the series $\sum_{n=0}^{\infty} f_n$ converges normally on the ball $\hat{B}(0, r')$ in $E_n$. Putting $\hat{f}(p_n(z)) = \sum_{n=0}^{\infty} f_n(z)$ for $z \in \hat{B}(0, r')$ we obtain the well defined analytic function on $p_n(z) + \hat{B}(0, r')$ such that $\hat{f} = f$ on $p_n(z) + \hat{B}(0, r')$. Denote by $\hat{p}_n$ the projection of $G$ onto $E_n$. We have $\hat{p}_n = p_n + \hat{B}(0, r')$. The set $V_n = \{z \in \hat{p}_n^{-1}(\hat{B}(0, r')) : z \in \hat{B}(0, r')\}$ is a neighborhood of $a$ in $E$. The function $f_n = \hat{f} \circ \hat{p}_n$ is analytic on $V_n$ and $f_n V_n \cap E = f$. Hence the function $\hat{f}(y)$ defined on the open in $\hat{B}$ set $\hat{U} = \bigcup_{x \in K} V_x$ by the formula $\tilde{f}(y) = f_n(y)$ for $y \in V_x$ is the required extension of $f$ over $\hat{U}$. (Since $V_n \cap U$ is dense in $V_n$ for every $x$, the above formula determines uniquely this extension.)

**Example 2.1.** Theorem 2.1 can be false if $E$ is not normed. Ph. Noverra [5] gave an example of a Banach space $E$ and its dense subspace $E^*$ such that for every $a \in \hat{E}^* \setminus E$ there exists an entire function $f_a : E \to C$ which cannot be extended onto any neighborhood of $a$. We take the function $\Phi : E \to C$, $C_a = C_a = C$, defined as follows:

$$\Phi(a) = (f_a(a))_{a \in \hat{E}^*}.$$

It is obvious that $\Phi$ is analytic and cannot be continued on any open set in $\hat{E}$.

Now we recall some known notions. Let $K$ be a compact subset of an l.o.s. $E$. The polynomially convex envelope $\hat{K}$ of $K$ is the set of all $a \in \hat{E}$ such that, for every complex-valued polynomial $P$ on $E$, $|P(a)| \leq \sup_{x \in K} |P(x)|$. It is easy to check that for every Banach space $E$ and every compact $\hat{K} \subset \hat{E}$, $\hat{K} = \hat{K}_E$. An open set $U \subset E$ is polynomially convex if for every compact subset $K$ of $U$, $\hat{K}$ is contained in $U$. A compact set $K \subset E$ is polynomially convex if $K = \hat{K}$.

**Definition 2.1.** Let $E$ be a l.o.s. over $C$ and let $F$ be a Banach space. We say that $E$ has the polynomial approximation property for $F$ if for every polynomially convex compact set $K \subset E$ and for every analytic function $f$ with values in $F$ defined on an open neighborhood of $K$ there exists a sequence of polynomials $(P_n)$ which converges to $f$ uniformly on $K$.

**Definition 2.2.** Let $E$ and $F$ be the same as above. We say that $E$ has the strong polynomial approximation property for $F$ if for every polynomially convex open set $U \subset E$ and for every analytic function $f$ on $U$ into $F$ there exists a sequence of polynomials $(P_n)$ which converges to $f$ uniformly on each compact set contained in $U$.

We now apply the method used by C. E. Rickart in the case of $E = C$ [9] to prove some facts concerning these notions.

**Lemma 2.1.** Let $E$ be a complete l.o.s. over $C$ with a basic system of l.o.s.'s $(\{E_k : k \in \omega\}, \{E_k : k \in \omega\})$. Let $K$ be a polynomially convex compact subset of $E$. Denote by $K_i$ the polynomially convex envelope of $p_i(K)$ in $E_i$. Then

1. $K_i$ is contained in $E_i$ for every $i \in I$.
2. $K = \bigcap_{i \in I} K_i$.
3. If $i \supseteq i'$ then $p_i^{-1}(K_i) = p_i^{-1}(K_{i'})$.
4. For every open set $U$ in $E_i$ containing $K$ and each $i \in I$ there exists an $i$, $i \supseteq i'$, such that $K_i = p_i(U)$.

**Proof.** Ad 1. Let $D_K$ denote the balanced, convex and compact envelope of $K$. Since $E$ is complete, $D_K$ is compact. The set $p_i(D_K)$ is balanced, convex and compact in $E_i$. It follows from the Hahn–Banach theorem that $K_i = p_i(D_K)$, because $p_i(K) = p_i(D_K)$. Hence $K_i = E_i$.

Ad 2. It is obvious that $K = \bigcap_{i \in I} p_i^{-1}(K_i)$. If $i \in K$ then there exists a continuous polynomial $P$ on $E$ for which $|P(x)| > \sup_{x \in K} |P(x)|$. It follows from Proposition 1.1 that there exists $i \succ i'$ and a continuous polynomial $P$ on $E_i$ such that $p_i \circ P$. We can extend $P$, by Theorem 2.1, to the continuous polynomial $\hat{P}$ on $\hat{E}_i$. We have $|\hat{P}(p_i(x))| > \sup_{x \in K} |P(x)|$, so $p_i(x) \in K_i$. Hence $K = \bigcap_{i \in I} p_i^{-1}(K_i)$.

Ad 3. If $i \in K$ then there exists a continuous polynomial $P$ on $E_i$ such that $P(p_i(x)) > \sup_{x \in K} |P(x)|$. Now, let $p_{i,c}$ denotes the mapping $p_{i,c}$ on $p_{i,c}(E_i)$.

**Proposition 2.1.** Let $K$ be a polynomially convex compact subset of a complete l.o.s. $E$ and let $U$ be an open set in $E$ containing $K$. Then there exists a polynomially convex open set $U_i$ such that $K \subset U_i \subset U$.

**Proof.** First we suppose that $E$ is a Banach space. Let $p_i$ be a continuous complex-valued polynomial on $E$. Denote by $V_p$ the set of all
\textbf{Local factorization of analytic functions} 

Proposition 1.1 There exist an $t \ni 1$ and an open set $V_f \subset E$, such that $(V_f) |f(U) = \phi \circ \chi| \subset U$. Let $D_{Kf}$ denote the closed, balanced and convex envelope of $K$. We have $(D_{Kf} \cup \Theta) \cap V_f = \Theta$, because $K = \cap V_f \subset U$. Since $D_{Kf} \setminus \Theta$ is compact, there exists a finite family of polynomials $P_1, \ldots, P_n$ such that $D_{Kf} \cap V_{P_1} \cap \ldots \cap V_{P_n} \subset U$. We denote by $D_{Kf}(a)$ the open $a$-envelope of $D_{Kf}$. It is a convex, balanced and open set in $E$. Now, we prove that there exists an $a > 0$ such that $D_{Kf}(a) \cap V_{P_1} \cap \ldots \cap V_{P_n} = U$. Suppose that this is not true. Then there exists a sequence $a_n \to \infty$, $a_n > 0$ such that for every $a_n$ there exists an $a_\xi \in D_{Kf}(a_n) \cap V_{P_1} \cap \ldots \cap V_{P_n}$ for every $a_n \in \mathbb{R}$. For every $a_n$ there exists a point $b_n \in D_{Kf}$ such that $||a_n - b_n|| < a_n$. Since $D_{Kf}$ is compact, we can choose a sequence $\{b_n\}$ converging to $b_n$. The corresponding sequence $\{a_n\}$ converges also to $b_n$. We have $b_n \in D_{Kf} \cap V_{P_1} \cap \ldots \cap V_{P_n}$ since $D_{Kf} \cap V_{P_1} \cap \ldots \cap V_{P_n}$ is compact and contained in $U$ and $a_n \in \mathbb{R}$ for every $a_n$, we obtain a contradiction. The set $U_n = D_{Kf}(a_n) \cap V_{P_1} \cap \ldots \cap V_{P_n}$ is open and polynomially convex. We have $K \subset U_n \subset U$. Now, $E$ be a complete l.c.s. We can treat $E$ as a subspace of the Cartesian product $G = \prod_{\xi \in \mathbb{N}} \tilde{K}_\xi$, where $\Gamma(E)$ is a basic system of seminorms in $E$. Denote by $\tilde{E}$ the projection of $E$ on $\tilde{K}_\xi$. Since $E$ is complete, it is closed in $G$. Let $E$ be a polynomially convex compact set in $E$ and let $U$ be an open set in $E$ containing $K$. Then, by condition 4 of Lemma 2.1, there exist a $g \in \Gamma(E)$ and an open set $V \subset E$ such that

\begin{align*}
K \subset U &= g^{-1} \in V, \\
K \subset V.
\end{align*}

Take for every $x \in V$ the ball $K(x', r) \subset K \subset V \subset \tilde{E}$ such that $K(x, r) = K(x', r) \cap \tilde{E}_\xi$ is open. Put $V' = \bigcup_{x \in V} K(x, r)$. We have $V \cap \tilde{E}_\xi = V'$. The set $E$ is a polynomially convex open set in $E$. Hence the set $U = g^{-1} \in V \cap \tilde{E}_\xi$ is open and polynomially convex in $E$. We have $K \subset U = \tilde{E}_\xi$. 

Corollary 2.1. If a complete l.c.s. $E$ has the strong polynomial approximation property for $P$, then $E$ has the polynomial approximation property for $P$.

Theorem 2.2. Let $P$ be a Banach space over $C$ and let $E$ be a complete l.c.s. over $C$ with a basic system of l.c.s.'s $(E_x, \mu)_x$ such that for every $x \in P$, $E_x$ has the polynomial approximation property for $P$. Then $E$ has the polynomial approximation property for $P$.

Proof. Let $K$ be a polynomially convex compact set in $E$ and let $f$ be an analytic function on an open neighbourhood $U$ of $K$ into $E$. By Proposition 1.1 there exist an $t \ni 1$ and an open set $V_f \subset E$, such that $K \subset U = g^{-1} \in V_f = U$ and $f|U = \phi \circ \chi|$. Where $\phi$ is an analytic function on $V_f$.

It follows from Lemma 2.1 that there exists an $a > 0$ such that $K = \cap V_f \subset U$. Since $D_{Kf} \setminus \Theta$ is compact, there exists a finite family of polynomials $P_1, \ldots, P_n$ such that $D_{Kf} \cap V_{P_1} \cap \ldots \cap V_{P_n} \subset U$. We denote by $D_{Kf}(a)$ the open $a$-envelope of $D_{Kf}$. It is a convex, balanced and open set in $E$. Now, we prove that there exists an $a > 0$ such that $D_{Kf}(a) \cap V_{P_1} \cap \ldots \cap V_{P_n} = U$. Suppose that this is not true. Then there exists a sequence $a_n \to \infty$, $a_n > 0$ such that for every $a_n$ there exists an $a_\xi \in D_{Kf}(a_n) \cap V_{P_1} \cap \ldots \cap V_{P_n}$ for every $a_n \in \mathbb{R}$. For every $a_n$ there exists a point $b_n \in D_{Kf}$ such that $||a_n - b_n|| < a_n$. Since $D_{Kf}$ is compact, we can choose a sequence $\{b_n\}$ converging to $b_n$. The corresponding sequence $\{a_n\}$ converges also to $b_n$. We have $b_n \in D_{Kf} \cap V_{P_1} \cap \ldots \cap V_{P_n}$ since $D_{Kf} \cap V_{P_1} \cap \ldots \cap V_{P_n}$ is compact and contained in $U$ and $a_n \in \mathbb{R}$ for every $a_n$, we obtain a contradiction. The set $U_n = D_{Kf}(a_n) \cap V_{P_1} \cap \ldots \cap V_{P_n}$ is open and polynomially convex. We have $K \subset U_n \subset U$. Now, $E$ be a complete l.c.s. We can treat $E$ as a subspace of the Cartesian product $G = \prod_{\xi \in \mathbb{N}} \tilde{K}_\xi$, where $\Gamma(E)$ is a basic system of seminorms in $E$. Denote by $\tilde{E}$ the projection of $E$ on $\tilde{K}_\xi$. Since $E$ is complete, it is closed in $G$. Let $E$ be a polynomially convex compact set in $E$ and let $U$ be an open set in $E$ containing $K$. Then, by condition 4 of Lemma 2.1, there exist a $g \in \Gamma(E)$ and an open set $V \subset E$ such that

\begin{align*}
K \subset U &= g^{-1} \in V, \\
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\end{align*}

Take for every $x \in V$ the ball $K(x, r) \subset K \subset V \subset \tilde{E}$ such that $K(x, r) = K(x', r) \cap \tilde{E}_\xi$ is open. Put $V' = \bigcup_{x \in V} K(x, r)$. We have $V \cap \tilde{E}_\xi = V'$. The set $E$ is a polynomially convex open set in $E$. Hence the set $U = g^{-1} \in V \cap \tilde{E}_\xi$ is open and polynomially convex in $E$. We have $K \subset U = \tilde{E}_\xi$.
Proposition 3.1. Suppose that $E$ is a t.v.s. over $R$ with a basic system $(E_i, p_i)_{i \in I}$. Then the family $(E_i, //\cdot//_i)_{i \in I}$ forms the basic system for $E$.

The proofs of the following two propositions will be omitted, since Proposition 3.2 follows immediately from Propositions 3.1 and 1.1 and Proposition 3.3 can be proved similarly to Proposition 1.2 in the complex case.

Proposition 3.2. Let $E$ be a t.v.s. over $R$ with a basic system $(E_i, p_i)_{i \in I}$. Let $f$ be a strongly analytic function on the open set $U \subseteq E$ into a normed space $F$. Then for every $a \in U$ there exists an $i \in I$ and a neighborhood of zero $V_a \subseteq E_i$ such that $a + p_i^{-1}(V_a) \subseteq U$ and $f(a) + f^* p_i(V_a) = f^* p_i(U)$, where $f^*$ is a strongly analytic function on $p_i(U) + V_a$ into $F$.

Proposition 3.3. Let $E$ be a t.v.s. over $R$ with a basic system $(E_i, p_i)_{i \in I}$. Assume that $f$ is a function defined on an open connected set $U \subseteq E$ with values in a normed space $F$, analytic on affine lines and strongly analytic on some open set $U_0 \subseteq U$. Then there exists an $i \in I$ such that for every $a \in U$ there exists a neighborhood of zero $V_a = V_{a}(x), V_a(x) \subseteq U$ for which:

1. $a + V_a(x) + V_a(x) = U$.
2. For every $a \in U$ and for every $m \in V_a(x) \cap \ker p_i$, $f(a + m) = f(a)$.

Theorem 3.1. Let $E$ be a t.v.s. over $R$ with a basic system $(E_i, p_i)_{i \in I}$. Assume that every $E_i$ is metrizable Baire space and every $p_i$ is open. Let $f$ be a weakly analytic function from an open connected set $U \subseteq E$ into a Banach space $F$. Suppose that $f$ is strongly analytic on an open set $U_0 \subseteq U$. Then $f$ is strongly analytic on the whole set $U$.

Outline of the proof. It follows from Lemma 7.1 [1] that $f$ is analytic on affine lines and hence $f$ satisfies the assumptions of Proposition 3.3. It can be proved in the same way as in the proof of Theorem 1.2 (by using Theorem 7.5 [1]) that $f$ is locally representable in the form $f = f_0 \cdot p_i$, where $f_0$ is a weakly analytic function on an open set $E_i$. By Theorem 7.4 [1] $f_0$ is analytic. It follows from Remark 7.1 [1], that $f_0$ is strongly analytic. Hence $f$ is strongly analytic.

Example 1. Theorem 3.1 holds for every subspace of $R^N$, since every continuous linear mapping onto a finite-dimensional t.v.s. is open.

Example 2. Theorem 3.1 holds if $E$ is a Cartesian product of complete linear metric spaces. Since such a product is always a Baire space, strong analyticity is in this case equivalent to analyticity in the usual sense [3].

Theorem 3.2. Let $E$ be a t.v.s. over $R$ with a basic system $(E_i, p_i)_{i \in I}$. Let $U$ be an open connected set in $E$ and let $S$ be a closed set in $E$ such that:

1. $U \cap S$ is connected.

2. For every $a \in U \cap S$ there exist a neighborhood $V_a$ of $a$ and an $i \in I$ such that $a + i \cdot V_a \cap S$ has the set $(a + i \cdot s + \ker p_i) \cap (U \cap S)$ non-empty and connected.

Then every strongly analytic function $f : U \cap S$ into a normed space $F$ can be extended to a strongly analytic function on the whole set $U$.

Proof. Let $a \in U \cap S$ and let $f$ be a strongly analytic function on $U \cap S$. Take $i_0$ as in Proposition 3.2 and let $a \in i_0 \cdot V_a$ such that there exists $a + i_0 \cdot V_a \subseteq U$ and $f(a) + f^* i_0 \cdot V_a = f^* i_0 \cdot V_a$. Since $S$ is closed, there exists a neighborhood $V_a$ of $a$ such that $V_a \subseteq W_a$ and $a + i_0 \cdot V_a \cap S$ has the set $(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ non-empty and connected.

Condition 2. It follows from Proposition 3.3 that the function $f(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ is locally constant, and so it is constant. Since $f$ is strongly analytic on $U \cap S$ and thus, for every $a \in U \cap S$ we put $f'(a) = f(a)$. Since $S$ is closed, there exists a neighborhood $V_a$ of $a$ such that $V_a \subseteq W_a$ and $a + i_0 \cdot V_a \cap S$ has the set $(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ non-empty and connected.

Condition 3. The function $f(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ is locally constant, and so it is constant. Since $f$ is strongly analytic on $U \cap S$ and thus, for every $a \in U \cap S$ we put $f'(a) = f(a)$. Since $S$ is closed, there exists a neighborhood $V_a$ of $a$ such that $V_a \subseteq W_a$ and $a + i_0 \cdot V_a \cap S$ has the set $(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ non-empty and connected.

Condition 4. The function $f(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ is locally constant, and so it is constant. Since $f$ is strongly analytic on $U \cap S$ and thus, for every $a \in U \cap S$ we put $f'(a) = f(a)$. Since $S$ is closed, there exists a neighborhood $V_a$ of $a$ such that $V_a \subseteq W_a$ and $a + i_0 \cdot V_a \cap S$ has the set $(a + i_0 \cdot s + \ker p_i) \cap (U \cap S)$ non-empty and connected.
Added in proof. S. Dinneen introduced in his paper * Fonctions analytiques dans les espaces vectoriels topologiques localement convexes* (C. R. Acad. Sci. Paris 274 (1972), A544–A546) the notion of $N$-projective limits being essentially the basic systems with open projections and studied the polynomial convexity and pseudoeconvexity in locally convex spaces with such systems.

Theorem 2.1 holds for every t.v.s. $E$ (not necessarily locally convex).

References


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On a functional representation of the lattice of projections on a Hilbert space

by

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Abstract. Let $(L, \leq, \vee, \wedge)$ be a $\sigma$-orthocomplemented partially ordered set with a full set of states $\mathcal{M}$. The dual $\mathcal{M}^{\ast}$ of $\mathcal{M}$ is defined as the set of functions $\delta: \mathcal{M} \to [0, 1]$, $\alpha \leq \beta$, where $\delta(m) = m(a)$ for all $m \in \mathcal{M}$. It is shown that $\mathcal{M}^{\ast}$ is isomorphic to $L$, and necessary and sufficient conditions are given in order that a set of functions $\mathcal{M} \subset [0, 1]$ be the dual of some full set of states on a $\sigma$-orthocomplemented poset. If $(L, \leq, \vee)$ is the $\sigma$-orthocomplemented lattice of projections on a Hilbert space $H$ and $\mathcal{M}$ the set of pure states induced by unit functionals in $H^{\ast}$, $\mathcal{M} = \{\varphi_{u}: u \in H^{\ast}, |\varphi_{u}| = 1\}$, then for each $\varphi \in H^{\ast}$ there is a unique continuous antilinear map $\varphi_{u}: H^{\ast} \to H^{\ast}$ such that $\varphi_{u}(u) = \varphi_{u}(u)(u)$ for all $u \in H^{\ast}, |\varphi_{u}| = 1$.

Let $L(H)$ be the set of orthogonal projections on a Hilbert space $H$. $L(H)$ is an orthomodular lattice with respect to the natural order $(P_{1} \leq P_{2})$ if and only if $R(P_{1}) \subseteq R(P_{2})$, where $R(P)$ denotes the range of $P$ with the orthogonality complementation $P \to P'$ (where $R(P') = R(P)^{\perp}$). This lattice belongs to a more general class of $\sigma$-orthocomplemented partially ordered sets which admit a full set of probability measures. Before we state a theorem about $L(H)$ we shall discuss some properties of this class of partially ordered sets.

Let $(L, \leq)$ be a partially ordered set (abbreviated to poset) with a one-to-one map $a \to a'$ of $L$ onto $L$. $(L_{1}, \leq, \vee, \wedge)$ is said to be a $\sigma$-orthocomplemented poset provided

(a) $a' = a$ for all $a \in L$.

(b) $a \leq b$ implies $b' \leq a'$.

(c) If $a_{1}, a_{2}, \ldots$ is a sequence of members of $L$ where $a_{i} \leq a_{j}$ for $i \neq j$, then the least upper bound $a_{1} \cup a_{2} \cup \ldots$ exists in $L$.

(d) $a \cup a' = b \cup b'$ for all $a$ and $b$ in $L$. (We denote $a \cup a'$ by $l$.)

A $\sigma$-orthocomplemented poset is said to be orthomodular (see [6]) if

(e) $a \leq b$ implies $b = a \cup (b' \cup a')$.

Let $L$ be a $\sigma$-orthocomplemented poset. A map $m: L \to [0, 1]$ is said to be a state on $L$ if $m$ is a probability measure, i.e. if $m(1) = 1$ and $m(a_{1} \cup a_{2} \cup \ldots) = m(a_{1}) + m(a_{2}) + \ldots$ whenever $a_{i} \leq a_{j}$ for $i \neq j$. 

[526]