$y_1, y_2, \ldots \in F$ such that $\sum_{i\geqslant 1} \|a_i\| \|y_i\| < \infty$ and, for all $V \in L(F, E)$, $\varphi(V) = \sum_{i\geqslant 1} \langle Vy_i, a_i\rangle$.

111. Remark. (E, E') is said to have *metric approximation property* (see [9]; V. 4) if $I_{E'}$ is in the $\mathscr{T}$-closure of $\{W: W \in L_0(E', E'),\ W$ is $w(E', E)$-continuous, $\|W\| \leqslant 1\}$. It has been proved in [11] that (E, E') has the metric approximation property $\Leftrightarrow E'$ has the metric approximation property. These considerations motivate Definition 112 and Theorem 113.

112. Definition. Let a be a totally splitting norm on L_0. We shall say that (E, E') has the *a-metric approximation property* (*a-m.a.p.*) if, for all F,

$$\{W: W \in L_0(E', F),\ W \text{ is } w(E', E)\text{-continuous},\ a^D(W) \leqslant 1\}$$

is $\mathscr{T}$-dense in $\{D_a(E', F): a^D \leqslant 1\}$.

If $a = g_{1,\infty} = d_{\infty,1}$ this concept is equivalent to that defined in Remark 111.

113. Theorem. *(E, E') has the a-m.a.p. $\Leftrightarrow E'$ has the a-m.a.p.*

Proof. ($\Rightarrow$) is trivial and ($\Leftarrow$) follows from Theorem 109 with a replaced by a^D.

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Received April 27, 1972 (519)

STUDIA MATHEMATICA T. XLVII. (1973)

A local factorization of analytic functions and its applications

by

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Abstract. The following notion is introduced: The family $(p_i: E \to E_i)_{i\in I}$ of linear epimorphisms is called a *basic system for E* iff the inverse images of neighbourhoods of zero in E_i form the base of neighbourhoods of zero in E and the family I is ordered in a suitable manner.

The local factorization, by some projection p_i, of analytic mappings $E \supset U \xrightarrow{f} F$, where U is a domain in E and F is a normed linear space, is proved and the following consequences are obtained:

If E and E_i are complex and every E has the Baire property, then every G-analytic mapping of a domain $U \subset E$, continuous at some point of U, is continuous on the whole set U.

The polynomial approximation property is studied in the case of locally convex complex E and E_i. Some results are also obtained in the case of real E and E_i.

INTRODUCTION

A. Hirschowitz has proved in [4] that every analytic complex-valued function on an open subset of the Cartesian product of a family of linear topological spaces can be locally factorized by the projection on a finite number of coordinates. (See also [3] for the case of C^N and [9] for the case of C^X.) An analogous fact was proved by L. Nachbin [7] for the case of a locally convex space E such that the canonical projections $E \to E_q$ are open. For the continuous seminorm q on E, E_q denotes the space $E/q^{-1}(0)$ with the norm induced by q.

The aim of this paper is to generalize this fact and to apply it to the proof of some theorems about analytic function on linear topological spaces. To obtain these results we introduce the notion of a basic system. A topological vector space endowed with a basic system generalizes both the Cartesian product of linear topological spaces and the locally convex space with its system of seminorms and it is also a special case of the projective limit in the sense of [10].

In part I of this paper we prove some fundamental facts concerning this notion and apply them to the proof of two theorems about the continuity of G-analytic and weakly analytic functions. Theorem 1.1 is a generalization of Theorem 6.1 [1]. The proofs of Propositions 1.1 and 1.2 are based on methods used by A. Hirschowitz in [3].

In part II the results of part I are applied to the study of analytic functions on locally convex spaces. There is proved a theorem about an extension of an analytic function over an open set in the completion of a locally convex space. This theorem was known in the case of a normed space. Next we generalize the Oka–Weil theorem about a polynomial approximation on polynomially convex compact sets on the large class of locally convex complete spaces containing all nuclear complete spaces.

Some results of part I are proved for the real case in part III. Theorem 3.2 is a generalization of the real analytic function extension theorem due to A. Hirschowitz [3].

All the notions and facts concerning analytic functions which are not defined here may be found in [1]. The notions concerning the functional analysis may be found in [10]. All spaces are assumed to be Hausdorff. We now recall some standard notations: "neighbourhood of zero" stands for "open, balanced neighbourhood of zero", "subspace" stands for "vector subspace". The abbreviations t.v.s. and l.c.s. stand for "topological vector space" and "locally convex space", respectively. The completion of E is always denoted by E and the complexification of E by $\tilde{E}$.

In the case where F is not complete we shall call a function f with values in F "an analytic function" iff it is analytic as a function with values in $\hat{F}$.

I would like to express my gratitude to Professor J. Siciak for his guidance and valuable remarks.

I. BASIC SYSTEM AND ANALYTIC FUNCTIONS ON TOPOLOGICAL VECTOR SPACES OVER C

DEFINITION 1.1. Let E be a t.v.s. and let $\{(E_i, p_i)\}_{i\in I}$ be a family of t.v.s.'s E_i and linear continuous mappings p_i of E onto E_i. We say that $\{(E_i, p_i)\}_{i\in I}$ is a *basic system for* E if the following conditions are satisfied:

(1) I is a directed set.

(2) The sets $p_i^{-1}(V_i)$, where $i\in I$ and V_i is a neighbourhood of zero in E_i, form the base of neighbourhoods of zero in E.

(3) If $i_1 \geqslant i_2$, then for every neighbourhood of zero $V_{i_2} \subset E_{i_2}$ there exists a neighbourhood of zero $V_{i_1} \subset E_{i_1}$ such that $p_{i_1}^{-1}(V_{i_1}) \subset p_{i_2}^{-1}(V_{i_2})$.

EXAMPLE 1.1. Let E be an l.c.s. and let E be a family of seminorms on E, corresponding to some fixed base of neighbourhoods of zero in E.

We shall name such a family "a basic system of seminorms". Denoting by E_q the space $E/q^{-1}(0)$ with the norm q and by p_q the canonical projection $p_q\colon E \to E_q$ we obtain the basic system $\{(E_q, p_q)\}_{q\in\Gamma(E)}$ for E.

EXAMPLE 1.2. Let $E = \underset{a\in A}{\boldsymbol{P}} E_a$ be a Cartesian product of t.v.s.'s E_a. Let I be the set of finite subsets of A ordered by inclusion. Denote by E_i the product $E_{a_1}\times\ldots\times E_{a_{n(i)}}$ for $i = \{a_1, \ldots, a_{n(i)}\}$ and by p_i the projection of E on E_i. The family $\{(E_i, p_i)\}_{i\in I}$ is a basic system for E.

EXAMPLE 1.3. For every t.v.s. E there exists a trivial basic system, where $I = \{i_0\}$, $E_{i_0} = E$ and $p_{i_0} = \mathrm{Id}_E$.

Remark 1.1. Let E be a t.v.s. with a basic system $\{(E_i, p_i)\}_{i\in I}$. Let F be a subspace of E. Then the family $\{(p_i(F), p_i|_F)\}_{i\in I}$ is a basic system for F.

Remark 1.2. Let E be a t.v.s. with a basic system $\{(E_i, p_i)\}_{i\in I}$. The following statements hold:

1. *If* $i_1 \geqslant i_2$, *then* $\ker p_{i_1} \subset \ker p_{i_2}$.

2. *Let* V_{i_0} *be an open set in* E_{i_0}. *Then for every* $i \geqslant i_0$ *the set*

$$V = p_i\big(p_{i_0}^{-1}(V_{i_0})\big)$$

is open in E_i.

Proof.

Ad 1. It follows from the condition 2 of Definition 1.1, that

$$\ker p_{i_2} = \bigcap_{U\in\mathscr{U}} p_{i_2}^{-1}(U) \supset \bigcap_{V\in\mathscr{V}} p_{i_1}^{-1}(V) = \ker p_{i_1}$$

where $\mathscr{U}$, $\mathscr{V}$ are the bases of neighbourhoods of zero in E_{i_2} and E_{i_1}, respectively.

Ad 2. Since $\ker p_{i_0} \supset \ker p_i$, the mapping $h = p_{i_0}\circ p_i^{-1}$ is well-defined and linear. Condition 2 of Definition 1.1 implies that h is continuous at zero, and so it is continuous. Hence $V = h^{-1}(V_{i_0})$ is open.

The following two results on factorization will be essential in the sequel.

PROPOSITION 1.1. *Let E be a t.v.s. over* $\mathbf{C}$ *with a basic system* $\{(E_i, p_i)\}_{i\in I}$. *Let f be an analytic function on an open set $U \subset E$ into a normed space F. Then for every $x\in U$ there exist $i\in I$ and a neighbourhood of zero $V_i \subset E_i$ such that $x+p_i^{-1}(V_i) \subset U$ and $f|x+p_i^{-1}(V_i) = f_i\circ p_i$, where f_i is an analytic function on $p_i(x)+V_i$ into F.*

Proof. Let $x\in U$, then there exist $i\in I$ and a neighbourhood $V_i \subset E_i$ such that $x+p_i^{-1}(V_i) \subset U$ and the set $f\big(x+p_i^{-1}(V_i)\big)$ is bounded in F. Take $x'\in x+p_i^{-1}(V_i)$ and put $f_0(m) = f(x'+m)$ for $m\in\ker p_i$. The function f_0 is analytic and bounded on $\ker p_i$, and so by the Liouville theorem it is constant. This implies that the function $f_i(y) = f(x')$, where $x'\in p_i^{-1}(y)$, is well-defined for $y \in p_i(x)+V_i$. Now, let $y_1, y_2 \in p_i(x)+V_i$ and $x_1 \in p_i^{-1}(y_1)$ $x_2 \in p_i^{-1}(y_2)$. The function $g(t) = f_i(y_1+ty_2) = f(x_1+tx_2)$, $t \in C$, is analytic in a neighbourhood of $t = 0$, and so f_i is G-analytic. Since f_i is bounded on $p_i(x)+V_i$, it is analytic on this set. We have $f = f_i \circ p_i$ on $x+p_i^{-1}(V_i)$.

PROPOSITION 1.2. *Let E be a t.v.s. over C with a basic system $\{(E_i, p_i)\}_{i\in I}$. Let f be a G-analytic function on an open connected set $U \subset E$ into a normed space F.*

Assume that f is continuous at a point $x_0 \in U$. We can then select some $i \in I$ such that for every $x \in U$ there exist neighbourhoods $V_1(x)$ and $V_2(x)$ of zero in E for which:

1) $x+V_1(x)+V_2(x) \subset U$.
2) *For every $x' \in x+V_1(x)$ and for every $m \in V_2(x) \cap \ker p_i$*

$$f(x'+m) = f(x').$$

Proof. Since F is normed, there exists a neighbourhood U_0 of x_0 such that $f(U_0)$ is bounded in F. By Proposition 1.1 there exist $i \in I$ and a neighbourhood V of x_0 for which $f(x) = f(x+m)$ if $x \in V$ and $m \in \ker p_i$. We denote by D the set of all $x \in U$ for which there exist neighbourhoods $V_1(x)$ and $V_2(x)$ satisfying conditions 1) and 2). It is clear that D is open and non-empty. It is enough to show that D is closed in U. Let $x' \in U$ be an accumulation point of D. Let $V_1(x')$ be a neighbourhood of zero in E such that $x'+V_1(x')+V_1'(x') \subset U$ and let $x'' \in (x'+V_1(x')) \cap D$. Put $V_2(x') = V_1(x') \cap V_2(x'')$ and fix $m \in V_2(x') \cap \ker p_i$. We define the function h on $x'+V_1(x')$ as follows $h(x) = f(x)-f(x+m)$. This function is G-analytic and vanishes on the set $(x'+V_1(x')) \cap (x''+V_1(x''))$, and so it is equal to zero on the whole set $x'+V_1(x')$. Hence $f(x) = f(x+m)$ for $x \in x'+V_1(x')$ and $m \in V_2(x') \cap \ker p_i$, and so $x' \in D$.

THEOREM 1.1. *Let E be a t.v.s. over C with a basic system $\{(E_i, p_i)\}_{i\in I}$. Suppose that every E_i is a Baire space. Let f be a G-analytic function on an open and connected set $U \subset E$ into an l.c.s. F, continuous at a point $x_0 \in U$. Then f is analytic on the whole set U.*

Proof. The mapping f into a l.c.s. F is continuous iff for every $q \in \Gamma(F)$, the composition $p_q \circ f$ is continuous. We can hence assume without loss of generality, that F is normed. Let D denote the set of all $x \in U$ such that f is continuous at x. The set D is non-empty. Since f is G-analytic and F is normed, D is open. It is enough to show that D is closed in U. Let x' be an accumulation point of D, $x' \in U$. Take $i \in I$ and $V_1(x')$ as in Proposition 1.2, and select $i_1 \geqslant i$ and a neighbourhood V_{i_1} of zero in E_{i_1} such that $p_{i_1}^{-1}(V_{i_1}) \subset V_1(x')$. We choose $x'' \in (x'+p_{i_1}^{-1}(V_{i_1})) \cap D$ and take $i_0 > i_1$ and a neighbourhood V_{i_0} of zero in E_{i_0} such that $x''+p_{i_0}^{-1}(V_{i_0}) \subset D \cap (x'+p_{i_1}^{-1}(V_{i_1}))$, and f is bounded on $x''+p_{i_0}^{-1}(V_{i_0})$. From Remark 1.2

and Proposition 1.2 we obtain:

(1) $\quad p_{i_0}(x')+p_{i_0}(p_{i_1}^{-1}(V_{i_1}))$ is open in E_{i_0}.

(2) $\quad f(x+m) = f(x)$ if $x \in x'+p_{i_1}^{-1}(V_{i_1})$ and $m \in \ker p_{i_0}$.

Condition (2) is satisfied, since the function $h(m) = f(x+m)$ is constant on $V_2(x') \cap \ker p_{i_0}$ and $x+\ker p_{i_0} \subset U$.

Hence $f = \bar{f} \circ p_{i_0}$ on $x'+p_{i_1}^{-1}(V_{i_1})$, where $\bar{f}$ is a G-analytic function on $p_{i_0}(x')+p_{i_0}(p_{i_1}^{-1}(V_{i_1}))$. We have

$$p_{i_0}(x'')+V_{i_0} \subset p_{i_0}(x')+p_{i_0}(p_{i_1}^{-1}(V_{i_1})).$$

The function $\bar{f}$ is bounded on $p_{i_0}(x)+V_{i_0}$. Hence $\bar{f}$ is continuous on this set. Since E_{i_0} is a Baire space, this implies by Theorem 6.1 [1] that $\bar{f}$ is continuous on the whole set $p_{i_0}(x')+p_{i_0}(p_{i_1}^{-1}(V_{i_1}))$. Hence f is continuous on $x'+p_{i_1}^{-1}(V_{i_1})$ as a composition of continuous mappings and $x' \in D$. So, $D = U$. Every G-analytic and continuous function is analytic, and so f is analytic on U.

EXAMPLE 1. Every subspace of C^X satisfies assumptions of Theorem 1.1. Particularly, Theorem 1.1 holds for every l.c.s. with a weak topology.

EXAMPLE 2. Let $E = \mathop{P}_{a\in A} E_a$, where for every a, E_a is a complete metric linear space. Let H be the subspace of E containing all those elements of E which have only finitely many coordinates different from zero. Theorem 1.1 holds for H.

EXAMPLE 3. Let X be a T_4-topological space. Let $C(X)$ denote the space of complex-valued continuous functions on X, with seminorms $q_K(f) = \sup_{x\in K}|f(x)|$, where K is a compact set in X. It follows from the Tietze–Urysohn theorem that $C(X)/_{q_K^{-1}(0)}$ is isomorphic to $C(K)$. Since $C(K)$ is a Banach space for every compact K, Theorem 1.1 holds for $C(X)$.

Remark 1.3. Theorem 1.1 remains true if the Baire property of E_i is replaced by the following property (B):

The t.v.s. E has the property (B) iff, for every open, connected set $U \subset E$ and for every G-analytic function f defined on U, the continuity of f on a non-empty open subset $U_0 \subset U$ implies the continuity of f on the whole set U. (Hence f is analytic on the whole set U.)

It is easy to check that the property (B) is equivalent to the following property: For every open, connected set $U \subset E$ and each G-analytic function f defined on U, if f is continuous at some point $x_0 \in U$ then f is analytic on the whole set U.

We can say that property (B) is an invariant of basic systems (If every E_i has the property (B) then E also has this property too.) For the application of this property see [5].

The examples 1 and 2 given above show that (B) is essentially weaker than the Baire property.

Remark 1.4. Theorem 1.1 can be false without the assumption that all E_i have property (B). A suitable example, due to A. Hirschowitz, can be found in the preprint of [1] published by I. H. E. S. 1970.

THEOREM 1.2. *Let E be a t.v.s. over C with a basic system $\{(E_i, p_i)\}_{i\in I}$. Assume that every E_i is metrizable and every p_i is open. Let f be a weakly analytic function on an open and connected set $U \subset E$ into a l.c.s. F, continuous at a point $x_0 \in U$. Then f is analytic on all U.*

Proof. We can assume, as above, without loss of generality, that F is a normed space. The weak analyticity of f implies that f is G-analytic. Let $x \in U$. Take $i_0 \in I$ and the neighbourhood $V_1(x)$ of zero as in Proposition 1.2. Choose $i \geqslant i_0$ and the neighbourhood V_i of zero in E_i such that $p_i^{-1}(V_i) \subset V_1(x)$. We infer from Remark 1.2 and Proposition 1.2 that $f(x') = f(x'+m)$ if $x' \in x + p_i^{-1}(V_i)$ and $m \in \ker p_i$. Hence $f = \bar{f} \circ p_i$ on $x + p_i^{-1}(V_i)$, where $\bar{f}$ is a G-analytic function on $p_i(x)+V_i$. For every $u \in F'$ the composition $u \circ f = u \circ \bar{f} \circ p_i$ is continuous on $x + p_i^{-1}(V_i)$. Since p_i is open, this implies, that $u \circ \bar{f}$ is continuous on $p_i(x)+V_i$ and therefore $\bar{f}$ is weakly analytic. Since E_i is metrizable, we infer by Theorem 6.3 [1] that $\bar{f}$ is analytic. This implies that f is analytic on $x + p_i^{-1}(V_i)$. This ends the proof.

COROLLARY 1.1. *Suppose that Theorem 1.2 holds for E. Let U be an open and connected subset of E and let f be an analytic function on an open subset V of U into a l.c.s. F such that for every $u \in F'$ the function $u \circ f$ can be extended to the analytic function f_u on U. It follows from Theorem 1 of [5] that f can be extended to a G-analytic function f on U into $\hat{F}$. Then, by Theorem 1.2, this extension is analytic.*

EXAMPLE. Theorem 1.2 holds if $E = \mathop{\boldsymbol{P}}_{a\in A} E_a$ where every E_a is a metrizable t.v.s.

For examples of l.c.s. E such that the canonical projections $E \to E_q$, $q \in \Gamma(E)$ are open see Nachbin [7].

Finally we give a proposition which is partially a converse of Theorem 1.1.

PROPOSITION 1.3. *Let E be a t.v.s. with a basic system $\{(E_i, p_i)\}_{i\in I}$. Suppose that E has property* (B) *(see Remark* 1.3) *and every p_i is open. Then every E_i has the property* (B).

Proof. Assume that there exists an $i \in I$ such that E_i does not have property (B). Then we can find an open connected set $U_i \subset E_i$, a G-analytic function f_i on U_i into a normed space F, an open subset V_i of U_i and a point $x_0 \in U_i \setminus V_i$ such that f_i is continuous on V_i and not continuous at x_0. Take $U = p_i^{-1}(U_i)$, $V = p_i^{-1}(V_i)$, $f = f_i \circ p_i$ and choose $y_0 \in p_i^{-1}(x_0)$.

It is clear that U is open and connected, V is open and f is G-analytic on U and continuous on V. It remains to show that f is not continuous at y_0. We shall prove that for every open neighbourhood W of y_0 the function f is not bounded on W. Since p_i is open, $p_i(W)$ is an open neighbourhood of x_0. The function f_i is not continuous at x_0, and so it is not bounded on $p_i(W)$. Hence $f = f_i \circ p_i$ is not bounded on W. This implies that f is not continuous at y_0. We obtain a contradiction of property (B).

COROLLARY 1.2. *A Cartesian product of t.v.s.'s E_a, $E = \mathop{\boldsymbol{P}}_{a\in A} E_a$ has property* (B) *iff every Cartesian product of a finite number of spaces E_a has this property.*

II. ANALYTIC FUNCTIONS ON LOCALLY CONVEX SPACES OVER C

THEOREM 2.1. *Let U be an open set in a l.c.s. E over C and let f map U analytically into a Banach space F. Then there exist an open set $\hat{U} \supset U$ in the completion $\hat{E}$ of E and an analytic extension $\hat{f}\colon \hat{U} \to F$ of f.*

Proof. We can treat E as a subspace of the cartesian product $G = \mathop{\boldsymbol{P}}_{q\in\Gamma(E)} \hat{E}_q$ where $\hat{E}_q$ is the completion of E_q and $\Gamma(E)$ is a basic system of seminorms on E (see Example 1.1). By the uniqueness of completion (see [10] p. 158) the completion $\hat{E}$ of E is the closure of E in G. Take $x \in U$. By Proposition 1.1 there exist $q \in \Gamma(E)$ and $r > 0$ such that $x + p_q^{-1}(K(0,r)) \subset U$ and $f = \bar{f} \circ p_q$, where $\bar{f}$ is an analytic function on $p_q(x)+K(0,r)$. ($K(0,r)$ denotes the ball with centre zero and radius r in E_q). Take the Taylor series of $\bar{f}$ at $p_q(x)$, $\bar{f}(p_q(x)+z) = \sum_{n=0}^{\infty} f_n(z)$. Let $\tilde{f}_n$ denote the symmetric n-linear mapping corresponding to f_n. We prove that f_n maps Cauchy sequences in E_q onto Cauchy sequences in F. Let $\{a_i\}$ be a Cauchy sequence in E_q. Clearly, there exists an $M > 0$ such that $q(a_i) < M$ for every i. We have

$$\|f_n(a_i) - f_n(a_j)\| = \|f_n(a_i) - f_n(a_i + a_j - a_i)\|$$
$$= \Big\| \sum_{k=1}^{n} \binom{n}{k} \tilde{f}_n\big(\underbrace{(a_j-a_i), \ldots, (a_j-a_i)}_{k \text{ times}}, \underbrace{a_i, \ldots, a_i}_{n-k \text{ times}}\big) \Big\|$$
$$\leqslant \sum_{k=1}^{n} \binom{n}{k} \|\tilde{f}_n\| \big(q(a_j-a_i)\big)^k \cdot M^{n-k}.$$

Therefore $\{f_n(a_i)\}$ is a Cauchy sequence in F. Hence f_n can be extended to the n-homogeneous polynomial $\hat{f}_n$ on $\hat{E}_q$. The norm of $\hat{f}_n$ is the same as the norm of f_n and therefore if $r' < r$ is less than the radius of convergence of $\sum_{n=0}^{\infty} f_n$ in E_q, then the series $\sum_{n=0}^{\infty} \hat{f}_n$ converges normally on the ball $\hat{K}(0, r')$ in $\hat{E}_q$. Putting $\hat{\bar{f}}(p_q(x)+z) = \sum_{n=0}^{\infty} \hat{f}_n(z)$ for $z \in \hat{K}(0, r')$ we obtain the well defined analytic function on $p_q(x)+\hat{K}(0, r')$ such that $\hat{\bar{f}} = \bar{f}$ on $p_q(x)+K(0, r')$. Denote by $\check{p}_q$ the projection of G onto E_q. We have $\check{p}_q = p_q$ on E. The set $V_x = \big(x+\check{p}_q^{-1}(\hat{K}(0, r'))\big) \cap \hat{E}$ is a neighbourhood of x in $\hat{E}$. The function $f_x = \hat{\bar{f}} \circ p_q$ is analytic on V_x and $f_x | V_x \cap E = f$. Hence the function $\hat{f}(y)$ defined on the open in $\hat{E}$ set $\hat{U} = \bigcup_{x \in E} V_x$ by the formula $\hat{f}(y) = f_x(y)$ for $y \in V_x$ is the required extension of f over $\hat{U}$. (Since $V_x \cap U$ is dense in V_x for every x, the above formula determines uniquely this extension.)

Example 2.1. Theorem 2.1 can be false if F is not normed. Ph. Noverraz [8] gave an example of a Banach space $\hat{E}$ and its dense subspace E such that for every $a \in \hat{E} \setminus E$ there exists an entire function $f_a \colon E \to \boldsymbol{C}$ which cannot be extended onto any neighbourhood of a. We take the function $\Phi \colon E \to \underset{a \in \hat{E} \setminus E}{\boldsymbol{P}} C_a$, $C_a = \boldsymbol{C}$, defined as follows

$$\Phi(x) = \{f_a(x)\}_{a \in \hat{E} \setminus E}.$$

It is obvious that Φ is analytic and cannot be continued on any open set in $\hat{E}$.

We now recall some known notions. Let K be a compact subset of an l.c.s. E. The polynomially convex envelope $\hat{K}$ of K is the set of all $x \in E$ such that, for every continuous complex-valued polynomial P on E, $|P(x)| \leqslant \sup_{z \in K} |P(z)|$. If E is complete then $\hat{K}$ is compact.

Now, let F be a Banach space. We can define the polynomially convex envelope $\hat{K}_F$ of K as the set of all $x \in E$ such that, for every continuous polynomial P on E with values in F, $\|P(x)\| \leqslant \sup_{z \in K} \|P(z)\|$. It is easy to check that for every Banach space F and each compact $K \subset E$, $\hat{K} = \hat{K}_F$. An open set $U \subset E$ is polynomially convex if for every compact subset K of U, $\hat{K}$ is contained in U. A compact set $K \subset E$ is polynomially convex if $K = \hat{K}$.

Definition 2.1. Let E be a l.c.s. over $\boldsymbol{C}$ and let F be a Banach space. We say that E has the *polynomial approximation property for F* iff for every polynomially convex compact set $K \subset E$ and for every analytic function f with values in F defined on an open neighbourhood of K there exists a sequence of polynomials $\{P_n\}$ which converges to f uniformly on K.

Definition 2.2. Let E and F be the same as above. We say that E has the *strong polynomial approximation property for F* iff for every polynomially convex open set $U \subset E$ and for every analytic function f on U into F there exists a sequence of polynomials $\{P_n\}$ which converges to f uniformly on each compact set contained in U.

We now apply the method used by C. E. Rickart in the case of $E = \boldsymbol{C}^X$ [9] to prove some facts concerning these notions.

Lemma 2.1. *Let E be a complete l.c.s. over $\boldsymbol{C}$ with a basic system of l.c.s.'s $\{(E_i, p_i)\}_{i \in I}$. Let K be a polynomially convex compact subset of E. Denote by K_i the polynomially convex envelope of $p_i(K)$ in $\hat{E}_i$. Then*

1) *K_i is contained in E_i for every $i \in I$.*

2) *$K = \bigcap_{i \geqslant i'} p_i^{-1}(K_i)$ for every $i' \in I$.*

3) *If $i' \geqslant i''$ then $p_{i'}^{-1}(K_{i'}) \subset p_{i''}^{-1}(K_{i''})$.*

4) *For every open set U in E containing K and each $i' \in I$ there exists an $i \in I$, $i \geqslant i'$, such that $K_i \subset p_i(U)$.*

Proof. Ad 1). Let D_K denote the balanced, convex and closed envelope of K. Since E is complete, D_K is compact. The set $p_i(D_K)$ is balanced, convex and compact in E_i. It follows from the Hahn–Banach theorem that $K_i \subset p_i(D_K)$, because $p_i(K) \subset p_i(D_K)$. Hence $K_i \subset E_i$.

Ad 2). It is obvious that $K \subset \bigcap_{i \geqslant i'} p_i^{-1} K_i$. If $x \notin K$ then there exists a continuous polynomial P on E for which $|P(x)| > \sup_{z \in K} |P(z)|$. It follows from Proposition 1.1 that there exist $i \geqslant i'$ and a continuous polynomial $\bar{P}$ on E_i such that $P = \bar{P} \circ p_i$. We can extend $\bar{P}$, by Theorem 2.1, to the continuous polynomial $\hat{P}$ on $\hat{E}_i$. We have $|\hat{P}(p_i(x))| > \sup_{z \in p_i(K)} |P(z)|$, so $p_i(x) \subset K_i$. Hence $K = \bigcap_{i \geqslant i'} p_i^{-1}(K_i)$.

Ad 3). If $x \notin p_{i''}^{-1}(K_{i''})$ then there exists a continuous polynomial P on $\hat{E}_{i''}$ such that $|P(p_{i''}(x))| > \sup_{z \in p_{i''}(K)} |P(z)|$. Now, let $p_{i',i''}$ denotes the mapping $p_{i''} \circ p_{i'}^{-1}$. This mapping is well defined, linear and continuous by the condition 2 of Definition 1.1. We have $p_{i',i''}(p_{i'}(x)) = p_{i''}(x)$. Taking $Q = P \circ \hat{p}_{i',i''}$ we obtain $|Q(p_{i'}(x))| > \sup_{z \in p_{i'}(K)} |Q(z)|$, and so $p_{i'}(x) \notin K_{i'}$.

Ad 4). We infer from 2) that $(D_K \setminus U) \cap \bigcap_{i \geqslant i'} p_i^{-1}(K_i) = \emptyset$. Since $D_K \setminus U$ is compact, there exists a finite family of indices $i_1, \ldots, i_n$ such that $D_K \cap p_{i_1}^{-1}(K_{i_1}) \cap \ldots \cap p_{i_n}^{-1}(K_{i_n}) \subset U$. Taking $i \geqslant i_j$, $j = 1, \ldots, n$ we obtain by 3) $D_K \cap p_i^{-1}(K_i) \subset U$. Hence $p_i(D_K) \cap K_i \subset p_i(U)$ and so $K_i \subset p_i(U)$.

Proposition 2.1. *Let K be a polynomially convex compact subset of a complete l.c.s. E and let U be an open set in E containing K. Then there exists a polynomially convex open set U_0 such that $K \subset U_0 \subset U$.*

Proof. First we suppose that E is a Banach space. Let P be a continuous complex-valued polynomial on E. Denote by V_P the set of all

$x \in E$ such that $|P(x)| \leqslant 1$. It is easy to check that K is equal to the intersection of all V_P for which $K \subset \{x \in E: |P(x)| < 1\} = U_P$. Let D_K denote the closed, balanced and convex envelope of K. We have $(D_K \setminus U) \cap \bigcap V_P = \emptyset$, because $K = \bigcap V_P \subset U$. Since $D_K \setminus U$ is compact, there exists a finite family of polynomials $P_1, \ldots, P_r$ such that $D_K \cap V_{P_1} \cap \ldots \cap V_{P_r} \subset U$. We denote by $D_K(\varepsilon)$ the open ε-envelope of D_K. It is a convex, balanced and open set in E. Now, we prove that there exists an $\varepsilon_0 > 0$ such that $D_K(\varepsilon_0) \cap V_{P_1} \cap \ldots \cap V_{P_r} \subset U$.

Suppose that this is not true. Then there exists a sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$ such that for every n there exists an $a_n \in D_K(\varepsilon_n) \cap V_{P_1} \cap \ldots \cap V_{P_r} \cap (E \setminus U)$. For every a_n there exists a point $b_n \in D_K$ such that $\|a_n - b_n\| < \varepsilon_n$. Since D_K is compact, we can choose a subsequence $\{b_{n_k}\}$, $b_{n_k} \to b_0$. The corresponding sequence $\{a_{n_k}\}$ converges also to b_0. We have $b_0 \in D_K \cap V_{P_1} \cap \ldots \cap V_{P_r}$. Since $D_K \cap V_{P_1} \cap \ldots \cap V_{P_n}$ is compact and contained in U and $a_{n_k} \in E \setminus U$ for every k, we obtain a contradiction. The set $U_0 = D_K(\varepsilon_0) \cap U_{P_1} \ldots \cap U_{P_r}$ is open and polynomially convex. We have $K \subset U_0 \subset U$.

Now, let E be a complete l.c.s. We can treat E as a subspace of the Cartesian product $G = \mathop{\boldsymbol{P}}_{q \in \Gamma(E)} \hat{E}_q$, where $\Gamma(E)$ is a basic system of seminorms in E. Denote by $\check{p}_q$ the projection of G on $\hat{E}_q$. Since E is complete, it is closed in G. Let K be a polynomially convex compact set in E and let U be an open set in E containing K. Then, by condition 4 of Lemma 2.1, there exist a $q \in \Gamma(E)$ and an open set $V \subset E_q$ such that

$$K \subset U_1 = p_q^{-1}(V) \subset U, \tag{1}$$

$$K_q \subset V. \tag{2}$$

Take for every $x \in V$ the ball $\hat{K}(x, r)$ in $\hat{E}_q$ such that $K(x, r) = \hat{K}(x, r) \cap E_q \subset V$. Put $\hat{V} = \bigcup_{x \in V} \hat{K}(x, r)$. We have $\hat{V} \cap E_q = V$. The set K_q is a polynomially compact set contained in $\hat{V}$. Since $\hat{E}_q$ is a Banach space, there exists a polynomially convex open set V_0 in $\hat{E}_q$ such that $K_q \subset V_0 \subset \hat{V}$. The set $\check{p}_q^{-1}(V_0)$ is a polynomially convex open set in G. Hence the set $U_0 = \check{p}_q^{-1}(V_0) \cap E$ is open and polynomially convex in E. We have $K \subset U_0 \subset U$.

Corollary 2.1. *If a complete l.c.s. E has the strong polynomial approximation property for F, then E has the polynomial approximation property for F.*

Theorem 2.2. *Let F be a Banach space over $\boldsymbol{C}$ and let E be a complete l.c.s. over $\boldsymbol{C}$ with a basic system of l.c.s.'s $\{(E_i, p_i)\}_{i \in I}$ such that for every $i \in I$ either E_i or $\hat{E}_i$ has the polynomial approximation property for F. Then E has the polynomial approximation property for F.*

Proof. Let K be a polynomially convex compact set in E and let f be an analytic function on an open neighbourhood U of K into F. By Proposition 1.1 there exist an $i' \in I$ and an open set $V_0 \subset E_{i'}$ such that $K \subset U_0 = p_{i'}^{-1}(V_0) \subset U$ and $f | U_0 = \bar{f} \circ p_{i'}$, where $\bar{f}$ is an analytic function on V_0. It follows from Lemma 2.1 that there exists an $i \geqslant i'$ such that $K_i \subset p_i(U_0)$. The set $p_i(U_0)$ is open in E_i by Remark 1.2. We have $f | U_0 = f_0 \circ p_i$, where $f_0 = \bar{f} \circ p_{i,i'}$ is analytic on $p_i(U_0)$. If E_i has the polynomial approximation property for F, then there exists a sequence of continuous polynomials $\{P_n\}$ on E_i into F converging to f_0 uniformly on K_i. Hence the sequence of polynomials $Q_n = P_n \circ p_i$ converges uniformly to f on K. Now, let $\hat{E}_i$ have the polynomial approximation property for F. By Theorem 2.1, f_0 can be extended over an open in $\hat{E}_i$ set W, $p_i(U_0) \subset W$ to the analytic function $\hat{f}_0$. Since $K_i \subset p_i(U_0)$, there exists a sequence of polynomials $\{\hat{P}_n\}$ on $\hat{E}_i$ such that $\hat{P}_n \to \hat{f}_0$ uniformly on K_i. Hence the sequence of polynomials $Q_n = \hat{P}_n \circ p_i$ converges uniformly on K to f.

Question. *Is it true that if $\hat{E}$ has the polynomial approximation property for F then E also has this property?*

C. Matyszczyk has proved in [6] that if E is a B_0-space with bounded approximation property, F is a B_0-space and f is an analytic function on an open and polynomially convex subset U of E, then for every locally bounded subset M of U there exists a sequence of polynomials $\{P_n\}$ which converges uniformly on compact subsets of M to f. Thus implies that every Banach space with a Schauder base has the strong polynomial approximation property for each Banach space. Using this result, Corollary 2.1 and Theorem 2.2, we obtain the following

Corollary 2.2. *If E is a complete l.c.s. with a basic system of seminorms $\Gamma(E)$ such that for every $q \in \Gamma(E)$, $\hat{E}_q$ is a Banach space with a Schauder base, then E has the polynomial approximation property for each Banach space F. This implies immediately that every complete nuclear space has this property.*

Remark. Ph. Noverraz has proved in [8] that every Banach space with a Schauder base has the polynomial approximation property for $\boldsymbol{C}$. (See also S. Dineen [2].) He announced also the following result: If E is a l.c.s. having the "strong approximation property", then E has the polynomial approximation property for C. Particularly, $C(K)$, $L^p(\mu)$ and every nuclear space possesses this property (see the author's review of [8] in Zentralblatt für Math. 216 p. 409).

III. ANALYTIC FUNCTIONS ON LINEAR TOPOLOGICAL SPACES OVER $\boldsymbol{R}$

Definition 3.1. Let E be a t.v.s. over $\boldsymbol{R}$ and let U be an open subset of E. We say that a function f defined on U with values in an l.c.s. F is *strongly analytic* iff it can be extended analytically over an open subset $\tilde{U} \supset U$ of the complexification $\tilde{E}$ of E.

PROPOSITION 3.1. *Suppose that E is a t.v.s. over $\mathbf{R}$ with a basic system $\{(E_i, p_i)_{i\in I}$. Then the family $\{(\tilde{E}_i, \tilde{p}_i)\}_{i\in I}$ forms the basic system for $\tilde{E}$.*

The proofs of the following two propositions will be omitted, since Proposition 3.2 follows immediately from Propositions 3.1 and 1.1 and Proposition 3.3 can be proved similarly to Proposition 1.2 in the complex case.

PROPOSITION 3.2. *Let E be a t.v.s. over $\mathbf{R}$ with a basic system $\{(E_i, p_i)\}_{i\in I}$. Let f be a strongly analytic function on the open set $U \subset E$ into a normed space F. Then for every $x \in U$ there exist an $i \in I$ and a neighbourhood of zero $V_i \subset E_i$ such that $x + p_i^{-1}(V_i) \subset U$ and $f|x+p_i^{-1}(V_i) = f_i \circ p_i$, where f_i is a strongly analytic function on $p_i(x)+V_i$ into F.*

PROPOSITION 3.3. *Let E be a t.v.s. over $\mathbf{R}$ with a basic system $\{(E_i, p_i)\}_{i\in I}$. Assume that f is a function defined on an open connected set $U \subset E$ with values in a normed space F, analytic on affine lines and strongly analytic on some open set $U_0 \subset U$. Then there exists an $i \in I$ such that for every $x \in U$ there exist neighbourhoods of zero $V_1(x)$, $V_2(x)$ in E for which:*

1) $x+V_1(x)+V_2(x) \subset U$.

2) *For every $x' \in x+V_1(x)$ and for every $m \in V_2(x) \cap \ker p_i$*

$$f(x'+m) = f(x').$$

THEOREM 3.1. *Let E be a t.v.s. over $\mathbf{R}$ with a basic system $\{(E_i, p_i)\}_{i\in I}$. Assume that every E_i is metrizable Baire space and every p_i is open. Let f be a weakly analytic function from an open connected set $U \subset E$ into a Banach space F. Suppose that f is strongly analytic on an open set $U_0 \subset U$. Then f is strongly analytic on the whole set U.*

Outline of the proof. It follows from Lemma 7.1 [1] that f is analytic on affine lines and hence f satisfies the assumptions of Proposition 3.3. It can be proved in the some way as in the proof of Theorem 1.2 (by using Theorem 7.5 [1]) that f is locally representable in the form $f = f_i \cdot p_i$, where f_i is a weakly analytic function on an open set in E_i. By Theorem 7.4 [1] f_i is analytic. It follows from Remark 7.1 [1], that f_i is strongly analytic. Hence f is strongly analytic.

EXAMPLE 1. Theorem 3.1 holds for every subspace of $\mathbf{R}^X$, since every continuous linear mapping onto a finite-dimensional t.v.s. is open.

EXAMPLE 2. Theorem 3.1 holds if E is a Cartesian product of complete linear metric spaces. Since such a product is always a Baire space, strong analyticity is in this case equivalent to analyticity in the usual sense [1].

THEOREM 3.2. *Let E be a t.v.s. over $\mathbf{R}$ with a basic system $\{(E_i, p_i)\}_{i\in I}$. Let U be an open connected set in E and let S be a closed set in E such that:*

1) *$U\setminus S$ is connected.*

2) *For every $x \in U\cap S$ there exist a neighbourhood V_x of x and an $i_0 \in I$ such that if $i \geqslant i_0$ and $x' \in V_x \cap S$, then the set $(x'+\ker p_i)\cap(U\setminus S)$ is non-empty and connected.*

Then every strongly analytic function from $U\setminus S$ into a normed space F can be extended to a strongly analytic function on the whole set U.

Proof. Let $x \in U\cap S$ and let f be a strongly analytic function on $U\setminus S$. Take i_1 as in Proposition 3.3 and $i_x \geqslant i_j$, $j = 0, 1$ such that there exists a $V_{i_x} \subset E_{i_x}$ for which $x \in W_x = p_{i_x}^{-1}(V_{i_x}) \subset V_x$. For $i \geqslant i_x$ and $x' \in W_x$ the set $(x'+\ker p_i)\cap(U\setminus S)$ is connected, since if $(x'+\ker p_i)\cap S$ contains some x_0 then the set $(x'+\ker p_i)\cap(U\setminus S) = (x_0+\ker p_i)\cap(U\setminus S)$ is non-empty and connected by the condition 2. It follows from Proposition 3.3 that the function $f|(x'+\ker p_i)\cap(U\setminus S)$ is locally constant, and so it is constant. Select for $y \in W_x \cap S$ a point $\bar{y} \in (y+\ker p_i)\cap(U\setminus S)$ and put $\bar{f}_x(y) = f(\bar{y})$. For $y \in W_x\setminus S$ we put $\bar{f}_x(y) = f(y)$. Since S is closed, there exists a neighbourhood V_0 of $\bar{y}$ such that $V_0 \subset W_x\setminus S$ and $y-\bar{y}+ +V_0 \subset W_x$. For every $y' \in y-\bar{y}+V_0$ we have $\bar{f}_x(y') = f(y'-y+\bar{y})$ and hence $\bar{f}_x$ is strongly analytic on some neighbourhood of y, and so it is a strongly analytic continuation of f over W_x. Observe that if $y \in W_x \cap W_{x''}$ then for $i > i_{x'}, i_{x''}$ $\bar{f}_{x'}(y) = f(\bar{y}) = \bar{f}_{x''}(y)$, where $\bar{y} \in (y+\ker p_i)\cap(U\setminus S)$. Put $\bar{f}(y) = \bar{f}_x(y)$ for $y \in W_x$ and $\bar{f}(y) = f(y)$ for $y \in U\setminus \bigcup_{x\in U\cap S} W_x$. The function $\bar{f}$ is the required continuation of f over U.

COROLLARY 3.1. *Suppose that E is a t.v.s. over $\mathbf{R}$ with a basic system $\{(E_i, p_i)\}_{i\in I}$ such that for every $i \in I$ $\ker p_i \neq \{0\}$. Let U be an open connected set in E and let K be a compact set in E. Then each strongly analytic function on $U\setminus K$ with values in a normed space F can be extended over the whole set U.*

Proof. First we prove that for every $i \in I$ $\dim\ker p_i = \infty$. Let $i \in I$. Take $a \neq 0$, $a \in \ker p_i$. Since E is a Hausdorff space, there exists, by conditions 2 and 3 of Definition 1.1, an $i_1 > i$ such that $a \notin \ker p_{i_1}$. Choose $a_1 \neq 0$, $a_1 \in \ker p_{i_1}$, and take an $i_2 > i_1$ such that $a_1 \notin \ker p_{i_2}$. Continuing this procedure we can construct a sequence $\{i_n\}$ such that $i_{n+1} > i_n > i$ and $\ker p_{i_{n+1}} \subsetneq \ker p_{i_n}$. This implies that $\dim\ker p_i = \infty$. Now, if U is an open connected subset of an infinite-dimensional t.v.s. and K is a compact subset of this t.v.s. then $U\setminus K$ is connected. It implies immediately that the assumptions of Theorem 3.2 are satisfied. This ends the proof.

The assumptions of Corollary 3.1 are satisfied for many t.v.s.'s Particularly, if E is a Cartesian product of infinitely many t.v.s.'s if E is the space of real continuous functions on a non-compact $T_{3\,1/2}$ space with a compact-open topology and if E is the space of functions of class C^∞ on the real line with a sequence of seminorms $\|f\|_n = \sup_{x\in\langle -n,n\rangle} (|f(x)|+ +|f'(x)|+\ldots+|f^{(n)}(x)|)$.

Added in proof. S. Dineen introduced in his paper *Fonctions analytiques dans les espaces vectoriels topologiques localement convexes* (C. R. Acad. Sci. Paris 274 (1972), A544–A546) the notion of N-projective limits being essentially the basic systems with open projections and studied the polynomial convexity and pseudoconvexity in locally convex spaces with such systems.

Theorem 2.1 holds for every t.v.s. E (not necessarily locally convex).

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Received May 10, 1972 (526)

STUDIA MATHEMATICA T. XLVII. (1973)

On a functional representation of the lattice of projections on a Hilbert space

by

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Abstract. Let $(L, \leqslant, ')$ be a σ-orthocomplemented partially ordered set with a full set of states M. The dual M' of M is defined as the set of functions $\bar{a}\colon M \to [0, 1]$, $a \in L$, where $\bar{a}(m) = m(a)$ for all $m \in M$. It is shown that M' is isomorphic to L, and necessary and sufficient conditions are given in order that a set of functions $M \subset [0, 1]^X$ be the dual of some full set of states on a σ-orthocomplemented poset. If $(L, \leqslant, ')$ is the σ-orthocomplemented lattice of projections on a Hilbert space H and M the set of pure states induced by unit functionals in H^*, $M = \{\psi(u)\colon u \in H^*, \|u\| = 1\}$, then for each $g \in M'$ there is a unique continuous antilinear map $\varphi_g\colon H^* \to H^{**}$ such that $g\psi(u) = \varphi_g(u)(u)$ for all $u \in H^*$, $\|u\| = 1$.

Let $L(H)$ be the set of orthogonal projections on a Hilbert space H. $L(H)$ is an orthomodular lattice with respect to the natural order ($P_1 \leqslant P_2$ if and only if $R(P_1) \subset R(P_2)$ where $R(P)$ denotes the range of P) with the orthogonal complementation $P \to P'$ (where $R(P') = R(P)^{\perp}$). This lattice belongs to a more general class of σ-orthocomplemented partially ordered sets which admit a full set of probability measures. Before we state a theorem about $L(H)$ we shall discuss some properties of this class of partially ordered sets.

Let $(L, \leqslant)$ be a partially ordered set (abbreviated to poset) with a one-to-one map $a \to a'$ of L onto L. $(L, \leqslant, ')$ is said to be a *σ-orthocomplemented poset* provided

(a) $a'' = a$ for all $a \in L$.

(b) $a \leqslant b$ implies $b' \leqslant a'$.

(c) If $a_1, a_2, \ldots$ is a sequence of members of L where $a_i \leqslant a_j'$ for $i \neq j$, then the least upper bound $a_1 \cup a_2 \cup \ldots$ exists in L.

(d) $a \cup a' = b \cup b'$ for all a and b in L. (We denote $a \cup a'$ by 1.)

A σ-orthocomplemented poset is said to be *orthomodular* (see [6]) if

(e) $a \leqslant b$ implies $b = a \cup (b' \cup a)'$.

Let L be a σ-orthocomplemented poset. A map $m\colon L \to [0, 1]$ is said to be a *state on L* if m is a probability measure, i.e. if $m(1) = 1$ and $m(a_1 \cup a_2 \cup \ldots) = m(a_1) + m(a_2) + \ldots$ whenever $a_i \leqslant a_j'$ for $i \neq j$.