Added in proof. Theorem 1.1 can be generalized to the following form:

THEOREM 1.1a. *Let X be a complemented subspace of $\sum_{i=1}^{n} l_{p_i}$. Then X is finite dimensional or X is isomorphic to $\sum_{k=1}^{r} l_{p_{i_k}}$ for some subset $\{p_{i_k}\}_{k=1}^{r}$ of the set $\{p_i\}_{i=1}^{n}$.*

The details will appear elsewhere.

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Splitting quasinorms and metric approximation properties

by

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Abstract. In this paper we consider quasinorms on the class of operators of finite rank between Banach spaces, the dual quasinorms that they define, and their connections with the Persson–Pietsch duality theory, maximal ideals and the metric approximation property.

INTRODUCTION

L stands for the class of all bounded operators between Banach spaces and L_0 stands for the subclass of L consisting of all operators of finite rank. In what follows, α is a quasinorm on L_0 (see Definition 3).

In Section 2 we consider three factorization conditions that can be imposed on α, namely that α be *left splitting*, *right splitting* or *splitting* (see Definition 7). (The second and third of these conditions were suggested by some comments of A. Pietsch. In particular, "splitting" was suggested by Pietsch's "upper semicontinuity".) We prove in Lemma 8 (c) that if α is left splitting then α' (see Notation 4) is right splitting and in Theorem 13 that if α is splitting then α' is splitting. We do not know whether if α is right splitting then α' is left splitting (see Problem 10).

In Section 3 we consider a general process by which splitting quasinorms on L_0 can be defined. In particular, we discuss the g_p and d_p norms of Saphar (see Remark 19).

Sections 4 and 5 are devoted to some technical results.

In Sections 6 and 7 we define a function $\alpha^D\colon L\to\overline{\mathscr{R}^+}$ and investigate some of its properties. In Section 8 we investigate the class D_α of operators for which $\alpha^D<\infty$. If α is *reasonable* (see Definition 38) then $(D_\alpha, \alpha^D|D_\alpha)$ is a normed ideal (see Lemma 42) even if α fails to be a norm on L_0. However, if α is a splitting norm on L_0 then we can prove a duality result (Theorem 44) which seems to be at the base of the Persson–Pietsch duality

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theory. In Section 14 we shall show that the theory of the objects $(D_\alpha, \alpha^D|D_\alpha)$ (where α is a splitting quasinorm on L_0) generalizes the theory of maximal ideals introduced by Pietsch (see Remark 107).

In Section 9 we continue the discussion of special cases that we started in Section 3. In particular (see Remark 56) the ideal of (r, s)-summing operators appears as a special case of $(D_\alpha, \alpha^D|D_\alpha)$. If α is a splitting norm then α can be recovered from α^D; however this fails if α is a splitting quasinorm (see Remark 63).

In Sections 10 and 11 we give conditions under which α^D (strictly $\alpha^D|L_0$) is splitting. It transpires from these that if $\alpha = g_p$ or d_p then α and α^D are both splitting, i.e., α is totally splitting (see Definition 78). We point out that $\|\cdot\|^D$ is splitting $\Leftrightarrow$ every Banach space has the m.a.p. (see Theorem 92).

In Section 12 we generalize the analysis of ([3]; § 5, No. 2) to metric approximation properties defined by totally splitting norms on L_0. Specifically, we define the α-m.a.p. in Definition 84. Theorem 83 generalizes parts of ([3]; § 5, No. 2, Proposition 39). Theorem 85 is an analog of ([3]; § 5, No. 2, Proposition 40). Theorem 86 gives some conditions for E' to have the (α')-m.a.p; even in the classical case (i.e., $\alpha = \alpha' = g_1$) our results seem to be new. In Section 15 we introduce the α-m.a.p. for a pair (E, E'), extending the concept introduced by Schwartz and we generalize [11] by proving that (E, E') has the α-m.a.p. $\Leftrightarrow E'$ has the α-m.a.p. (see Theorem 113). In Corollary 90 we generalize the main Lemma of the Persson–Pietsch paper (see the discussion in Remark 93).

Section 13 is the only part of this paper in which we use any measure theory. In particular, we define *right p-integral maps* (see Definition 101) which bear the same relation to d_p as the integral maps do to g_p.

1. IDEALS OF OPERATORS AND QUASINORMS

1. Notation. We write $\mathscr{K}$ for the real or complex field and all linear spaces will be over the field $\mathscr{K}$. We write L for the class of all bounded linear operators between Banach spaces. We use the symbols E, F, G and H to represent Banach spaces. If $A \subset L$ we write

$$A(E, F) = \{T\colon T \text{ is a bounded linear operator from } E \text{ into } F \text{ and } T \in A\}.$$

We write E_1 for $\{x\colon x \in E,\ \|x\| \leqslant 1\}$ and J_E for the canonical map from E into E''.

2. Definition. Let $A \subset L$. We say that A is an *ideal* if
(a) whenever $T \in A(E, F)$ and $S \in L(F, G)$ then $ST \in A(E, G)$,
(b) whenever $T \in L(E, F)$ and $S \in A(F, G)$ then $ST \in A(E, G)$,
(c) whenever $a \in E'$ and $y \in F$ then $\langle \cdot, a\rangle y \in A(E, F)$,
(d) whenever S and $T \in A(E, F)$ then $S + T \in A(E, F)$.

3. Definition. Let A be an ideal and $\alpha\colon A \to \mathscr{R}^+$. We say that α is a *quasinorm* on A if, in (a) and (b) of Definition 2, $\alpha(ST) \leqslant \|S\|\alpha(T)$ and $\alpha(ST) \leqslant \alpha(S)\|T\|$, respectively.

4. Notation. If $A \subset L$ we define a subclass A' of L by the rule: if $T \in L(E, F)$ then $T \in A' \Leftrightarrow T' \in A(F', E')$. If A is an ideal then A' is an ideal. If $\alpha\colon A \to \mathscr{R}$ we define $\alpha'\colon A' \to \mathscr{R}$ by the rule $\alpha'(T) = \alpha(T')$ $(T \in A')$. If A is an ideal and α is a quasinorm on A then α' is a quasinorm on A'.

2. QUASINORMS ON L_0

5. Notation. We write L_0 for the subclass of L consisting of all operators of finite rank. If $T \in L(E, F)$ and $T' \in L_0(F', E')$ then $T \in L_0(E, F)$, that is, $L_0' = L_0$. L_0 is an ideal.

6. Notation. For the rest of this paper, unless otherwise stated, α will be a quasinorm on L_0.

7. Definition. We say that α is *left splitting* if, for all $T \in L_0(E, F)$ and $\varepsilon > 0$ there exist G, $P \in L_0(E, G)$ and $Q \in L_0(G, F)$ such that

$$T = QP \quad \text{and} \quad \alpha(Q)\|P\| \leqslant \alpha(T) + \varepsilon. \tag{1}$$

We say that α is *right splitting* if, for all $T \in L_0(E, F)$ and $\varepsilon > 0$ there exist G, $P \in L_0(E, G)$ and $Q \in L_0(G, F)$ such that

$$T = QP \quad \text{and} \quad \|Q\|\alpha(P) \leqslant \alpha(T) + \varepsilon. \tag{2}$$

We say that α is *splitting* if α is both left splitting and right splitting

8. Lemma. *We suppose that α is left splitting.*

(a) *We can, in fact, specify in the definition of left splitting that $P(E) = G$ (in which case G is finite dimensional).*

(b) *For all $T \in L_0$, $\alpha(T'') \leqslant \alpha(T)$.*

(c) *α' is right splitting.*

Proofs. (a) If G, P and Q are as in (1) then so are $\hat{G}$, $\hat{P}$ and $\hat{Q}$, where $\hat{G} = P(E)$, $\hat{Q} = Q|P(E)$ and $\hat{P}$ is the element of $L_0(E, P(E))$ such that, for all $x \in E$, $\hat{P}x = Px$.

(b) Let $T \in L_0(E, F)$ and $\varepsilon > 0$. We choose G, P and Q as in (1). Since $P \in L_0(E, G)$, there exists $R \in L_0(E'', G)$ such that $P'' = J_G R$ and $\|R\| = \|P''\| = \|P\|$. Thus $T'' = Q''P'' = Q''J_G R = J_F QR$ and so $\alpha(T'') \leqslant \|J_F\|\alpha(Q)\|R\| = \alpha(Q)\|P\| \leqslant \alpha(T) + \varepsilon$. The result follows since ε is arbitrary.

(c) We suppose that $T\in L_0(E,F)$ and $\varepsilon>0$. Since α is left splitting, there exist G, $P\in L_0(F',G)$ and $Q\in L_0(G,E')$ such that $T'=QP$ and

$$\alpha(Q)\|P\|\leqslant\alpha(T')+\varepsilon=\alpha'(T)+\varepsilon. \tag{3}$$

We write $\hat{P}$ for the element of $L_0(G',P'(G'))$ such that, for all $y\in G'$, $\hat{P}y=P'y$. Then $\|\hat{P}\|=\|P'\|=\|P\|$. From ([5]; Theorem 3.1) or ([10]; Theorem 6) there exists $U\in L_0(P'(G'),F)$ such that $\|U\|\leqslant 1+\varepsilon$ and, for all $y\in P'(G')\cap J_F(F)$, $J_FUy=y$. Now, for all $x\in E$,

$$J_FTx=T''J_Ex=P'Q'J_Ex=\hat{P}Q'J_Ex\in P'(G')\cap J_F(F)$$

hence

$$J_FTx=J_FU\hat{P}Q'J_Ex$$

from which

$$T=U\hat{P}Q'J_E.$$

However

$$\begin{aligned}\|U\hat{P}\|\alpha'(Q'J_E)&\leqslant(1+\varepsilon)\|\hat{P}\|\alpha'(Q')\|J_E\|\\&\leqslant(1+\varepsilon)\|P\|\alpha(Q'')\\&\leqslant(1+\varepsilon)\|P\|\alpha(Q)\quad\text{from (b)}\\&\leqslant(1+\varepsilon)\big(\alpha'(T)+\varepsilon\big)\quad\text{from (3)}.\end{aligned}$$

The result follows since ε is arbitrary.

9. Lemma. *We suppose that α is right splitting.*

(a) *In the definition of right splitting we can, in fact, specify that G is a finite dimensional subspace of F and that Q is the inclusion map of G into F.*

(b) *For all $T\in L_0(E,F)$, $\alpha(T)=\alpha(J_FT)$.*

(c) *For all $T\in L_0$, $\alpha(T)\leqslant\alpha(T'')$.*

Proofs. (a) If G, P and Q are as in (2) then so are $\hat{G}$, $\hat{P}$ and $\hat{Q}$, where $\hat{G}=Q(G)$, $\hat{Q}$ is the inclusion map from $Q(G)$ into F and $\hat{P}$ is the element of $L_0(E,Q(G))$ such that, for all $x\in E$, $\hat{P}x=QPx$.

(b) Let $\varepsilon>0$. From (a), there exists a finite dimensional subspace H of F'' and $R\in L_0(E,H)$ such that $J_FT=IR$ and $\alpha(R)\leqslant\alpha(J_FT)+\varepsilon$, where $I\in L_0(H,F'')$ is the inclusion map. From ([5]; Theorem 3.1) or ([10]; Theorem 6) there exists $S\in L_0(H,F)$ such that $\|S\|\leqslant 1+\varepsilon$ and, for all $y\in H\cap J_F(F)$, $J_FSy=y$. Now, for all $x\in E$,

$$J_FTx=Rx\in H\cap J_F(F)$$

hence

$$J_FTx=J_FSRx$$

from which $T=SR$. Hence $\alpha(T)\leqslant\|S\|\alpha(R)\leqslant(1+\varepsilon)\big(\alpha(J_FT)+\varepsilon\big)$. The result follows since ε is arbitrary. (This result was suggested to us by A. Pietsch.)

(c) If $T\in L_0(E,F)$ then, from (b), $\alpha(T)=\alpha(J_FT)=\alpha(T''J_E)\leqslant\alpha(T'')$.

10. Problem. *If α is right splitting is α' necessarily left splitting?*

11. Theorem. *If α is splitting then $\alpha''=\alpha$.*

Proof. Immediate from Lemma 8 (b) and Lemma 9 (c).

12. Corollary. *We suppose that α is splitting.*

(a) *Let $a_1,\dots,a_m\in E'$, $y_1,\dots,y_m\in F$ and $U\in L(E',G')$. Write $T=\sum_i\langle\cdot,a_i\rangle y_i\in L_0(E,F)$ and $V=\sum_i\langle\cdot,Ua_i\rangle y_i\in L_0(G,F)$. Then $\alpha(V)\leqslant\|U\|\alpha(T)$.*

(b) *Let $T\in L_0(F,E')$. Then $\alpha(T'J_E)=\alpha(T')$.*

Proof. (a) $V'=UT'$ hence, from Theorem 11, $\alpha(V)=\alpha(V'')=\alpha(T''U')\leqslant\alpha(T'')\|U'\|=\alpha(T)\|U\|$.

(b) $(T'J_E)''=J_{F'}T'$ hence, from Lemma 9(b) and Theorem 11, $\alpha(T'J_E)=\alpha\big((T'J_E)''\big)=\alpha(J_{F'}T')=\alpha(T')$.

13. Theorem. *If α is splitting then α' is splitting.*

Proof. We suppose that $T\in L_0(E,F)$ and $\varepsilon>0$. From Definition 7, there exist G, H, $P\in L_0(F',G)$, $Q\in L_0(G,H)$ and $R\in L_0(H,E')$ such that $T'=RQP$ and $\|R\|\alpha(Q)\|P\|\leqslant\alpha(T')+\varepsilon=\alpha'(T)+\varepsilon$. We define $\hat{P}$ and U as in the proof of Lemma 8(c). Exactly as therein, $T=U\hat{P}Q'R'J_E$ and, further, $\|U\hat{P}\|\alpha'(Q')\|R'J_E\|\leqslant(1+\varepsilon)\big(\alpha'(T)+\varepsilon\big)$. Since ε is arbitrary, it follows that α' is splitting.

14. Remark. If the solution to Problem 10 is "yes" then Theorem 13 is immediate from Lemma 8(c).

3. EXAMPLES OF SPLITTING QUASINORMS

15. Lemma. *Let $m\geqslant 1$ and $N\colon\mathscr{K}^m\to\mathscr{R}^+$ be a continuous function such that, for all $\lambda\in\mathscr{K}$ and $t\in\mathscr{K}^m$, $N(\lambda t)=|\lambda|N(t)$.*

(a) *If $y_1,\dots,y_m\in F$ and $G=\operatorname{lin}\{y_1,\dots,y_m\}\subset F$ then*

$$\sup_{b\in G_1'}N(\langle y_1,b\rangle,\dots,\langle y_m,b\rangle)=\sup_{b\in F_1'}N(\langle y_1,b\rangle,\dots,\langle y_m,b\rangle)$$

and

$$\sup_{b_1,\dots,b_m\in G_1'}N(\langle y_1,b_1\rangle,\dots,\langle y_m,b_m\rangle)=\sup_{b_1,\dots,b_m\in F_1'}N(\langle y_1,b_1\rangle,\dots,\langle y_m,b_m\rangle).$$

(b) *Let* $a_1, \ldots, a_m \in E'$. *We define* $P\colon E \to \mathscr{K}^m$ *by*

$$Px = (\langle x, a_1\rangle, \ldots, \langle x, a_m\rangle) \quad (x \in E)$$

and write $G = P(E) \subset \mathscr{K}^m$. *If* $y \in G$ *we write* $\|y\| = \inf\|P^{-1}y\|$ — *then* $\|\cdot\|$ *is a norm on* G. *We write* $pr_1, \ldots, pr_m$ *for the projection maps*: $G \to \mathscr{K}$. *Then*

$$\sup_{y\in G_1} N(\langle y, pr_1\rangle, \ldots, \langle y, pr_m\rangle) = \sup_{x\in E_1} N(\langle x, a_1\rangle, \ldots, \langle x, a_m\rangle)$$

and

$$\sup_{y_1,\ldots,y_m\in G_1} N(\langle y_1, pr_1\rangle, \ldots, \langle y_m, pr_m\rangle) = \sup_{x_1,\ldots,x_m\in E_1} N(\langle x_1, a_1\rangle, \ldots, \langle x_m, a_m\rangle).$$

(c) *If* $a_1, \ldots, a_m \in E'$ *then*

$$\sup_{\xi\in E_1''} N(\langle a_1, \xi\rangle, \ldots, \langle a_m, \xi\rangle) = \sup_{x\in E_1} N(\langle x, a_1\rangle, \ldots, \langle x, a_m\rangle)$$

and

$$\sup_{\xi_1,\ldots,\xi_m\in E_1''} N(\langle a_1, \xi_1\rangle, \ldots, \langle a_m, \xi_m\rangle) = \sup_{x_1,\ldots,x_m\in E_1} N(\langle x_1, a_1\rangle, \ldots, \langle x_m, a_m\rangle).$$

Proof. We first observe that the continuity of N ensures that all the suprema above are finite. We shall prove the second equality in each case and leave the proof of the first to the reader.

(a) "$\geqslant$" is trivial and "$\leqslant$" follows from the Hahn–Banach Theorem.

(b) For all $i = 1, \ldots, m$, $a_i = pr_i \circ P$. "$\geqslant$" is immediate since $\|P\| \leqslant 1$. Let $y_1, \ldots, y_m \in G_1$ and $\varepsilon > 0$. Choose $z_1, \ldots, z_m \in E$ such that, for all $i = 1, \ldots, m$, $Pz_i = y_i$ and $\|z_i\| \leqslant 1+\varepsilon$. Then

$$N(\langle y_1, pr_1\rangle, \ldots, \langle y_m, pr_m\rangle) = (1+\varepsilon) N\left(\left\langle \frac{z_1}{1+\varepsilon}, a_1\right\rangle, \ldots, \left\langle \frac{z_m}{1+\varepsilon}, a_m\right\rangle\right)$$
$$\leqslant (1+\varepsilon) \sup_{x_1,\ldots,x_m\in E_1} N(\langle x_1, a_1\rangle, \ldots, \langle x_m, a_m\rangle)$$

and "$\leqslant$" follows on taking the supremum over $y_1, \ldots, y_m$ and letting $\varepsilon \to 0$.

(c) "$\geqslant$" is trivial. Let $\xi_1, \ldots, \xi_m \in E_1''$. Then there exist nets $\varphi_1, \ldots, \varphi_m$ in E_1 such that, for all $i = 1, \ldots, m$, $J_E \circ \varphi_i \to \xi_i$ in $w(E'', E')$ and, by passing to product nets, we can suppose that $\varphi_1, \ldots, \varphi_m$ are indexed by the same directed set. Then

$$(\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_m, a_m\rangle) \to (\langle a_1, \xi_1\rangle, \ldots, \langle a_m, \xi_m\rangle) \text{ in } \mathscr{K}^m.$$

Since N is continuous,

$$N(\langle a_1, \xi_1\rangle, \ldots, \langle a_m, \xi_m\rangle) \leqslant \sup_{x_1,\ldots,x_m\in E_1} N(\langle x_1, a_1\rangle, \ldots, \langle x_m, a_m\rangle)$$

and "$\leqslant$" follows on taking the supremum over $\xi_1, \ldots, \xi_m \in E_1''$.

16. Theorem. *We suppose that* $M, N\colon \bigcup_{m\geqslant 1}\mathscr{K}^m \to \mathscr{R}^+$ *and, for all* $m \geqslant 1$, $M|\mathscr{K}^m$ *and* $N|\mathscr{K}^m$ *satisfy the conditions of Lemma* 15. *If* $T \in L_0(E, F)$ *we write*

$$g_{M,N}(T) = \inf\{\sup_{x_1,\ldots,x_m\in E_1, b\in F_1'} M(\langle x_1, a_1\rangle, \ldots, \langle x_m, a_m\rangle) N(\langle y_1, b\rangle, \ldots, \langle y_m, b\rangle)\}$$

$$d_{M,N}(T) = \inf\{\sup_{x\in E_1, b_1,\ldots,b_m\in F_1'} M(\langle x, a_1\rangle, \ldots, \langle x, a_m\rangle) N(\langle y_1, b_1\rangle, \ldots, \langle y_m, b_m\rangle)\}$$

$$s_{M,N}(T) = \inf\{\sup_{x_1,\ldots,x_m\in E_1, b_1,\ldots,b_m\in F_1'} M(\langle x_1, a_1\rangle, \ldots, \langle x_m, a_m\rangle) \times N(\langle y_1, b_1\rangle, \ldots, \langle y_m, b_m\rangle)\}$$

$$i_{M,N}(T) = \inf\{\sup_{x\in E_1, b\in F_1'} M(\langle x, a_1\rangle, \ldots, \langle x, a_m\rangle) N(\langle y_1, b\rangle, \ldots, \langle y_m, b\rangle)\}$$

where, in each case, the "inf" is taken over all finite sets $\{a_1, \ldots, a_m\} \subset E'$ *and* $\{y_1, \ldots, y_m\} \subset F$ *such that* $T = \sum_{i=1}^{m} \langle \cdot, a_i\rangle y_i$. *Then*

(a) $g_{M,N}$, $d_{M,N}$, $s_{M,N}$ *and* $i_{M,N}$ *are all splitting quasinorms on* L_0,

(b) $g'_{M,N} = d_{N,M}$, $d'_{M,N} = g_{N,M}$, $s'_{N,M} = s_{N,M}$ *and* $i'_{M,N} = i_{N,M}$.

Proofs. (a) In all the cases the two inequalities of Definition 3 are easily verified and the two inequalities of Definition 7 are established using Lemma 15(a) and (b).

(b) It follows easily from Lemma 15(c) that, if $T \in L_0$, then $d_{N,M}(T') \leqslant g_{M,N}(T)$ and $g_{M,N}(T') \leqslant d_{N,M}(T)$. From the first of these relationships, with T replaced by T', $d_{N,M}(T'') \leqslant g_{M,N}(T')$. Thus

$$d_{N,M}(T'') \leqslant g_{M,N}(T') \leqslant d_{N,M}(T).$$

Hence, from Lemma 9(c), $g'_{M,N} = d_{N,M}$. The other relationships are proved with similar arguments.

17. Remark. We shall continue the study of the above four quasinorms in Theorem 48.

18. Corollary. *We suppose that* $0 < p \leqslant \infty$ *and* $0 < q \leqslant \infty$. *If* $T \in L_0(E, F)$ *we write (with the usual conventions about the cases* $p = \infty$

and $q = \infty$)

$$g_{p,q}(T) = \inf\Big\{\Big(\sum_i \|a_i\|^p\Big)^{1/p} \sup_{b\in F_1'}\Big(\sum_i |\langle y_i, b\rangle|^q\Big)^{1/q}\Big\},$$

$$d_{p,q}(T) = \inf\Big\{\sup_{x\in E_1}\Big(\sum_i |\langle x, a_i\rangle|^p\Big)^{1/p}\Big(\sum_i \|y_i\|_q\Big)^{1/q}\Big\},$$

$$s_{p,q}(T) = \inf\Big\{\Big(\sum_i \|a_i\|^p\Big)^{1/p}\Big(\sum_i \|y_i\|^q\Big)^{1/q}\Big\},$$

$$i_{p,q}(T) = \inf\Big\{\sup_{x\in E_1, b\in F_1'}\Big(\sum_i |\langle x, a_i\rangle|^p\Big)^{1/p}\Big(\sum_i |\langle y_i, b\rangle|^q\Big)^{1/q}\Big\}.$$

Then $g_{p,q}$, $d_{p,q}$, $s_{p,q}$ *and* $i_{p,q}$ *are splitting quasinorms on* L_0, $g'_{p,q} = d_{q,p}$, $d'_{p,q} = g_{q,p}$, $s'_{p,q} = s_{q,p}$ *and* $i'_{p,q} = i_{q,p}$.

Proof. Immediate from Theorem 16 with $M(t) = \big(\sum_i |t_i|^p\big)^{1/p}$ and $N(t) = \big(\sum_i |t_i|^q\big)^{1/q}$ ($m \geq 1$, $t \in \mathcal{K}^m$).

19. Remark. $g_{p,p'}$ and $d_{p',p}$ are identical with g_p and d_p as defined in ([8]; Section 3) if we identify $L_0(E, F)$ with $E'\otimes F$. One can easily see that $g_{1,\infty}(T) = d_{\infty,1}(T) = \inf\{\sum_i \|a_i\|\,\|y_i\|\}$. We shall continue the study of the above four quasinorms in Lemma 51.

4. THE TRACE

20. Definition. If $T \in L_0(E, E'')$ and $T = \sum_i \langle\cdot, a_i\rangle \xi_i$ ($a_1, \ldots, a_m \in E'$ and $\xi_1, \ldots, \xi_m \in E''$) then $\sum_i \langle a_i, \xi_i\rangle$, which is independent of the representation of T chosen, is known as the *trace* of T and is written $\mathrm{tr}\, T$.

21. Lemma. (a) *If* $T \in L_0(E, F)$ *and* $V \in L(F, E)$ *then* $\mathrm{tr} J_E VT = \mathrm{tr} J_F TV = \mathrm{tr} J_{E'} T' V'$.

(b) *If* $T \in L_0(F', E')$ *and* $V \in L(F, E)$ *then* $\mathrm{tr} J_{E'} TV' = \mathrm{tr} T' J_E V = \mathrm{tr} J_{E''} V'' T'$.

(c) *If* $T \in L_0(E'', F)$ *and* $V \in L(F, E'')$ *then* $\mathrm{tr} J_F TV = \mathrm{tr} T' V' J_{E'}$.

(d) *If* $a_1, \ldots, a_m \in E'$ *and* $y_1, \ldots, y_m \in F$ *we write* $T = \sum_i \langle\cdot, a_i\rangle y_i \in L_0(E, F)$ *and* $\hat{T} = \sum_i \langle a_i, \cdot\rangle y_i \in L_0(E'', F)$. *Then, for all* $V \in L(F, E'')$, $\mathrm{tr}\, VT = \mathrm{tr} J_F \hat{T} V$.

(e) *If* $a_1, \ldots, a_m \in E'$ *and* $\eta_1, \ldots, \eta_m \in F''$ *we write* $T = \sum_i \langle\cdot, a_i\rangle \eta_i \in L_0(E, F'')$ *and* $\hat{T} = \sum_i \langle a_i, \cdot\rangle \eta_i \in L_0(E'', F'')$. *Then, for all* $V \in L(E', F')$, $\mathrm{tr}\, V'T = \mathrm{tr}\, \hat{T} V' J_F$.

Proofs. Immediate from Definition 20.

5. AN APPROXIMATION RESULT

22. Lemma. *We suppose that* α *is right splitting,* $V \in L(F, E)$, $T \in L_0(E, F'')$, $\alpha(T) > 0$ *and* $\varepsilon > 0$. *Then there exists* $W \in L_0(E, F)$ *such that*

$$|\mathrm{tr} J_F WV - \mathrm{tr}\, TV| < \varepsilon \tag{4}$$

and

$$\alpha(W) \leq \alpha(T). \tag{5}$$

Proof. We choose $\delta > 0$ such that $2\delta|\mathrm{tr}\, TV| < (1+\delta)\varepsilon$. From Lemma 9(a) there exist a finite dimensional subspace H of F'' and $R \in L_0(E, H)$ such that $T = IR$ and $\alpha(R) \leq (1+\delta)\alpha(T)$, where $I \in L_0(H, F'')$ is the inclusion map. Then there exist $a_1, \ldots, a_m \in E'$ and $z_1, \ldots, z_m \in H$ such that $T = \sum_i \langle\cdot, a_i\rangle z_i$. From ([10]; Corollary 7) (see also [7]; Lemma 6), there exists $U \in L_0(H, F)$ such that $\|U\| \leq 1$ and

$$\Big|\sum_i \langle V' a_i, J_F U z_i - z_i\rangle\Big| < \varepsilon/2,$$

i.e.,

$$|\mathrm{tr} J_F URV - \mathrm{tr}\, TV| < \varepsilon/2.$$

We write $W = (1+\delta)^{-1} UR \in L_0(E, F)$. Then

$$|\mathrm{tr} J_F WV - \mathrm{tr}\, TV| \leq (1+\delta)^{-1}|\mathrm{tr} J_F URV - \mathrm{tr}\, TV| + \delta(1+\delta)^{-1}|\mathrm{tr}\, TV| < \varepsilon$$

and so (4) is true. On the other hand, $\alpha(W) \leq (1+\delta)^{-1}\|U\|\alpha(R)$ hence (5) is true.

6. THE FUNCTION α^D

23. Definition. If $V \in L(F, E)$ we write

$$\alpha^D(V) = \sup\{|\mathrm{tr} J_F TV| \colon T \in L_0(E, F),\ \alpha(T) \leq 1\}.$$

24. Lemma. *If* α *is right splitting and* $V \in L(F, E)$ *then*

$$\alpha^D(V) = \sup\{|\mathrm{tr}\, TV| \colon T \in L_0(E, F''),\ \alpha(T) \leq 1\}.$$

Proof. "$\leq$" is immediate and "$\geq$" follows from Lemma 22.

25. Theorem. *If* α *is left splitting then* $\alpha'^D = \alpha^{D'}$.

Proof. Let $V \in L(F, E)$. If $T \in L_0(E, F)$ and $\alpha'(T) \leq 1$ then $T' \in L_0(F', E')$ and $\alpha(T') \leq 1$ hence, from Lemma 21(a)

$$|\mathrm{tr} J_F TV| = |\mathrm{tr} J_{E'} T' V'| \leq \alpha^D(V') = \alpha^{D'}(V)$$

hence, taking the supremum over T, $\alpha'^{D}(V) \leqslant \alpha^{D'}(V)$. If, on the other hand, $T \in L_0(F', E')$ and $\alpha(T) \leqslant 1$ then, from Lemma 8 (b), $T' J_E \in L_0(E, F'')$ and $\alpha'(T' J_E) \leqslant 1$. From Lemma 8 (c), α' is right splitting hence, from Lemma 24 (with α replaced by α') and Lemma 21 (b)

$$\alpha'^{D}(V) \geqslant |\mathrm{tr}\, T' J_E V| = |\mathrm{tr}\, J_{E'} T V'|.$$

Hence, taking the supremum over T,

$$\alpha'^{D}(V) \geqslant \alpha^{D}(V') = \alpha^{D'}(V).$$

26. Remark. The result of the following theorem should be compared with that of Theorem 11.

27. Theorem. *If α is splitting then $\alpha^{D''} = \alpha^{D}$.*

Proof. Since α is left splitting, from Theorem 25, $\alpha^{D''} = \alpha'^{D'}$. From Theorem 13, α' is left splitting hence, from Theorem 25 again, $\alpha'^{D'} = \alpha''^{D}$. From Theorem 11, $\alpha''^{D} = \alpha^{D}$.

7. MORE APPROXIMATION RESULTS

28. Lemma. *We suppose that α is right splitting, $V \in L(F, E'')$, $T \in L_0(F', E')$, $\alpha(T') > 0$ and $\varepsilon > 0$. Then there exists $W \in L_0(E, F)$ such that $|\mathrm{tr}\, VW - \mathrm{tr}\, J_{E'} T V' J_{E'}| < \varepsilon$ and $\alpha(W) \leqslant \alpha(T')$.*

Proof. We proceed as in the proof of Lemma 22, choosing δ, H and R with T replaced by T' and E by E''. Then there exist $a_1, \ldots, a_m \in E'$ and $z_1, \ldots, z_m \in H$ such that $T' = \sum_i \langle a_i, \cdot \rangle z_i$. We then continue as in the proof of Lemma 22, choosing U with V' replaced by $V' J_{E'}$. We can then take $W = (1+\delta)^{-1} U R J_E$.

29. Lemma. *We suppose that α is right splitting and $V \in L(F, E'')$. Then*

$$\alpha'^{D}(V' J_{E'}) \leqslant \sup\{|\mathrm{tr}\, VT|:\ T \in L_0(E, F),\ \alpha(T) \leqslant 1\}.$$

Proof. Immediate from Lemma 28 and the definition of $\alpha'^{D}(V' J_{E'})$.

30. Lemma. *We suppose that α is left splitting and $V \in L(F, E'')$. Then*

$$\alpha^{D}(V) \leqslant \alpha'^{D}(V' J_{E'}).$$

Proof. If $T \in L_0(E'', F)$ and $\alpha(T) \leqslant 1$ then $T' \in L_0(F', E''')$ and, from Lemma 8 (b), $\alpha'(T') \leqslant 1$. Hence, from Lemma 21 (c),

$$\begin{aligned}\alpha^{D}(V) &= \sup\{|\mathrm{tr}\, J_F T V|:\ T \in L_0(E'', F),\ \alpha(T) \leqslant 1\}\\ &= \sup\{|\mathrm{tr}\, T' V' J_{E'}|:\ T \in L_0(E'', F),\ \alpha(T) \leqslant 1\}\\ &\leqslant \sup\{|\mathrm{tr}\, T V' J_{E'}|:\ T \in L_0(F', E'''),\ \alpha'(T) \leqslant 1\}.\end{aligned}$$

From Lemma 8 (c), α' is right splitting. The result follows from Lemma 24 with α, E, F and V replaced by α', F', E' and $V' J_{E'}$, respectively.

31. Theorem. *If α is splitting and $V \in L(F, E'')$ then*

$$\alpha^{D}(V) = \sup\{|\mathrm{tr}\, VT|:\ T \in L_0(E, F),\ \alpha(T) \leqslant 1\}.$$

Proof. If $T \in L_0(E, F)$ and $\alpha(T) \leqslant 1$ we define $\hat{T} \in L_0(E'', F)$ as in Lemma 21 (d). Then $|\mathrm{tr}\, VT| = |\mathrm{tr}\, J_F \hat{T} V|$. On the other hand, from Lemma 9 (b), $\alpha(\hat{T}) = \alpha(J_F \hat{T}) = \alpha(T'')$ and, from Theorem 11, $\alpha(T'') = \alpha(T) \leqslant 1$. Hence $\alpha(\hat{T}) \leqslant 1$ and so $|\mathrm{tr}\, VT| \leqslant \alpha^{D}(V)$. On taking the supremum over T, we obtain "$\geqslant$". "$\leqslant$" is immediate from Lemma 29 and Lemma 30.

32. Remark. The result of Corollary 33 should be compared with that of Lemma 9 (b).

33. Corollary. *If α is splitting and $V \in L(F, E)$ then $\alpha^{D}(J_E V) = \alpha^{D}(V)$.*

Proof. From Theorem 31, $\alpha^{D}(J_E V) = \sup\{|\mathrm{tr}\, J_E V T|:\ T \in L_0(E, F),\ \alpha(T) \leqslant 1\}$. The result follows from Lemma 21 (a).

34. Remark. The result of Corollary 35 should be compared with that of Corollary 12 (b).

35. Corollary. *If α is splitting and $V \in L(E', F')$ then $\alpha^{D}(V' J_F) = \alpha^{D}(V')$.*

Proof. From Theorem 31 with F and V replaced by F' and V', respectively,

$$\alpha^{D}(V') = \sup\{|\mathrm{tr}\, V' T|:\ T \in L_0(E, F''),\ \alpha(T) \leqslant 1\}.$$

If $T \in L_0(E, F'')$ and $\alpha(T) \leqslant 1$ we define $\hat{T} \in L_0(E'', F'')$ as in Lemma 21 (e). Then $|\mathrm{tr}\, V' T| = |\mathrm{tr}\, \hat{T} V' J_F|$. Arguing as in Theorem 31, $\alpha(\hat{T}) \leqslant 1$. Hence

$$\alpha^{D}(V') \leqslant \sup\{|\mathrm{tr}\, T V' J_F|:\ T \in L_0(E'', F''),\ \alpha(T) \leqslant 1\}.$$

Thus, from Lemma 24 with E and V replaced by E'' and $V' J_F$, respectively, $\alpha^{D}(V') \leqslant \alpha^{D}(V' J_F)$. The reverse inequality follows since $\alpha^{D}(V' J_F) \leqslant \alpha^{D}(V') \|J_F\|$.

8. D_α, REASONABLE QUASINORMS AND NORMS

36. Definition. We write $D_\alpha(F, E) = \{V:\ V \in L(F, E),\ \alpha^{D}(V) < \infty\}$.

37. Theorem. *If α is left splitting then $D_{\alpha'} = D'_\alpha$. If α is splitting then $D''_\alpha = D_\alpha$.*

Proof. Immediate from Theorem 25 and Theorem 27.

38. Definition. We say that a quasinorm α on an ideal A is *reasonable* if, in (c) of Definition 2, $\alpha(\langle \cdot, a \rangle y) = \|a\| \|y\|$.

39. Lemma. *If α is a reasonable quasinorm on an ideal A then, for all $T\in A$, $\|T\|\leqslant \alpha(T)$. In particular, $\alpha(T)=0 \Rightarrow T=0$. Further, α' is a reasonable quasinorm on A'.*

Proof. We prove only the first assertion. (This proof was suggested by some remarks of A. Pietsch.) Let $T\in A(E,F)$ and $x\in E$. Define $S\in L(\mathcal{K},E)$ by $S(\lambda)=\lambda x$ $(\lambda\in\mathcal{K})$. Then $\|S\|=\|x\|$. Further, $TS(\lambda)=\lambda(Tx)$ $(\lambda\in\mathcal{K})$. Hence $\alpha(TS)=\|Tx\|$. Thus

$$\|Tx\|=\alpha(TS)\leqslant\alpha(T)\|S\|=\alpha(T)\|x\|.$$

The result follows.

40. Definition. We say that a quasinorm α on an ideal A is a *norm* if α is reasonable and, further, in (d) of Definition 2, $\alpha(S+T)\leqslant\alpha(S)+\alpha(T)$. If α is a norm on an ideal A then α' is a norm on A'.

41. Remark. We now return to the convention of Notation 6.

42. Lemma. *If α is reasonable then D_α is an ideal and $\alpha^D|D_\alpha$ is a norm on D_α (even if α fails to be a norm on L_0).*

Proof. Let $b\in F'$, $x\in E$ and $V=\langle\cdot,b\rangle x\in L(F,E)$. From Lemma 39,

$$\alpha^D(V)\leqslant\sup\{|\mathrm{tr}\,J_F TV|:\ T\in L_0(E,F),\ \|T\|\leqslant 1\}$$
$$=\sup\{|\langle Tx,b\rangle|:\ T\in L_0(E,F),\ \|T\|\leqslant 1\}\leqslant\|x\|\,\|b\|.$$

On the other hand, if $a\in E_1'$, $y\in F_1$ and $T=\langle\cdot,a\rangle y\in L(E,F)$ then, since $\alpha(T)\leqslant 1$, $\alpha^D(V)\geqslant|\mathrm{tr}\,J_F TV|=|\langle x,a\rangle\langle y,b\rangle|$ and so, taking the supremum over a and y, $\alpha^D(V)\geqslant\|x\|\,\|b\|$. We leave the rest of the verification to the reader.

43. Remark. If α is a norm on L_0, by abuse of notation we shall write $(L_0(E,F),\alpha)$ for the normed space $(L_0(E,F),\alpha|L_0(E,F))$.

44. Theorem. *If α is a splitting norm and $V\in D_\alpha(F,E'')$ then th map φ of $L_0(E,F)$ into $\mathcal{K}$ defined by*

$$\varphi(T)=\mathrm{tr}\,VT\qquad(T\in L_0(E,F))\tag{6}$$

is in $(L_0(E,F),\alpha)'$ and, conversely, any element φ of $(L_0(E,F),\alpha)'$ can be put in the form (6) for a unique $V\in D_\alpha(F,E'')$. Further $\|\varphi\|=\alpha^D(V)$.

Proof. This is immediate from Theorem 31 and the usual techniques.

45. Corollary. *Let α be a reasonable splitting quasinorm. Then α is a norm $\Leftrightarrow$ for all $T\in L_0(E,F)$,*

$$\alpha(T)=\sup\{|\mathrm{tr}\,VT|:V\in D_\alpha(F,E''),\ \alpha^D(V)\leqslant 1\}.\tag{7}$$

Proof. $(\Leftarrow)$ is immediate and $(\Rightarrow)$ follows from Theorem 44 and the Hahn–Banach Theorem.

9. CONTINUATION OF SECTION 3

46. Definition. We suppose that $m\geqslant 1$ and $M:\mathcal{K}^m\to\mathcal{R}^+$. We say that M is *increasing* if $t,u\in\mathcal{K}^m$ and $|t|\leqslant|u|$ (i.e., for all $i=1,\ldots,m$, $|t_i|\leqslant|u_i|$) imply that $M(t)\leqslant M(u)$.

47. Lemma. *We suppose that $m\geqslant 1$ and $M:\mathcal{K}^m\to\mathcal{R}^+$ is continuous and increasing. Then*

$$\sup_{x_1,\ldots,x_m\in E_1} M(\langle x_1,a_1\rangle,\ldots,\langle x_m,a_m\rangle)=M(\|a_1\|,\ldots,\|a_m\|).$$

Proof. "$\leqslant$" follows since M is increasing and "$\geqslant$" since M is continuous.

48. Theorem. *We suppose that M and N are as in Theorem 16. We define $M'\colon\bigcup_{m\geqslant 1}\mathcal{K}^m\to\mathcal{R}^+$ by*

$$M'(u)=\sup\Big\{\Big|\sum_{i=1}^m t_iu_i\Big|:\ t\in\mathcal{K}^m,\ M(t)\leqslant 1\Big\}\qquad(m\geqslant 1,\ u\in\mathcal{K}^m).$$

(a) *Suppose that, for all $m\geqslant 1$, $M|\mathcal{K}^m$ is increasing and $V\in L(F,E)$, and write $\alpha=g_{M,N}$. Then $V\in D_\alpha(F,E)\Leftrightarrow$ there exists $\varrho\in\mathcal{R}^+$ such that*

$$M'(\|Vy_1\|,\ldots,\|Vy_m\|)\leqslant\varrho\sup_{b\in F_1'}N(\langle y_1,b\rangle,\ldots,\langle y_m,b\rangle)\quad\text{for all } m\geqslant 1 \text{ and } y_1,\ldots,y_m\in F,\tag{8}$$

and, further, $\alpha^D(V)=\inf\{\varrho:\ \varrho\in\mathcal{R}^+,\ (8)\text{ is satisfied}\}$.

(b) *As in (a) with $\alpha=s_{M,N}$ but with (8) replaced by for all $m\geqslant 1$ and $y_1,\ldots,y_m\in F$,*

$$M'(\|Vy_1\|,\ldots,\|Vy_m\|)\leqslant\varrho\sup_{b_1,\ldots,b_m\in F_1'}N(\langle y_1,b_1\rangle,\ldots,\langle y_m,b_m\rangle).$$

(c) *Suppose that, for all $m\geqslant 1$, $N|\mathcal{K}^m$ is increasing and $V\in L(F,E)$, and write $\alpha=d_{M,N}$. Then $V\in D_\alpha(F,E)\Leftrightarrow$ there exists $\varrho\in\mathcal{R}^+$ such that*

$$N'(\|V'a_1\|,\ldots,\|V'a_m\|)\leqslant\varrho\sup_{x\in E_1}M(\langle x,a_1\rangle,\ldots,\langle x,a_m\rangle)\quad\text{for all } m\geqslant 1 \text{ and } a_1,\ldots,a_m\in E',\tag{9}$$

and, further, $\alpha^D(V)=\inf\{\varrho:\ \varrho\in\mathcal{R}^+,\ (9)\text{ is satisfied}\}$.

Proof. (a) Let $\varrho\in\mathcal{R}^+$. Then $\alpha^D(V)\leqslant\varrho\Leftrightarrow$

$$\text{for all } T\in L_0(E,F),\qquad|\mathrm{tr}\,J_F TV|\leqslant\varrho\alpha(T)$$

or equivalently, using Lemma 47, for all $m\geqslant 1$, $a_1,\ldots,a_m\in E'$ and $y_1,\ldots,y_m\in F$

$$\Big|\sum_{i=1}^m\langle Vy_i,a_i\rangle\Big|\leqslant\varrho M(\|a_1\|,\ldots,\|a_m\|)\sup_{b\in F_1'}N(\langle y_1,b\rangle,\ldots,\langle y_m,b\rangle)$$

or equivalently, for all $m \geqslant 1$, $a_1, \ldots, a_m \in E_1'$, $t \in \mathscr{K}^m$ and $y_1, \ldots, y_m \in F$

$$\Big|\sum_{i=1}^{m} t_i \langle V y_i, a_i\rangle\Big| \leqslant \varrho M(t) \sup_{b \in F_1'} N(\langle y_1, b\rangle, \ldots, \langle y_m, b\rangle)$$

and this is equivalent to (8).

We leave to the reader the proof of (b), which is very similar to that of (a).

(c) From Theorem 16(a), Theorem 11, Theorem 27 and Theorem 25, $\alpha^D(V) = \alpha^D(V'') = \alpha'^D(V')$. However, from Theorem 16(b), $\alpha' = g_{N,M}$. The result follows from (a) and Lemma 15(c).

49. Problem. *Under suitable conditions, find an analog of Theorem 48 for* $\alpha = i_{M,N}$.

50. Theorem. *We suppose that M and N are as in Theorem 16, $M(1)N(1) = 1$ and, for all $m \geqslant 1$ and $t, u \in \mathscr{K}^m$, $M(t)N(u) \geqslant \big|\sum_{i=1}^{m} t_i u_i\big|$. If $\alpha = g_{M,N}$, $d_{M,N}$, $s_{M,N}$ or $i_{M,N}$ then α is a reasonable splitting quasinorm on L_0, D_α is an ideal and $\alpha^D|D_\alpha$ is a norm on D_α.*

Proof. If $a \in E'$, $y \in F$ and $T = \langle\cdot, a\rangle y \in L_0(E, F)$ then

$$\alpha(T) \leqslant \sup_{x \in E_1, b \in F_1'} M(\langle x, a\rangle) N(\langle y, b\rangle)$$

$$= \|a\|\,\|y\| M(1) N(1) = \|a\|\,\|y\|.$$

On the other hand, if $T = \sum_{i=1}^{m} \langle\cdot, a_i\rangle y_i$ then

$$\|T\| = \sup_{x \in E_1, b \in F_1'} \Big|\sum_{i=1}^{m} \langle x, a_i\rangle \langle y_i, b\rangle\Big|$$

$$\leqslant \sup_{x \in E_1, b \in F_1'} M(\langle x, a_1\rangle, \ldots, \langle x, a_m\rangle) N(\langle y_1, b\rangle, \ldots, \langle y_m, b\rangle).$$

On taking the infimum, $\|T\| \leqslant i_{M,N}(T) \leqslant \alpha(T)$. Thus α is reasonable. The rest follows from Theorem 16(a) and Lemma 42.

51. Lemma. *We suppose that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} > 0$ and that $\alpha = g_{p,q}$, $d_{p,q}$, $s_{p,q}$ or $i_{p,q}$ (see Corollary 18). If $S \in L_0(E, F)$ and $T \in L_0(E, F)$ then*

$$[\alpha(S+T)]^r \leqslant [\alpha(S)]^r + [\alpha(T)]^r.$$

Proof. We prove the result for the case $\alpha = s_{p,q}$ and leave the other ones to the reader. Let $\varepsilon > 0$. There exist $a_1, \ldots, a_m, a_{m+1}, \ldots, a_n \in E'$ and $y_1, \ldots, y_m, y_{m+1}, \ldots, y_n \in F$ such that $S = \sum_{i=1}^{m} \langle\cdot, a_i\rangle y_i$, $T = \sum_{i=m+1}^{n} \langle\cdot, a_i\rangle y_i$, $\sum_{i=1}^{m} \|a_i\|^p \leqslant [\alpha(S)+\varepsilon]^r$, $\sum_{i=1}^{m} \|y_i\|^q \leqslant [\alpha(S)+\varepsilon]^r$, $\sum_{i=m+1}^{n} \|a_i\|^p \leqslant [\alpha(T)+\varepsilon]^r$ and $\sum_{i=m+1}^{n} \|y_i\|^q \leqslant [\alpha(T)+\varepsilon]^r$. Thus

$$[\alpha(S+T)]^r \leqslant [\alpha(S)+\varepsilon]^r + [\alpha(T)+\varepsilon]^r$$

and the desired result follows since ε is arbitrary.

52. Theorem. *We suppose that $\alpha = g_{p,q}$, $d_{p,q}$, $s_{p,q}$ or $i_{p,q}$.*

(a) *If $\frac{1}{p} + \frac{1}{q} < 1$ then $\alpha = 0$.*

(b) *If $\frac{1}{p} + \frac{1}{q} \geqslant 1$ then α is a reasonable splitting quasinorm on L_0. D_α is an ideal and $\alpha^D|D_\alpha$ is a norm on D_α.*

(c) *If $\frac{1}{p} + \frac{1}{q} = 1$ then α is a splitting norm on L_0.*

Proof. (a) We define $r > 1$ by $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $\frac{1}{r} = 0$ then $p = q = \infty$ and the result follows easily from (e.g.) the case where $p = q = 1$. So we suppose that $\frac{1}{r} > 0$. Let $T \in L_0$. From Corollary 18 and Lemma 51, $[\alpha(T)]^r \leqslant 2\left[\alpha\left(\frac{T}{2}\right)\right]^r = 2^{1-r}[\alpha(T)]^r$. The result follows since $2^{1-r} < 1$. (b) is immediate from Theorem 50. (c) is immediate from (b) and Lemma 51.

53. Problem. *To find cases other than that of Theorem 52(c) where the function α of Theorem 50 is a (splitting) norm on L_0.*

54. Problem. *To find a splitting norm on L_0 that is not equal to any of $g_{p,p'}$, $d_{p,p'}$, $s_{p,p'}$ or $i_{p,p'}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$. We conjecture that $\|\cdot\||L_0$ (which is obviously a splitting norm on L_0) has the required properties.*

55. Theorem. *We suppose that $\frac{1}{p} + \frac{1}{q} \geqslant 1$ and $V \in L(F, E)$.*

(a) *Let $p \geqslant 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\alpha = g_{p,q}$. Then $V \in D_\alpha \Leftrightarrow$ there exists $\varrho \in \mathscr{R}^+$ such that for all $m \geqslant 1$ and $y_1, \ldots, y_m \in F$,*

$$\Big(\sum_i \|V y_i\|^{p'}\Big)^{1/p'} \leqslant \varrho \sup_{b \in F_1'} \Big(\sum_i |\langle y_i, b\rangle|^q\Big)^{1/q}$$

and $\alpha^D(V) = \inf\{\varrho\}$.

(b) *Let* $q \geqslant 1$, $\frac{1}{q}+\frac{1}{q'} = 1$ *and* $\alpha = d_{p,q}$. *Then* $V \in D_\alpha \Leftrightarrow$ *there exists* $\varrho \in \mathscr{R}^+$ *such that for all* $m \geqslant 1$ *and* $a_1, \ldots, a_m \in E'$,

$$\Big(\sum_i \|V' a_i\|^{q'}\Big)^{1/q'} \leqslant \varrho \sup_{x \in E_1} \Big(\sum_i |\langle x, a_i\rangle|^p\Big)^{1/p}$$

and $\alpha^D(V) = \inf\{\varrho\}$.

(c) *If* $0 < p \leqslant 1$ *and* $\alpha = g_{p,q}$ *then* $V \in D_\alpha$ *and* $\alpha^D(V) = \|V\|$.

(d) *If* $0 < q \leqslant 1$ *and* $\alpha = d_{p,q}$ *then* $V \in D_\alpha$ *and* $\alpha^D(V) = \|V\|$.

Proofs. (a) and (b) are immediate from Theorem 48(a) and (c).

(c) If $m \geqslant 1$ and $y_1, \ldots, y_m \in F$ then

$$\sup_i \|Vy_i\| \leqslant \|V\| \sup_i \|y_i\| \leqslant \|V\| \sup_{b \in F_1'} \Big(\sum_i |\langle y_i, b\rangle|^q\Big)^{1/q}$$

and it follows from Theorem 48(a) that $V \in D_\alpha$ and $\alpha^D(V) \leqslant \|V\|$. On the other hand, it follows from Theorem 52(b) and Lemma 39 (with α and A replaced by α^D and D_α) that $\|V\| \leqslant \alpha^D(V)$. The proof of (d) is similar to that of (c).

56. Remark. The result of Theorem 55(a) is known in the case $\frac{1}{p}+\frac{1}{q} = 1$ (see [8]; Theorem 3.2) and gives that V be q-absolutely summing. The result of Theorem 55(b) is also related to known results in the case $\frac{1}{p}+\frac{1}{q} = 1$: here $\alpha = d_{p,q}$; we write $\beta = g_{p,q}$. Then, from Theorem 37 and Corollary 18, $D_\alpha = D_{\beta'} = (D_\beta)'$ so (b) gives that V' be p-absolutely summing (see [2]; Prop. 3.2.7, p. 51).

In general, Theorem 55(a) gives that V be (p', q)-absolutely summing and Theorem 55(b) that V' be (q', p)-absolutely summing. It thus follows from Theorem 27, Corollary 33 and Corollary 35 that if β is either the (r, s)-absolutely summing norm or its "′" then for all $V \in L(F, E)$,

$$\beta(V'') = \beta(J_E V) = \beta(V) \quad \text{(see [10]; Theorem 17)},$$

and for all $V \in L(E', F')$,

$$\beta(V') = \beta(V' J_F).$$

57. Theorem. *We suppose that* $\frac{1}{p}+\frac{1}{q} = \frac{1}{r} > 0$ *and* $T \in L_0(E, F)$. *Then*

$$s_{p,q}(T) = \inf\Big\{\Big(\sum_i \|a_i\|^r \|y_i\|^r\Big)^{1/r}\Big\}.$$

Proof. If $T = \sum_i \langle \cdot, a_i\rangle y_i$ then, from Lemma 51,

$$s_{p,q}(T)^r \leqslant \sum_i s_{p,q}(\langle \cdot, a_i\rangle y_i)^r \leqslant \sum_i \|a_i\|^r \|y_i\|^r$$

and "$\leqslant$" is immediate. On the other hand, as is well known,

$$\Big(\sum_i \|a_i\|^p\Big)^{1/p}\Big(\sum_i \|y_i\|^q\Big)^{1/q} \geqslant \Big(\sum_i \|a_i\|^r \|y_i\|^r\Big)^{1/r}$$

and so "$\geqslant$" also follows.

58. Corollary. *If* $\frac{1}{p}+\frac{1}{q} = 1$ *then* $s_{p,q} = g_{1,\infty} = d_{\infty,1}$.

Proof. It follows from Theorem 57 that if $T \in L_0(E, F)$ then

$$s_{p,q}(T) = \inf\Big\{\sum_i \|a_i\| \|y_i\|\Big\}.$$

It is easily seen that $g_{1,\infty}(T)$ and $d_{\infty,1}(T)$ are given by the same formula.

59. Theorem. *We suppose that* $p \geqslant 1$, $q \geqslant 1$ *and* $\frac{1}{p}+\frac{1}{q} = \frac{1}{r} \geqslant 1$ *and we write* T *for the identity map from* $l_{p'}^n$ *onto* $l_q^n\Big(\frac{1}{p}+\frac{1}{p'} = 1\Big)$. Then $s_{p,q}(T) = n^{1/r}$.

Proof. We suppose that $T = \sum_{i=1}^m \langle \cdot, a_i\rangle y_i$ $(a_1, \ldots, a_m \in (l_{p'}^n)' = l_p^n$ and $y_1, \ldots, y_m \in l_q^n)$. Then

$$\begin{aligned}
\sum_{i=1}^m \|a_i\|^r \|y_i\|^r &= \sum_{i=1}^m \Big(\sum_{j=1}^n |a_{i,j}|^p\Big)^{r/p} \Big(\sum_{j=1}^n |y_{i,j}|^q\Big)^{r/q} \\
&\geqslant \sum_{i=1}^m \Big(\sum_{j=1}^n |a_{i,j}|^r |y_{i,j}|^r\Big) \\
&= \sum_{j=1}^n \Big(\sum_{i=1}^m |a_{i,j}|^r |y_{i,j}|^r\Big) \\
&\geqslant \sum_{j=1}^n \Big(\sum_{i=1}^m |a_{i,j}| |y_{i,j}|\Big)^r \quad \text{since } r \leqslant 1 \\
&\geqslant \sum_{j=1}^n \Big|\sum_{i=1}^m a_{i,j} y_{i,j}\Big|^r \\
&= \sum_{j=1}^n |\langle Te^{(j)}, b^{(j)}\rangle|^r
\end{aligned}$$

where $e^{(1)}, \ldots, e^{(n)}$ are the usual basic elements of $l_{p'}^n$ and $b^{(1)}, \ldots, b^{(n)}$ are the coordinate functionals on l_q^n. Hence

$$\sum_{i=1}^{m} \|a_i\|^r \|y_i\|^r \geqslant \sum_{j=1}^{n} 1^r = n$$

and it follows from Theorem 57 that $s_{p,q}(T) \geqslant n^{1/r}$. "$\leqslant$" is immediate from the standard representation of T.

60. Corollary. *We suppose that $p \geqslant 1$, $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q} = \frac{1}{r} \geqslant 1$ and we write U for the identity map from l_∞^n onto l_1^n. Then $s_{p,q}(U) = n^{1/r}$.*

Proof. We write V for the identity map from $l_{p'}^n$ onto l_∞^n and W for the identity map from l_1^n onto l_q^n. Then the map T of Theorem 59 can be written as WUV hence

$$n^{1/r} = s_{p,q}(T) = s_{p,q}(WUV) \leqslant \|W\| s_{p,q}(U) \|V\| \leqslant s_{p,q}(U).$$

"$\leqslant$" is immediate from the standard representation of U.

61. Remark. The result of Theorem 62 should be compared with those of Theorem 55.

62. Theorem. *We suppose that $p \geqslant 1$, $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q} \geqslant 1$ and $V \in L(F, E)$. We write $\alpha = s_{p,q}$. Then $V \in D_\alpha(F, E)$ and $\alpha^D(V) = \|V\|$.*

Proof. Since $\frac{1}{q} \geqslant 1 - \frac{1}{p} = \frac{1}{p'}$, we have $p' \geqslant q$. Hence, for all $m \geqslant 1$ and $y_1, \ldots, y_m \in F$,

$$\Big(\sum_i \|Vy_i\|^{p'}\Big)^{1/p'} \leqslant \Big(\sum_i \|Vy_i\|^q\Big)^{1/q} \leqslant \|V\| \Big(\sum_i \|y_i\|^q\Big)^{1/q}$$

and it follows from Theorem 48(b) that $V \in D_\alpha(F, E)$ and $\alpha^D(V) \leqslant \|V\|$. On the other hand, it follows from Theorem 52(b) and Lemma 39 (with α and A replaced by α^D and D_α) that $\|V\| \leqslant \alpha^D(V)$.

63. Remark. We have already observed in Corollary 45 that the inversion formula (7) fails if α is a reasonable splitting quasinorm but not a norm. Corollary 60 and Theorem 62 give the even stronger result that it is impossible to recapture α from α^D.

10. LIFTABLE QUASINORMS

64. Definition. We say that a quasinorm α on L_0 is *liftable* if, whenever $T \in L_0(H, F)$, $I: H \to E$ is an isometry (into) and $\varepsilon > 0$ then there exists $U \in L_0(E, F)$ such that $T = UI$ and $\alpha(U) \leqslant \alpha(T) + \varepsilon$.

65. Lemma. *We suppose that M, N are as in Theorem 48(a). Then $g_{M,N}$ and $s_{M,N}$ are liftable. In particular, $g_{p,q}$ and $s_{p,q}$ are liftable.*

Proof. These results are immediate from Lemma 47 and the Hahn–Banach Theorem.

66. Theorem. *Let α be liftable, $W \in L(F, H)$ and $I: H \to E$ be an isometry. Then $\alpha^D(W) = \alpha^D(IW)$.*

Proof. Let $\varepsilon > 0$, $T \in L_0(H, F)$ and $\alpha(T) \leqslant 1$. Then there exists $U \in L_0(E, F)$ such that $T = UI$ and $\alpha(U) \leqslant 1+\varepsilon$. Thus

$$|\mathrm{tr} J_F TW| = |\mathrm{tr} J_F UIW| \leqslant \alpha(U)\alpha^D(IW) \leqslant (1+\varepsilon)\alpha^D(IW).$$

Taking the supremum over T and letting $\varepsilon \to 0$, $\alpha^D(W) \leqslant \alpha^D(IW)$. The reverse inequality is trivial.

67. Notation. By abuse of notation we write α^D for $\alpha^D | L_0$.

68. Corollary. *If α is a reasonable liftable quasinorm on L_0 then α^D is a right splitting norm on L_0.*

Proof. Immediate from Theorem 66 and Lemma 42.

69. Theorem. *Let α be a right splitting reasonable liftable quasinorm on L_0 and β a splitting norm on L_0. Then* (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

(a) *If $N \geqslant 1$ and $T \in L_0(l_\infty^N, F)$ then $\alpha(T) = \beta^D(T)$.*

(b) *If $N \geqslant 1$ and $V \in L_0(F, l_\infty^N)$ then $\alpha^D(V) = \beta(V)$.*

(c) *If E is a $\mathscr{L}_{\infty,\lambda}$-space for all $\lambda > 1$ and $V \in L_0(F, E)$ then $\alpha^D(V) = \beta(V)$.*

(d) *α^D is splitting.*

Proof. ((a) $\Rightarrow$ (b)). If $V \in L_0(F, l_\infty^N)$ then, from Lemma 24,

$$\begin{aligned}
\alpha^D(V) &= \sup\{|\mathrm{tr}\, TV|:\ T \in L_0(l_\infty^N, F''),\ \alpha(T) \leqslant 1\} \\
\text{from (a)} \quad &= \sup\{|\mathrm{tr}\, TV|:\ T \in L_0(l_\infty^N, F''),\ \beta^D(T) \leqslant 1\} \\
&= \sup\{|\mathrm{tr}\, TV|:\ T \in D_\beta(l_\infty^N, F''),\ \beta^D(T) \leqslant 1\}.
\end{aligned}$$

(b) now follows from Corollary 45 (with α replaced by β). (We note, in passing, that if α is a liftable splitting norm then (a) $\Leftrightarrow$ (b).)

((b) $\Leftrightarrow$ (c)). β is given to be right splitting and we know from Corollary 68 that α^D is also right splitting. It follows easily from the definition of a $\mathscr{L}_{\infty,\lambda}$-space (see [5]; p. 326) that (b) $\Rightarrow$ (c) and it is trivial that (c) $\Rightarrow$ (b).

((c) $\Rightarrow$ (d)). Let $T \in L_0(E, F)$ and $\varepsilon > 0$. There exists an isometry $I: F \to H$, where H is a $C(K)$-space and hence a $\mathscr{L}_{\infty,\lambda}$-space for all $\lambda > 1$. From Lemma 8(a), there exist G, a surjection $P \in L_0(E, G)$ and $Q \in L_0(G, H)$ such that

$$IT = QP \quad \text{and} \quad \beta(Q)\|P\| \leqslant \beta(IT) + \varepsilon. \tag{10}$$

We write S for the identity map from $I(F)$ into H. Since $Q(G) = QP(E) = IT(E) \subset I(F)$, we can define $\hat{Q} \in L_0(G, I(F))$ so that $Q = S\hat{Q}$. Finally, we define $U \in L(I(F), F)$ so that $IU = S$. Then $IT = QP = S\hat{Q}P = IU\hat{Q}P$ hence $T = U\hat{Q}P$. Further

$$\alpha^D(U\hat{Q})\|P\| \leqslant \alpha^D(\hat{Q})\|P\|.$$

From Theorem 66,

$$\alpha^D(\hat{Q})\|P\| = \alpha^D(S\hat{Q})\|P\| = \alpha^D(Q)\|P\|$$

hence, using (c) twice and (10),

$$\alpha^D(U\hat{Q})\|P\| \leqslant \alpha^D(Q)\|P\| = \beta(Q)\|P\| \leqslant \beta(IT)+\varepsilon = \alpha^D(IT)+\varepsilon \leqslant \alpha^D(T)+\varepsilon.$$

Thus α^D is left splitting. It follows from Corollary 68 that α^D is splitting.

70. Remark. We shall apply Theorem 69 in Theorem 76.

11. A PROPERTY OF $g_{p,p'}$

71. Lemma. *Let $x_1, \ldots, x_n \in E$, $a_1, \ldots, a_m \in E'$ and $p \geqslant 1$. Then*

$$\Big(\sum_{i,j} |\langle x_j, a_i\rangle|^p\Big)^{1/p} \leqslant \Big(\sum_i \|a_i\|^p\Big)^{1/p} \sup_{a\in E_1'} \Big(\sum_j |\langle x_j, a\rangle|^p\Big)^{1/p}.$$

Proof. Immediate.

72. Lemma. *If $T \in L_0(E, F)$, $V \in L_0(F, E)$, $p \geqslant 1$, $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q} = 1$ then*

$$|\operatorname{tr} J_E VT| \leqslant g_{p,q}(T) g_{q,p}(V).$$

Proof. Let $a_1, \ldots, a_m \in E'$, $y_1, \ldots, y_m \in F$, $b_1, \ldots, b_n \in F'$ and $x_1, \ldots, x_n \in E$ be such that $T = \sum_i \langle\cdot, a_i\rangle y_i$ and $V = \sum_j \langle\cdot, b_j\rangle x_j$. Then

$$|\operatorname{tr} J_F TV| = \Big|\sum_{i,j} \langle x_j, a_i\rangle \langle y_i, b_j\rangle\Big| \leqslant \Big(\sum_{i,j} |\langle x_j, a_i\rangle|^p\Big)^{1/p} \Big(\sum_{i,j} |\langle y_i, b_j\rangle|^q\Big)^{1/q},$$

from Lemma 71

$$\leqslant \Big(\sum_i \|a_i\|^p\Big)^{1/p} \sup_{b\in F_1'} \Big(\sum_i |\langle y_i, b\rangle|^q\Big)^{1/q} \Big(\sum_j \|b_j\|^q\Big)^{1/q} \sup_{a\in E_1'} \Big(\sum_j |\langle x_j, a\rangle|^p\Big)^{1/p}.$$

The required result follows on taking the infimum.

73. Corollary. *If $T \in L_0$, $p \geqslant 1$, $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q} = 1$ then*

$$g^D_{q,p}(T) \leqslant g_{p,q}(T).$$

Proof. This is immediate from Lemma 72 and Definition 23.

74. Theorem. *If $T \in L_0(l_\infty^N, F)$, $p \geqslant 1$, $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q} = 1$ then*

$$g^D_{q,p}(T) = g_{p,q}(T).$$

Proof. We prove the case $p < \infty$ and leave the case $p = \infty$ to the reader. From ([4]; Proposition 3.1) (or, alternatively, by using a minimax theorem) there exist $\theta_1, \ldots, \theta_N \geqslant 0$ such that $\sum_k \theta_k = (g^D_{q,p}(T))^p$ and

$$\text{for all } x \in l_\infty^N, \quad \|Tx\|^p \leqslant \sum_k \theta_k |x_k|^p.$$

In what follows, l runs over those $k \in \{1, \ldots, N\}$ for which $\theta_k \neq 0$. For each such l we write $e^{(l)}$ for the lth unit vector of l_∞^N and $a^{(l)}$ for the lth coordinate functional on l_∞^N. Clearly

$$T = \sum_l \langle\cdot, \theta_l^{1/p} a^{(l)}\rangle \theta_l^{-1/p} Te^{(l)}.$$

If $b \in F_1'$ and $x \in l_\infty^N$ then

$$\Big|\sum_l x_l \langle Te^{(l)}, b\rangle\Big| = |\langle Tx, b\rangle| \leqslant \|Tx\| \leqslant \Big(\sum_l \theta_l |x_l|^p\Big)^{1/p}.$$

If we take x_l to be $\theta_l^{-1}|\langle Te^{(l)}, b\rangle|^{q/p}$ multiplied by a suitable complex number of absolute value 1 we obtain that

$$\Big(\sum_l |\langle \theta_l^{-1/p} Te^{(l)}, b\rangle|^q\Big)^{1/q} \leqslant 1.$$

We have proved that

$$\sup_{b\in F_1'} \Big(\sum_l |\langle \theta_l^{-1/p} Te^{(l)}, b\rangle|^q\Big)^{1/q} \leqslant 1.$$

On the other hand

$$\Big(\sum_l \|\theta_l^{1/p} a^{(l)}\|^p\Big)^{1/p} = \Big(\sum_l \theta_l\Big)^{1/p} = g^D_{q,p}(T).$$

Hence $g_{p,q}(T) \leqslant g^D_{q,p}(T)$. The result now follows from Corollary 73.

75. Remark. The result of Theorem 74 can be deduced easily from ([6]; Satz 46, [6]; Lemma 7 and [6]; Lemma 11). We have preferred to

give an independent proof since we shall be proving a stronger result than ([6]; Lemma 7) and ([6]; Lemma 11) later. See Remark 93.

76. Theorem. *If* $p \geqslant 1$, $q \geqslant 1$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$ *then* $g_{p,q}^D$ *is a splitting norm on* L_0.

Proof. This is immediate from Theorem 52(c), Lemma 65, Theorem 69 (with $\alpha = g_{p,q}$ and $\beta = g_{q,p}$) and Theorem 74.

77. Problems.

(a) *If* $r \geqslant s$ *is the restriction to* L_0 *of the* (r, s)*-absolutely summing norm splitting?* (See the discussion in Remark 56.)

(b) *To find cases other than that of Theorem* 76 *where Theorem* 69 *can be applied.*

12. TOTALLY SPLITTING NORMS AND THEIR METRIC APPROXIMATION PROPERTIES

78. Definition. We shall say that a norm α on L_0 is *totally splitting* if both α and α^D are splitting.

79. Lemma. *If* α *is totally splitting then so is* α'.

Proof. Immediate from Theorem 13 and Theorem 25.

80. Theorem. *If* $p \geqslant 1$, $q \geqslant 1$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$ *then* $g_{p,q}$ *and* $d_{p,q}$ *are totally splitting norms on* L_0.

Proof. The $g_{p,q}$ case follows from Theorem 52(c) and Theorem 76. The $d_{p,q}$ case follows from Corollary 18 and Lemma 79.

81. Notation. We shall suppose for the rest of this section that α is a totally splitting norm on L_0.

We write $\mathscr{T}$ for the topology of simple convergence on $L(F, E)$.

82. Lemma. *If* $T \in L_0(E, F)$ *then the map* φ *of* $L(F, E)$ *into* $\mathscr{K}$ *defined by*

$$\varphi(V) = \operatorname{tr} J_F T V \quad \big(V \in L(F, E)\big) \tag{11}$$

is a $\mathscr{T}$*-continuous linear functional on* $L(F, E)$ *and, conversely, any* $\mathscr{T}$*-continuous linear functional* φ *on* $L(F, E)$ *can be put in the form* (11) *for a unique* $T \in L_0(E, F)$.

Proof. See [1]; § 2 No. 9 Prop. 11 p. 77.

83. Theorem. *The conditions* (a)–(d) *on* F *are equivalent.*

(a) *For all* E, $\{L_0(F, E)\colon \alpha^D \leqslant 1\}$ *is* $\mathscr{T}$*-dense in* $\{D_\alpha(F, E)\colon \alpha^D \leqslant 1\}$.

(b) *For all* E, $\{L_0(F, E'')\colon \alpha^D \leqslant 1\}$ *is* $\mathscr{T}$*-dense in* $\{D\ (F, E'')\colon \alpha^D \leqslant 1\}$.

(c) *For all* E *and* $T \in L_0(E, F)$, $\alpha(T) \leqslant \alpha^{DD}(T)$.

(d) *For all* E *and* $T \in L_0(E'', F)$, $\alpha(T) \leqslant \alpha^{DD}(T)$.

Proof. It is trivial that (a) $\Rightarrow$ (b). If (b) is true then, from Lemma 82, for all $T \in L_0(E'', F)$

$$\sup\{|\operatorname{tr} J_F T V|\colon V \in D_\alpha(F, E''),\ \alpha^D(V) \leqslant 1\} \leqslant \sup\{|\operatorname{tr} J_F T V|\colon V \in L_0(F, E''),\ \alpha^D(V) \leqslant 1\}.$$

Hence, from Lemma 21(d), for all $T \in L_0(E, F)$

$$\sup\{|\operatorname{tr} V T|\colon V \in D_\alpha(F, E''),\ \alpha^D(V) \leqslant 1\} \leqslant \sup\{|\operatorname{tr} V T|\colon V \in L_0(F, E''),\ \alpha^D(V) \leqslant 1\}$$

and (c) follows from Corollary 45 and Lemma 24 (with E, F and α replaced by F, E and α^D, respectively). If (c) is true then, from Corollary 45 and Definition 23 (with E, F and α again replaced by F, E and α^D, respectively), for all $T \in L_0(E, F)$,

$$\sup\{|\operatorname{tr} V T|\colon V \in D_\alpha(F, E''),\ \alpha^D(V) \leqslant 1\} \leqslant \sup\{|\operatorname{tr} J_F T V|\colon V \in L_0(F, E),\ \alpha^D(V) \leqslant 1\}.$$

If $V \in L(F, E)$ then $J_E V \in L(F, E'')$ and, from Lemma 21(a), $\operatorname{tr}(J_E V) T = \operatorname{tr} J_F T V$. Since $\alpha^D(J_E V) \leqslant \alpha^D(V)$

$$\sup\{|\operatorname{tr} J_F T V|\colon V \in D_\alpha(F, E),\ \alpha^D(V) \leqslant 1\} \leqslant \sup\{|\operatorname{tr} J_F T V|\colon V \in L_0(F, E),\ \alpha^D(V) \leqslant 1\} \tag{12}$$

and (a) follows from Lemma 82 and the bipolar theorem. We have proved that (a), (b) and (c) are equivalent. It is trivial that (c) $\Rightarrow$ (d) and we can prove that (d) $\Rightarrow$ (b) by analogy with the proof already given that (c) $\Rightarrow$ (a).

84. Definition. We say that a Banach space F has the α*-metric approximation property* (α*-m.a.p.*) if (a), (b), (c) or (d) of Theorem 83 is satisfied.

85. Theorem. *If* F'' *has the* α*-m.a.p. then so does* F.

Proof. If $T \in L_0(E, F)$ then, from Lemma 9(b) and Corollary 33 (with E, F and α replaced by F, E and α^D, respectively)

$$\alpha(T) = \alpha(J_F T) \leqslant \alpha^{DD}(J_F T) = \alpha^{DD}(T).$$

86. Theorem. *The conditions* (a)–(d) *on a Banach space* E *are equivalent.*

(a) E' *has the* α'*-m.a.p.*

(b) *For all* F, $\{L_0(E', F')\colon \alpha'^D \leqslant 1\}$ *is* $\mathscr{T}$*-dense in* $\{D_\alpha(E', F')\colon \alpha'^D \leqslant 1\}$.

(c) *For all* F *and* $T \in L_0(F', E')$, $\alpha'(T) \leqslant \alpha'^{DD}(T)$.

(d) *For all* F *and* $T \in L_0(E, F)$, $\alpha(T) \leqslant \alpha^{DD}(T)$.

Proof. It follows from (a) and (b) of Theorem 83 that (a) $\Leftrightarrow$ (b), and from conditions (c) and (d) of Theorem 83 that (a) $\Leftrightarrow$ (c). If (c) is true and $T \in L_0(E, F)$ then $\alpha'(T') \leqslant \alpha'^{DD}(T')$ and (d) follows from Theorem 11 and Theorem 25. If (d) is true and $T \in L_0(F', E')$ then, from (d) (with F replaced by F''), $\alpha(T'J_E) \leqslant \alpha^{DD}(T'J_E)$. It follows from Corollary 12(b) (with F replaced by F') and Theorem 25 that (c) is true.

87. THEOREM. *If E' has the α'-m.a.p. then, for all F*

$$\{V' : V \in L_0(F, E),\ \alpha^D(V) \leqslant 1\} \text{ is } \mathcal{T}\text{-dense in } \{V' : V \in D_\alpha(F, E),\ \alpha^D(V) \leqslant 1\}.$$

Proof. Let $T \in L_0(F', E')$. Then, from Lemma 21(b), Theorem 27, Corollary 45, Theorem 86((a) $\Rightarrow$ (c)), Theorem 25, Corollary 35 (with E, F, α and V replaced by F, E, α^D and T, respectively), Theorem 31 (with E, F, α and V replaced by F, E, α^D and $T'J_E$, respectively) and Lemma 21(b) (in that order)

$$\begin{aligned}
\sup\{|\operatorname{tr} J_{E'}TV'| : V \in D_\alpha(F, E),\ \alpha^D(V) \leqslant 1\}
&= \sup\{|\operatorname{tr} J_{E''}V''T'| : V \in D_\alpha(F, E),\ \alpha^D(V) \leqslant 1\}\\
&\leqslant \sup\{|\operatorname{tr} VT'| : V \in D_\alpha(F'', E''''),\ \alpha^D(V) \leqslant 1\}\\
&= \alpha(T') \leqslant \alpha'^{DD}(T) = \alpha^{DD}(T') = \alpha^{DD}(T'J_E)\\
&= \sup\{|\operatorname{tr} T'J_E V| : V \in L_0(F, E),\ \alpha^D(V) \leqslant 1\}\\
&= \sup\{|\operatorname{tr} J_{E'}TV'| : V \in L_0(F, E),\ \alpha^D(V) \leqslant 1\}.
\end{aligned}$$

The result follows from Lemma 82 and the bipolar theorem.

88. DEFINITION. F is said to have the *metric approximation property* (*m.a.p.*) if 1_F is in the $\mathcal{T}$-closure of $\{L_0(F, F) : \|\cdot\| \leqslant 1\}$. The m.a.p. is equivalent to the α-m.a.p. where $\alpha = g_{1,\infty} = d_{\infty,1}$. (See [3]; Proposition 39, p. 179.) ($\alpha^D = \|\cdot\|$ by virtue of Theorem 55(c) and (d).)

89. THEOREM. *If F has the m.a.p. then F has the α-m.a.p.*

Proof. Let $V \in D_\alpha(F, E)$ and $\alpha^D(V) \leqslant 1$. If $T \in L_0(E, F)$ then $TV \in L_0(F, F)$ hence, from Definition 88 and Lemma 82

$$|\operatorname{tr} J_F(TV)1_F| \leqslant \sup\{|\operatorname{tr} J_F(TV)U| : U \in L_0(F, F),\ \|U\| \leqslant 1\}$$

hence

$$|\operatorname{tr} J_F TV| \leqslant \sup\{|\operatorname{tr} J_F TW| : W \in L_0(F, E),\ \alpha^D(W) \leqslant 1\}.$$

We have established inequality (12) and the result follows from the proof of Theorem 83.

90. COROLLARY. *If E' or F has the m.a.p. then, for all $T \in L_0(E, F)$, $\alpha(T) = \alpha^{DD}(T)$.*

Proof. Immediate from Lemma 79, Theorem 89, Theorem 83 and Theorem 86.

91. COROLLARY. *If E' has the m.a.p. then*

$$\{V' : V \in L_0(F, E),\ \alpha^D(V) \leqslant 1\} \text{ is } \mathcal{T}\text{-dense in } \{V' : V \in D_\alpha(F, E),\ \alpha^D(V) \leqslant 1\}.$$

Proof. Immediate from Lemma 79, Theorem 89 and Theorem 87.

92. THEOREM. *The following conditions on α are equivalent.*

(a) *Every Banach space has the α-m.a.p.*

(b) *α^{DD} is splitting.*

(c) *α^{DD} is right splitting.*

(d) *α^{DD} is left splitting.*

Proof. *If* (a) is true then, from condition (c) of Theorem 83, $\alpha = \alpha^{DD}$ (on L_0) hence α^{DD} is splitting, i.e., (b) is true. It is trivial that (b) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d).

If (c) is true, $T \in L_0(E, F)$ and $\varepsilon > 0$ then, from Lemma 9(a), there exists a finite dimensional subspace H of F and $R \in L_0(E, H)$ such that $T = IR$ and $\alpha^{DD}(R) \leqslant \alpha^{DD}(T) + \varepsilon$, where $I \in L_0(H, F)$ is the inclusion map. Since H is finite dimensional, from Theorem 83 ((a) $\Rightarrow$(c)), $\alpha(R) \leqslant \alpha^{DD}(R)$. Thus $\alpha(T) = \alpha(IR) \leqslant \alpha(R) \leqslant \alpha^{DD}(R) \leqslant \alpha^{DD}(T) + \varepsilon$. Since ε is arbitrary, we have established condition (c) of Theorem 83. Hence (a) is true.

If (d) is true, $T \in L_0(E, F)$ and $\varepsilon > 0$ then, from Lemma 8(a), there exists a finite dimensional space G, $P \in L_0(E, G)$ and $Q \in L_0(G, F)$ such that $T = QP$ and $\alpha^{DD}(Q)\|P\| \leqslant \alpha^{DD}(T) + \varepsilon$. Since G is finite dimensional, from Theorem 86 ((b) $\Rightarrow$ (d)), $\alpha(Q) \leqslant \alpha^{DD}(Q)$. Hence $\alpha(T) = \alpha(QP) \leqslant \alpha(Q)\|P\| \leqslant \alpha^{DD}(Q)\|P\| \leqslant \alpha^{DD}(T) + \varepsilon$. Since ε is arbitrary, we have again established condition (c) of Theorem 83. Hence (a) is true.

93. Remarks. If we take $\alpha = g_{p,p'}$ in Corollary 90 (where $1 \leqslant p \leqslant \infty$) we obtain: if E' or F has the m.a.p. then, for all $T \in L_0(E, F)$,

$$g_{p,p'}(T) = g_{p,p'}{}^{DD}(T). \tag{13}$$

We shall see in Corollary 96 that, in the notation of [6], (13) can be rewritten

$$g_{p,p'}(T) = \iota_p(J_F T)$$

thus this case of Corollary 90 generalizes ([6]; Lemmas 7 and 11) in which it is proved that

$$g_{p,p'}(T) \leqslant \iota_p(T).$$

If we take $\alpha = g_{1,\infty}$ in Corollary 91 we obtain: if E' has the m.a.p. then

$\{V'\colon V \in L_0(F, E), \|V\| \leqslant 1\}$ is $\mathscr{T}$-dense in $\{V'\colon V \in L(F, E), \|V\| \leqslant 1\}$.

This generalizes the footnote on ([6]; p. 38) in which $F = E$ and the density is applied at $I'_E = I_{E'} \in L(E', E')$.

Theorem 92 shows that it is not entirely trivial to decide whether ${g_{1,\infty}}^{DD} = \|\cdot\|^D$ is splitting. It is, however, not hard to prove that ${g_{2,2}}^{DD}$ and ${d_{2,2}}^{DD}$ are splitting.

Finally, in all the "density" conditions in this section we can replace $\mathscr{T}$ by the topology of compact convergence. (See [3]; § 5, No. 2, Lemma 20, p. 178.)

13. CONNECTIONS WITH p-INTEGRAL MAPS AND RIGHT p-INTEGRAL MAPS

94. THEOREM. *We suppose that the notation is as in Theorem 69 and that condition* (c) *therein is satisfied. Let F be a dual Banach space and $T \in L(E, F)$. Then $T \in D_{\alpha^D}(E, F) \Leftrightarrow$ there exists $Q \in D_\beta(C(E_1'), F)$ such that $T = QP$, where $P \in L(E, C(E_1'))$ is the canonical map. Further, for any such factorization $\alpha^{DD}(T) \leqslant \beta^D(Q)$ and there exists such a factorization for which $\alpha^{DD}(T) = \beta^D(Q)$.*

Proof. It follows from condition (c) of Theorem 69 that if G is a $\mathscr{L}_{\infty,\lambda}$-space for all $\lambda > 1$ then, for all $Q \in L(G, F)$, $\alpha^{DD}(Q) = \beta^D(Q)$. Thus if $T = QP$, where P and Q are as in the statement, then $T \in D_{\alpha^D}(E, F)$ and $\alpha^{DD}(T) \leqslant \|P\|\alpha^{DD}(Q) = \|P\|\beta^D(Q) = \beta^D(Q)$. Conversely, we suppose that $T \in D_{\alpha^D(E,F)}$. From Definition 23 (with E, F and α replaced by F, E and α^D, respectively), the map φ of $L_0(F, E)$ into $\mathscr{K}$ defined by

$$\varphi(V) = \operatorname{tr} J_E VT \qquad (V \in L_0(F, E))$$

is in $(L_0(F, E), \alpha^D)'$ and $\|\varphi\| = \alpha^{DD}(T)$. From Theorem 66 and condition (c) of Theorem 69, the map $V \to PV$ is an isometry of $(L_0(F, E), \alpha^D)$ into $(L_0(F, C(E_1')), \beta)$. From the Hahn–Banach Theorem, there exists $\hat{\varphi} \in L_0((F, C(E_1')), \beta)'$ such that $\|\hat{\varphi}\| = \|\varphi\|$ and, for all $V \in L_0(F, E)$, $\hat{\varphi}(PV) = \varphi(V)$. From Theorem 44 (with E, F and α replaced by F, $C(E_1')$ and β, respectively), there exists $\hat{T} \in D_\beta(C(E_1'), F'')$ such that $\beta^D(\hat{T}) = \|\hat{\varphi}\|$ and, for all $W \in L_0(F, C(E_1'))$, $\hat{\varphi}(W) = \operatorname{tr}\hat{T}W$. Thus, for all $V \in L_0(F, E)$

$$\operatorname{tr} J_F TV = \operatorname{tr} J_E VT = \varphi(V) = \hat{\varphi}(PV) = \operatorname{tr}\hat{T}PV.$$

Hence $J_F T = \hat{T}P$. Since F is a dual Banach space, there exists $U \in L(F'', F)$ such that $\|U\| \leqslant 1$ and $UJ_F = 1_F$. Hence $T = U\hat{T}P$ and the result follows since $\beta^D(U\hat{T}) \leqslant \beta^D(\hat{T}) = \|\hat{\varphi}\| = \|\varphi\| = \alpha^{DD}(T)$.

95. THEOREM. *Let $1 \leqslant p \leqslant \infty$ and $\alpha = g_{p,p'}$. Let F be a dual Banach space and $T \in L(E, F)$. Then (with I_p and ι_p as in* ([6]; Sections 3 and 9)

$$T \in D_{\alpha^D}(E, F) \Leftrightarrow T \in I_p(E, F) \quad \textit{and then } \alpha^{DD}(T) = \iota_p(T).$$

Proof. The result is immediate from Theorem 74, Theorem 69, Theorem 94, Theorem 55(a) and ([5]; Satz 46).

96. COROLLARY. *Let $1 \leqslant p \leqslant \infty$ and $\alpha = g_{p,p'}$. Let F be any Banach space and $T \in L(E, F)$. Then $T \in D_{\alpha^D}(E, F) \Leftrightarrow J_F T \in I_p(E, F'')$ and then $\alpha^{DD}(T) = \iota_p(J_F T)$.*

Proof. This result is immediate from Corollary 33 (with E, F and α replaced by F, E and α^D, respectively) and Theorem 95.

97. Remarks. Corollary 96 justifies the comments made about Corollary 90 in Remark 93.

It seems to be an open question whether $T \in L(E, F)$ and $J_F T \in I_p(E, F'') \Rightarrow T \in I_p(E, F)$ $(p \neq 2)$.

The proof of Theorem 94 is a generalization of that of ([6]; Satz 53). Both ([6]; Satz 53) and ([3]; Proposition 27, p. 124) can be deduced from Theorem 44 and Theorem 95.

98. DEFINITION. In what follows $(\Omega, \mathscr{A}, \mu)$ will stand for a measure space such that $0 < \mu(\Omega) < \infty$ and, if $1 \leqslant q \leqslant p \leqslant \infty$, $S_{p,q}$ for the canonical map $L_p(\Omega, \mathscr{A}, \mu) \to L_q(\Omega, \mathscr{A}, \mu)$.

We suppose that $T \in L(E, F)$. If $1 \leqslant p < \infty$, a *left p-setup for T* is a quintuple $(\Omega, \mathscr{A}, \mu, P, Q)$ where $P \in L(E, L_\infty)$, $Q \in L(L_p, F)$ and $T = QS_{\infty,p}P$. A *left ∞-setup for T* is a triple (G, P, Q) where $G = C(K)$ for some compact Hausdorff K, $P \in L(E, G)$, $Q \in L(G, F)$ and $T = QP$. If $1 \leqslant p < \infty$ a *right p-setup for T* is a quintuple $(\Omega, \mathscr{A}, \mu, P, Q)$ where $P \in L(E, L_{p'})$, $Q \in L(L_1, F)$ and $T = QS_{p',1}P$. A *right ∞-setup for T* is a triple (G, P, Q) where G is a L_1 space (for a nonzero but not necessarily finite measure), $P \in L(E, G)$, $Q \in L(G, F)$ and $T = QP$.

99. LEMMA. *Let $T \in L(E, F)$.*

(a) *If $1 \leqslant p < \infty$ and $(\Omega, \mathscr{A}, \mu, P, Q)$ is a left p-setup for $J_F T$ then $(\Omega, \mathscr{A}, \mu, Q'J_{F'}, P'J_{L_1})$ is a right p-setup for $T' \in L(F', E')$.*

(b) *If $1 \leqslant p < \infty$ and $(\Omega, \mathscr{A}, \mu, P, Q)$ is a right p-setup for $J_F T$ then $(\Omega, \mathscr{A}, \mu, Q'J_{F'}, P'J_{L_p})$ is a left p-setup for $T' \in L(F', E')$.*

(c) *If (G, P, Q) is a left (resp. right) ∞-setup for $J_F T$ then $(G', Q'J_{F'}, P')$ is a right (resp. left) ∞-setup for $T' \in L(F', E')$.*

Proofs. (a) and (b) follow from the Riesz representation theorem and (c) from Kakutani's results that the dual of a $C(K)$ space is an L_1 space and vice versa.

100. Remark. We can prove, with only minor modifications to the proof given in ([6]; Satz 18), that, if $1 \leqslant p < \infty$ and $T \in L(E, F)$ then

$T \in I_p(E, F) \Leftrightarrow$ there exists a left p-setup for T and then $\iota_p(T) = \inf\{\|Q\|\,\|S_{\infty,p}\|\,\|P\|$: all left p-setups for $T\}$. The corresponding result for $p = \infty$ is stated explicitly in ([6]; Section 9). These considerations motivate us to make the following definition

101. Definition. We suppose that $1 \leqslant p \leqslant \infty$ and $T \in L(E, F)$. We shall say that T is *right-p-integral* if there exists a right p-setup for T. If T is right-p-integral we write

$$\varrho_p(T) = \inf\{\|Q\|\,\|S_{p',1}\|\,\|P\|: \text{ all right } p\text{-setups for } T\} \ (1 \leqslant p < \infty),$$
$$\varrho_\infty(T) = \inf\{\|Q\|\,\|P\|: \text{ all right } \infty\text{-setups for } T\}.$$

We write $R_p(E, F)$ for the family of all right-p-integral maps from E into F. It can be proved easily that (R_p, ϱ_p) is a normed ideal. (Cf. [6]; Section 3.)

102. Theorem. (a) *Let $1 \leqslant p \leqslant \infty$ and $\alpha = d_{p',p}$. Let F be a dual Banach space and $T \in L(E, F)$. Then*

$$T \in D_{\alpha^D}(E, F) \Leftrightarrow T \in R_p(E, F) \quad \textit{and then } \alpha^{DD}(T) = \varrho_p(T).$$

(b) *Let $1 \leqslant p \leqslant \infty$ and $\alpha = d_{p',p}$. Let F be any Banach space and $T \in L(E, F)$. Then $T \in D_{\alpha^D}(E, F) \Leftrightarrow J_F T \in R_p(E, F'')$ and then $\alpha^{DD}(T) = \varrho_p(J_F T)$.*

(c) *Let $1 \leqslant p \leqslant \infty$. If E' or F has the m.a.p. then, for all $T \in L_0(E, F)$*

$$d_{p',p}(T) = \varrho_p(J_F T).$$

Proofs. We shall prove (a). (b) and (c) then follow exactly as in the "left" cases we have already established.

We write $\beta = g_{p,p'}$. From Corollary 18, $\alpha = \beta'$. Hence $D_{\alpha^D}(E, F) = D'_{\beta^D}(E, F)$. (from Theorem 25) $= D_{\beta^{D'}}(E, F)$ (from Theorem 76 and Theorem 37) $= D_{\beta^{D'}}(E, F)$. Further, if $T \in D_{\alpha^D}(E, F)$ then $\alpha^{DD}(T) = \beta'^{DD}(T) = \beta^{DD}(T')$. Hence, from Theorem 95,

$$T \in D_{\alpha^D}(E, F) \text{ and } \alpha^{DD}(T) < \lambda \Leftrightarrow T' \in I_p(F', E') \text{ and } \iota_p(T') < \lambda.$$

If this latter condition is satisfied then there exists a left p-setup for T' such that $\{\ldots\} < \lambda$. From Lemma 99(a) or (c), there exists a right p-setup for $T'' \in L(E'', F'')$ such that $\{\ldots\} < \lambda$, i.e., $T'' \in R_p(E'', F'')$ and $\varrho_p(T'') < \lambda$. Since F is a dual Banach space, there exists $U \in L(F'', F)$ such that $\|U\| \leqslant 1$ and $UJ_F = 1_F$. Hence

$$T = UJ_F T = UT'' J_E \in R_p(E, F) \text{ and } \varrho_p(T) \leqslant \|U\|\,\varrho(T'')\,\|J_E\| < \lambda.$$

It follows that

$$\text{if } T \in D_{\alpha^D}(E, F) \text{ then } T \in R_p(E, F) \text{ and } \varrho_p(T) \leqslant \alpha^{DD}(T).$$

The reverse implication and inequality are somewhat simpler and use Lemma 99(b) or (c). We leave the details to the reader.

103. Problem. *Is it true that $T \in L(E, F)$ and $J_F T \in R_p(E, F'') \Rightarrow T \in R_p(E, F)$?* (Cf. Remark 97.)

14. CONNECTIONS WITH MAXIMAL IDEALS

104. Remark. If B is an ideal and β is a norm on B then the pair (B, β) is said to be *perfect* or *maximal* (see [7]; Section 4) if the following is true:

If $V \in L(F, E)$ and $\sup\{\beta(PVQ)$: $\dim G < \infty$, $\dim H < \infty$, $P \in L_0(E, G)$, $Q \in L_0(H, F)$, $\|P\| \leqslant 1$, $\|Q\| \leqslant 1\} = \lambda < \infty$ then $V \in B(F, E)$ and $\beta(U) \leqslant \lambda$.

We shall show in this section that the theory of maximal normed ideals is equivalent to the theory of the normed ideals $(D_\alpha, \alpha^D|D_\alpha)$ (see Lemma 42) where α is a reasonable splitting quasinorm on L_0. This discussion will be continued in Remark 107.

105. Lemma. (a) *Let γ: $L_0 \to \mathscr{R}^+$. If $T \in L_0(E, F)$ we write*

$$(14) \quad \gamma^S(T) = \inf\{\|Q\|\,\gamma(\hat{T})\,\|P\|: \dim G < \infty, \dim H < \infty, P \in L_0(E, G), \hat{T} \in L_0(G, H), Q \in L_0(H, F) \textit{ and } T = Q\hat{T}P\}.$$

Then γ^S is a splitting quasinorm on L_0.

(b) *If γ is a reasonable quasinorm on L_0 then γ^S is a reasonable splitting quasinorm on L_0.*

Proof. (a) We leave to the reader the details of the proof that γ^S is a quasinorm on L_0. Clearly, if E and F are finite dimensional and $T \in L_0(E, F)$ then $\gamma^S(T) \leqslant \gamma(T)$ from which for all $T \in L_0(E, F)$,

$$\gamma^S(T) \geqslant \inf\{\|Q\|\,\gamma^S(\hat{T})\,\|P\|: \ldots\}$$

and it follows easily from this that γ^S is splitting.

(b) Let $a \in E' \setminus \{0\}$, $y \in F \setminus \{0\}$ and $T = \langle \cdot, a\rangle y \in L_0(E, F)$. Then $\gamma^S(T) \geqslant \gamma(T) = \|a\|\,\|y\|$. On the other hand, we can write $T = Q\hat{T}P$, where $G = \mathscr{K}$, $H = \mathscr{K}y \subset F$, $P = a \in L_0(E, G)$, $Q \in L_0(H, F)$ is the inclusion map and $\hat{T} \in L_0(G, H)$ is defined by $\hat{T}\lambda = \lambda y$ $(\lambda \in G = \mathscr{K})$. Hence $\gamma^S(T) \leqslant \|Q\|\,\gamma(\hat{T})\,\|P\| = \|a\|\,\|y\|$.

106. Theorem. (a) *If α is a reasonable splitting quasinorm on L_0 then $(D_\alpha, \alpha^D|D_\alpha)$ is a maximal normed ideal.*

(b) *If (B, β) is a maximal normed ideal and β^*: $L_0 \to \mathscr{R}^+$ is as in* ([7]; Section 3) *we write $\alpha = \beta^{*S}$. Then α is a reasonable splitting quasinorm on L_0 and $(B, \beta) = (D_\alpha, \alpha|D_\alpha)$.*

Proof. (a) We suppose that $V \in L(F, E)$ and that $\sup\{\alpha^D(PVQ): \ldots\} = \lambda < \infty$. Let $\varepsilon > 0$ and $T \in L_0(E, F)$ be such that $\alpha(T) \leqslant 1$. Arguing as in Lemma 8(a) and Lemma 9(a), there exist finite dimensional G and H, $P \in L_0(E, G)$, $Q \in L_0(H, F)$, $\hat{T} \in L_0(G, H)$ such that $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$, $\alpha(\hat{T}) \leqslant 1 + \varepsilon$ and $T = Q\hat{T}P$. Then

$$|\operatorname{tr} J_F T V| = |\operatorname{tr} J_F Q\hat{T}PV| = |\operatorname{tr} J_H \hat{T}PVQ| \leqslant \alpha(\hat{T})\,\alpha^D(PVQ) \leqslant (1+\varepsilon)\lambda .$$

On taking the supremum over T and letting $\varepsilon \to 0$ it follows that $V \in D_\alpha(F, E)$ and $\alpha^D(V) \leqslant \lambda$.

(b) It follows from Lemma 105 that α is a reasonable splitting quasinorm on L_0.

We suppose first that $V \in B(F, E)$. Let $T \in L_0(E, F)$ and $\alpha(T) \leqslant 1$. For all $\varepsilon > 0$ there exist $G, H, P, \hat{T}$ and Q as in (14) such that $\|Q\|\,\beta^*(\hat{T})\,\|P\| \leqslant 1 + \varepsilon$. Then

$$|\operatorname{tr} J_F T V| = |\operatorname{tr} J_F Q\hat{T}PV| = |\operatorname{spur} \hat{T}PVQ| \leqslant \beta^{**}(V)\,\beta^*(\hat{T})\,\|P\|\,\|Q\|$$
$$\leqslant (1+\varepsilon)\beta^{**}(V) \leqslant (1+\varepsilon)\beta(V)$$

from ([7]; Lemma 1 and Satz 2). On taking the supremum over T and letting $\varepsilon \to 0$, it follows that $V \in D_\alpha(F, E)$ and $\alpha^D(V) \leqslant \beta(V)$.

We next suppose that $V \in D_\alpha(F, E)$. Let $\dim G < \infty$, $\dim H < \infty$, $P \in L_0(E, G)$, $\hat{T} \in L_0(G, H)$ and $Q \in L_0(H, F)$. From ([7]; Lemma 1 and Satz 2) again,

$$|\operatorname{spur} \hat{T}PVQ| = |\operatorname{tr} J_F Q\hat{T}PV| \leqslant \alpha(Q\hat{T}P)\,\alpha^D(V) \leqslant \|Q\|\,\alpha(\hat{T})\,\|P\|\,\alpha^D(V)$$
$$\leqslant \alpha^D(V)\,\beta^*(\hat{T})\,\|P\|\,\|Q\|$$

hence $\beta^{**}(V) \leqslant \alpha^D(V)$. Since (B, β) is perfect, $V \in B(F, E)$ and $\beta(V) \leqslant \alpha^D(V)$.

107. Remark. Let (B, β) be a maximal ideal and α be a reasonable splitting quasinorm on L_0. We shall say that (B, β) is *determined by* α if $(B, \beta) = (D_\alpha, \alpha^D|D_\alpha)$. (We observe from Remark 63 that (B, β) may be determined by many different α's.) The following results are then immediate from Theorem 25, Theorem 27, Corollary 33, Corollary 35 and Theorem 106(b): we suppose that (B, β) is determined by α.

(a) $(B, \beta)'$ is maximal and determined by α': (See [7]; Satz 11.)
(b) $(B, \beta)'' = (B, \beta)$. (See [7]; Satz 10.)
(c) If $V \in L(F, E)$ and $J_E V \in B$ then $V \in B$ and $\beta(V) = \beta(J_E V)$.
(d) If $V \in L(E', F')$ and $V' J_F \in B$ then $V' \in B$ and $\beta(V') = \beta(V' J_F)$.

Our results are more general, in that we do not assume that α be reasonable.

15. (E, E') METRIC APPROXIMATION PROPERTIES

108. Lemma. *We suppose that α is a splitting quasinorm on L_0, $V \in L(F, E')$, $T \in L_0(E', F)$, $\alpha(T) > 0$ and $\varepsilon > 0$. Then there exists $W \in L_0(E', F)$, W $w(E', E)$-continuous, such that*

$$|\operatorname{tr} J_{E'} VW - \operatorname{tr} J_{E'} VT| < \varepsilon \quad \textit{and} \quad \alpha(W) \leqslant \alpha(T).$$

Proof. We choose $\delta > 0$ such that $2\delta|\operatorname{tr} J_{E'} VT| < (1+\delta)\varepsilon$. From Lemma 8(c), Lemma 9(a) and Theorem 11, there exists a finite dimensional subspace H of E'' and $R \in L_0(F', H)$ such that $T' = IR$ and $\alpha'(R) \leqslant (1+\delta)\alpha'(T') = (1+\delta)\alpha(T)$, where $I \in L_0(H, E'')$ is the inclusion map. Since T' is $w(F', F)$-continuous, the same is true of R hence there exist $y_1, \ldots, y_m \in F$ and $z_1, \ldots, z_m \in H$ such that $R = \sum_i \langle y_i, \cdot\rangle z_i \in L_0(F', H)$. It follows from this that $T = \sum_i \langle \cdot, z_i\rangle y_i \in L_0(E', F)$. From ([10]; Corollary 7) (see also [7]; Lemma 6) there exists $U \in L_0(H, E)$ such that $\|U\| \leqslant 1$ and

$$\Big|\sum_i \langle V y_i, J_E U z_i - z_i\Big| < \varepsilon/2 .$$

We write

$$W = (1+\delta)^{-1} \sum_i \langle U z_i, \cdot\rangle y_i \in L_0(E', F).$$

W is clearly $w(E', E)$-continuous. We can show exactly as in the proof of Lemma 22 that $|J_{E'} VW - \operatorname{tr} J_{E'} VT| < \varepsilon$. Finally (cf. Corollary 12(a)), $W' = (1+\delta)^{-1} J_E UR$ hence, from Theorem 11,

$$\alpha(W) = \alpha'(W') \leqslant (1+\delta)^{-1}\alpha'(R) \leqslant \alpha(T).$$

109. Theorem. *If α is a splitting quasinorm on L_0 then*

$$\{W:\ W \in L_0(E', F),\ W \textit{ is } w(E', E)\textit{-continuous},\ \alpha(W) \leqslant 1\}$$

is $\mathcal{T}$-dense in $\{T: T \in L_0(E', F), \alpha(T) \leqslant 1\}$.

Proof. This is immediate from Lemma 82, Lemma 108 and the bipolar theorem.

110. Remark. The result corresponding to that of Theorem 109 with $\mathcal{T}$ replaced by the topology of compact convergence is also true. If α is reasonable this is immediate from Lemma 39 and ([3]; § 5, No. 2, Lemma 20, p. 178). If α is not reasonable we can proceed as in Theorem 109, except replacing Lemma 82 by the fact (see [3]; Corollaire, p. 111] (essentially)) that if φ is a linear functional on $L(F, E)$, continuous with respect to the topology of compact convergence, then there exist $a_1, a_2, \ldots \in E'$ and

$y_1, y_2, \ldots \in F$ such that $\sum_{i \geq 1} \|a_i\| \|y_i\| < \infty$ and, for all $V \in L(F, E)$, $\varphi(V) = \sum_{i \geq 1} \langle V y_i, a_i \rangle$.

111. Remark. (E, E') is said to have *metric approximation property* (see [9]; V. 4) if $I_{E'}$ is in the $\mathscr{T}$-closure of $\{W: W \in L_0(E', E'),\ W$ is $w(E', E)$-continuous, $\|W\| \leq 1\}$. It has been proved in [11] that (E, E') has the metric approximation property $\Leftrightarrow E'$ has the metric approximation property. These considerations motivate Definition 112 and Theorem 113.

112. DEFINITION. Let a be a totally splitting norm on L_0. We shall say that (E, E') has the *a-metric approximation property* (*a-m.a.p.*) if, for all F,

$$\{W: W \in L_0(E', F),\ W \text{ is } w(E', E)\text{-continuous},\ a^D(W) \leq 1\}$$

is $\mathscr{T}$-dense in $\{D_a(E', F):\ a^D \leq 1\}$.

If $a = g_{1,\infty} = d_{\infty,1}$ this concept is equivalent to that defined in Remark 111.

113. THEOREM. *(E, E') has the a-m.a.p. $\Leftrightarrow E'$ has the a-m.a.p.*

Proof. $(\Rightarrow)$ is trivial and $(\Leftarrow)$ follows from Theorem 109 with a replaced by a^D.

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A local factorization of analytic functions and its applications

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Abstract. The following notion is introduced: The family $(p_i: E \to E_i)_{i \in I}$ of linear epimorphisms is called a *basic system for E* iff the inverse images of neighbourhoods of zero in E_i form the base of neighbourhoods of zero in E and the family I is ordered in a suitable manner.

The local factorization, by some projection p_i, of analytic mappings $E \supset U \xrightarrow{f} F$, where U is a domain in E and F is a normed linear space, is proved and the following consequences are obtained:

If E and E_i are complex and every E has the Baire property, then every G-analytic mapping of a domain $U \subset E$, continuous at some point of U, is continuous on the whole set U.

The polynomial approximation property is studied in the case of locally convex complex E and E_i. Some results are also obtained in the case of real E and E_i.

INTRODUCTION

A. Hirschowitz has proved in [4] that every analytic complex-valued function on an open subset of the Cartesian product of a family of linear topological spaces can be locally factorized by the projection on a finite number of coordinates. (See also [3] for the case of C^N and [9] for the case of C^X.) An analogous fact was proved by L. Nachbin [7] for the case of a locally convex space E such that the canonical projections $E \to E_q$ are open. For the continuous seminorm q on E, E_q denotes the space $E/q^{-1}(0)$ with the norm induced by q.

The aim of this paper is to generalize this fact and to apply it to the proof of some theorems about analytic function on linear topological spaces. To obtain these results we introduce the notion of a basic system. A topological vector space endowed with a basic system generalizes both the Cartesian product of linear topological spaces and the locally convex space with its system of seminorms and it is also a special case of the projective limit in the sense of [10].

In part I of this paper we prove some fundamental facts concerning this notion and apply them to the proof of two theorems about the conti-