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On complemented subspaces and unconditional bases in l_p+l_q

by

P. WOJTASZCZYK (Warszawa)

Abstract. The isomorphic types of complemented subspaces in l_p+l_q, $1<p,q\leqslant\infty$ are described. The form of unconditional bases in l_p+l_q, $1<p\leqslant 2\leqslant q<\infty$ is established. By l_∞ we mean the space usually denoted by c_0. This paper generalises the results of Edelstein [4].

Introduction. It seems to be an important question in the Banach space theory to describe all complemented subspaces of a given Banach space X. In general very little is known in this direction. Pełczyński [13] described complemented subspaces of the space l_p. Namely he proved the following

THEOREM A (Pełczyński [13]). *Any complemented subspace of l_p, $1\leqslant p\leqslant\infty$, is isomorphic to the whole space or is finite-dimensional.*

An analogous theorem for the space m was proved by Lindenstrauss [8].

All complemented subspaces of the Cartesian product l_p+l_2 were described by Edelstein [4]. In Section 1 we prove the following.

THEOREM 1.1. *A complemented subspace of l_p+l_q, $1<p,q\leqslant\infty$ is isomorphic to l_p, l_q, l_p+l_q or is finite-dimensional.*

In Section 2 we prove the following

THEOREM 2.1. *Let (z_n) be an unconditional seminormalized basis in l_p+l_q, $1<p\leqslant 2\leqslant q<\infty$. Then one can divide the set of natural numbers into two disjoint subsets N_1 and N_2 in such a way that $\overline{sp}\{z_n\}_{n\in N_1}\sim l_p$ and $\overline{sp}\{z_n\}_{n\in N_2}\sim l_q$.*

These theorems for l_p+l_2, $1\leqslant p\leqslant\infty$ were proved by Edelstein [4].

The main ideas of this paper are taken from Edelstein [4]. In some places we repeat Edelstein's argument. The proof of Theorem 1.1 is a generalisation of the proof of Edelstein. The proof of Theorem 2.1 uses the same ideas as Edelstein's proof but is much simpler. However, it does not cover the most interesting cases, namely l_1+l_2, l_1+l_∞, l_2+l_∞.

In this paper we employ the notation commonly used in the Banach space theory. *The one exception is that the symbol l_∞ denotes the space of* *sequences tending to zero equipped with the supremum norm, usually denoted by* c_0. If we have a decomposition of the space X into the direct sum $X = X_1+X_2$, by P_{X_1} we always denote the projection onto X_1 annihilating X_2 and by P_{X_2} we always denote the projection onto X_2 annihilating X_1. The term *operator* always means "bounded linear operator" and the term *subspace* always means "closed linear submanifold". The symbol $\sim$ denotes an isomorphism (i.e. a linear homeomorphism). For any subset Z of a Banach space X by $\mathrm{sp}Z$ we denote the linear span of Z and by $\overline{\mathrm{sp}}Z$ we denote the closure of $\mathrm{sp}Z$.

All concepts and symbols not defined in this paper can be found in [3] and [16].

1. Complemented subspaces. Let us begin with some definitions.

DEFINITION 1.1 (cf. [14]). Banach spaces X and Y are *totally incomparable* if X and Y have no isomorphic infinite-dimensional subspaces.

Such spaces were considered by Gurarij [5] and Rosenthal [14]. The following lemma is in fact contained in those papers.

LEMMA 1.1 [5], [14]. *Let X_1 be a subspace of a Banach space $X+Y$ such that X_1 and Y are totally incomparable and $X_1\cap Y = \{0\}$. Then $P_X|X_1$ is an isomorphism.*

The next definition was introduced by Whitley [17] in the context of strictly singular operators.

DEFINITION 1.2 (cf. [17]). A Banach space X is said to be *superprojective* if for every subspace $Y\subset X$ with $\dim X/Y = \infty$ there exists a complemented subspace $Y_1\subset X$ such that $Y\subset Y_1$ and $\dim X/Y_1 = \infty$.

If s is an integer and X is a Banach space, then by $X^{(s)}$ we denote $X+E_s$ if $s\geqslant 0$ (E_s means an s-dimensional space) or a subspace of X of codimension $-s$ if $s\leqslant 0$ (cf. [18]).

LEMMA 1.2. *Let E be a Banach space and let us have two decompositions of E into direct sums, $E\sim X_1+Y_1$ and $E\sim X_2+Y_2$, where (X_1, Y_1) and (X_1, Y_2) are pairs of totally incomparable spaces and X_1 is superprojective. Then there exist an integer s and a subspace $Z\subset Y_1^{(s)}$ such that $X_2\sim X_1+Z$.*

Proof. Let us write $U = X_1\cap Y_2$. Since X_1 and Y_2 are totally incomparable, we have $\dim U<\infty$. Hence there is a decomposition $X_1 = U+\tilde{X}$. By Lemma 1.1 $P_{X_2}|\tilde{X}$ is an isomorphism.

Let $U_1 = (P_{X_2}|\tilde{X})^{-1}(P_{X_2}(\tilde{X})\cap Y_1)$ and $\overline{U} = \mathrm{sp}\{U_1\cup U\}$. Since X_1 and Y_1 are totally incomparable and $P_{X_2}(\tilde{X})$ is isomorphic to a subspace of X_1, we get $\dim U_1<\infty$ and so $\dim\overline{U}<\infty$. Hence we have a decomposition $X_1 = \overline{X}+\overline{U}$.

Put $T = P_{Y_1}+P_{X_2}P_{\overline{X}}$ ($P_{\overline{X}}$ is a projection from E onto $\overline{X}$ annihilating $\overline{U}$ and Y_1). Since Y_1 and $P_{X_2}(\overline{X})$ are totally incomparable and $Y_1\cap P_{X_2}(\overline{X})$ $= 0$, we infer (by [14] Theorem 1 or [5] Theorem 14) that T is an isomorphism from $\overline{X}+Y_1$ into E. Obviously $T|Y_1 = \mathrm{id}$ and $T(\overline{X})\subset X_2$.

Now we are going to show that $T(Y_1+\overline{X})$ is of finite codimension in E. Suppose it is not so. Then $T(Y_1+\overline{X})\cap\overline{X}$ is of infinite codimension in X. Since $\overline{X}$ is a subspace of finite codimension of a superprojective space X_1, $\overline{X}$ is superprojective. So we have $\overline{X} = M_1+M_2$, where M_1 and M_2 are infinite-dimensional spaces and $T(Y_1+\overline{X})\cap\overline{X}\subset M_1$.

Take $m\in M_2$. Then we have

a) $m-P_{X_1}P_{X_2}(m)\in X_1$ and $P_{Y_1}P_{X_2}(m)\in Y_1$,

b) $P_{X_1}P_{X_2}(m)\in T(Y_1+\overline{X})\cap X_1$.

This statement is true because

$$P_{X_1}P_{X_2}(m) = P_{X_2}(m)-P_{Y_1}P_{X_2}(m) \quad\text{and}$$

$$P_{X_2}(m) = P_{Y_1}(m)+P_{X_2}P_{\overline{X}}(m) = T(m)\in T(Y_1+\overline{X}) \quad\text{and}$$

$$P_{Y_1}P_{X_2}(m)\in Y_1\subset T(Y_1+\overline{X}).$$

Using a) and b) and the definition of M_2, we have

$$\|P_{Y_2}(m)\| = \|m-P_{X_2}(m)\| = \|m-P_{X_1}P_{X_2}(m)-P_{Y_1}P_{X_2}(m)\| \geqslant a\|m-P_{X_1}P_{X_2}(m)\|\geqslant a\beta\|m\|,$$

where a and β are positive constants not depending on m. So $P_{Y_2}|M_2$ is an isomorphism. But M_2 is an infinite-dimensional subspace of X_1 and this contradicts the total incomparability of X_1 and Y_2. This contradiction proves that $T(Y_1+\overline{X})$ is of finite codimension in E.

Thus we have $T(\overline{X})+Y_1^{(k)} = E$, $T(\overline{X})\subset X_2$, $T(\overline{X})\sim X_1^{(r)}$ so $X_2\sim X_1^{(r)}+Z$ and $Z^{(r)}$ is isomorphic to a subspace of $Y_1^{(k+r)}$. This completes the proof of the Lemma.

Remark 1.1. It is unknown whether there exists a Banach space X such that $X^{(r)}$ is not isomorphic to X for some integer r. For subspaces of l_p, $1\leqslant p\leqslant\infty$ we have $X^{(r)}\sim X$ for $r = \pm 1, \pm 2, \pm 3, \ldots$ In this case the statement of the Lemma can be simplified in an obvious way.

Now we are going to state some known results which we will use later. The next Lemma is a special case of Theorem 1 of Zaharyuta [18].

LEMMA 1.3. *Let X_1, X_2, Y_1, Y_2 be Banach spaces such that any operator from X_1 into Y_2 and from X_2 into Y_1 is compact. Then $X_1+Y_1\sim X_2+Y_2$ iff there exists an integer s such that $X_1\sim X_2^{(s)}$ and $Y_1\sim Y_2^{(-s)}$.*

LEMMA 1.4. *Let X be a subspace of l_p and Y be a subspace of l_q where $p\neq q$, $1\leqslant p, q\leqslant\infty$. Then X and Y are totally incomparable Banach spaces.*

This Lemma goes back to Banach [1] (cf. also [13]).

LEMMA 1.5. *The spaces l_p for $1<p\leqslant\infty$ are superprojective.*

For $1 < p < \infty$ this Lemma was proved by Whitley [17] Corollary 4.8 and for $p = \infty$ by Lindenstrauss and Rosenthal [9] Lemma 1.c.

LEMMA 1.6. *Let $p < q$. Then any operator from l_q into l_p is compact.*

This Lemma was in fact proved by Banach [1]. For more general results see Rosenthal [15], Theorem A.2.

LEMMA 1.7. *Let $X \subset l_p + l_q$ be an infinite-dimensional, complemented subspace isomorphic to a subspace of l_q where $p, q > 1$. Then $X \sim l_q$.*

Proof. By our assumption there exists a subspace $V \subset l_p + l_q$ such that $l_p + l_q = X + V$. By Lemmas 1.4, 1.5, and 1.2, $V \sim l_p + \overline{V}$ where $\overline{V}$ is isomorphic to a subspace of l_q. So $l_p + l_q \sim l_p + \overline{V} + X$ and by Lemma 1.3 and Remark 1.1 we have $\overline{V} + X \sim l_q$. Then by Theorem A we obtain $X \sim l_q$.

LEMMA 1.8. *Let X be a complemented subspace of $E = l_p + l_q$ such that there exists a complemented subspace $Z \subset X$ and $Z \sim E$. Then $X \sim E$.*

Proof. In view of Theorem A we can assume without loss of generality that $p < q$. Let Q_1 be a projection onto X and put $Q_2 = I - Q_1$ and $Q_2(E) = Y$.

By $\hat{E}$ we denote the Banach space of all sequences $(x_n)_{n=1}^{\infty} \subset E$ such that

$$\|(x_n)\| = \Big(\sum_{n=1}^{\infty} \|P_{l_p}(x_n)\|^p\Big)^{\frac{1}{p}} + \Big(\sum_{n=1}^{\infty} \|P_{l_q}(x_n)\|^q\Big)^{\frac{1}{q}}$$

is finite. Clearly $\hat{E} \sim E$. For a sequence of subspaces $Y_n \subset E$ we put

$$\sum_{n=1}^{\infty} Y_n = \{(y_n) \colon y_n \in Y_n \text{ and } (y_n) \in \hat{E}\}.$$

It is a closed subspace of $\hat{E}$. If $Y_n = Y$ for $n = 1, 2, 3, \ldots$ we will denote $\sum_{n=1}^{\infty} Y_n$ by $\sum Y$.

We denote the canonical unit vector basis in $l_q \subset E$ by $(0, f_i)_{i=1}^{\infty}$ and put $L_k = \operatorname{sp}\{(0, f_i)\}_{i=1}^{k}$ and $\tilde{E}_k = l_p + \overline{\operatorname{sp}}\{(0, f_i)\}_{i>k}$. By Lemma 1.6 we have

$$\lim_k \|P_{l_p} Q_r | \overline{\operatorname{sp}}\{(0, f_i)\}_{i=k}^{\infty}\| = 0 \quad \text{for } r = 1, 2.$$

So we can choose a sequence of natural numbers (n_i) such that

$$\|P_{l_p} Q_r | \overline{\operatorname{sp}}\{(0, f_j)\}_{j>n_i}\| \leqslant \frac{1}{i^2} \quad \text{for } r = 1, 2.$$

Obviously we have $E = \tilde{E}_{n_i} + L_{n_i}$ and $\hat{E} = \sum_{i=1}^{\infty} \tilde{E}_{n_i} + \sum_{i=1}^{\infty} L_{n_i}$. Let us denote $\sum_{i=1}^{\infty} \tilde{E}_{n_i}$ by $\tilde{E}$ and $\sum_{i=1}^{\infty} L_{n_i}$ by L. Obviously we have $\tilde{E} \sim E$ and $L \sim l_q$.

We claim that $\sum X + \sum Y \supset \tilde{E}$. This inclusion means the natural isomorphic embedding.

Let us take $(z_n) \in \tilde{E}$. We want to show that $(Q_1(z_n)) \in \sum X$ and $(Q_2(z_n)) \in \sum Y$.

a) Suppose that $z_n = P_{l_p}(z_n)$ for $n = 1, 2, 3, \ldots$ Then $\sum_{n=1}^{\infty} \|z_n\|^p < \infty$, so $\sum_{n=1}^{\infty} \|P_{l_p} Q_r(z_n)\|^p < \infty$ for $r = 1, 2$ and moreover

$$\Big(\sum_{n=1}^{\infty} \|P_{l_q} Q_r(z_n)\|^q\Big)^{\frac{1}{q}} \leqslant \|P_{l_q} Q_r\| \Big(\sum_{n=1}^{\infty} \|z_n\|^q\Big)^{\frac{1}{q}} \leqslant \|P_{l_q} Q_r\| \Big(\sum_{n=1}^{\infty} \|z_n\|^p\Big)^{\frac{1}{p}}$$

for $r = 1, 2$.

b) Suppose $z_n = P_{l_q}(z_n)$ for $n = 1, 2, 3, \ldots$ Then $\sum_{n=1}^{\infty} \|z_n\|^q < \infty$ so

$$\Big(\sum_{n=1}^{\infty} \|P_{l_q} Q_r(z_n)\|^q\Big)^{\frac{1}{q}} \leqslant \|P_{l_q} Q_r\| \Big(\sum_{n=1}^{\infty} \|z_n\|^q\Big)^{\frac{1}{q}} < \infty \quad \text{for } r = 1, 2 \text{ and}$$

$$\Big(\sum_{n=1}^{\infty} \|P_{l_p} Q_r(z_n)\|^p\Big)^{\frac{1}{p}} = \Big[\sum_{n=1}^{\infty} \Big(\|z_n\| \Big\| P_{l_p} Q_r\Big(\frac{z_n}{\|z_n\|}\Big)\Big\|\Big)^p\Big]^{\frac{1}{p}} \leqslant \Big(\sum_{n=1}^{\infty} \|z_n\|^p \frac{1}{n^{2p}}\Big)^{\frac{1}{p}} < \infty$$

because $z_n \in \overline{\operatorname{sp}}\{(0, f_j)\}_{j>n_n}$.

Since for $(z_n) \in \tilde{E}$, we have $(z_n) = (P_{l_p}(z_n)) + (P_{l_q}(z_n))$, we obtain our claim.

Hence $\hat{E} \supset \sum X + \sum Y \supset \tilde{E}$ and $\hat{E} = \tilde{E} + L$. This implies that there exists a $\bar{Z} \subset L$ such that $\sum X + \sum Y = \tilde{E} + \bar{Z}$. From our assumptions it follows that there exists a Z_1 such that $X = Z + Z_1$. Thus we have

$$\begin{aligned} \bar{Z} + E &\sim X + Y + \bar{Z} \sim (Z + Z_1) + Y + \bar{Z} \sim E + Z_1 + Y + \bar{Z} \\ &\sim E + Z_1 + Y + \bar{Z} \sim \tilde{E} + Z_1 + Y + L + \bar{Z} \\ &\sim \textstyle\sum X + \sum Y + Z_1 + Y + L \sim \sum X + \sum Y + Z_1 + L \\ &\sim \tilde{E} + \bar{Z} + Z_1 + L \sim E + Z_1 + \bar{Z} \sim Z + Z_1 + \bar{Z} \sim X + \bar{Z}. \end{aligned}$$

Hence $l_p + (l_q + \bar{Z}) \sim X + \bar{Z}$. Since $\bar{Z} \subset l_q$, the product $l_q + \bar{Z}$ is isomorphic to a subspace of l_q. Thus by Lemma 1.2, in view of Lemmas 1.4 and 1.5, we obtain $X \sim U + l_p$ where U is isomorphic to a subspace of l_q. Thus U is isomorphic to a complemented subspace of $l_p + l_q$ and by Lemma 1.7, we have $U \sim l_q$. Hence $X \sim l_p + l_q$. This completes the proof of the Lemma.

Remark 1.2. As was pointed out by Edelstein [4], it is not true that if $E = X_1 + X_2$ then $\hat{E} = \sum X_1 + \sum X_2$.

Lemma 1.9. *Let X be a complemented subspace of $l_p + l_q$, $1 < p, q \leqslant \infty$. Then there are only three possibilities:*

a) *X is isomorphic to a subspace of l_p.*

b) *X is isomorphic to a subspace of l_q.*

c) *X contains a complemented subspace isomorphic to $l_p + l_q$.*

Proof. Suppose there exists a subspace $Y \subset X$, $\dim X/Y < \infty$ such that $P_{l_p}|Y$ is an isomorphism. Then obviously X is isomorphic to a subspace of l_p (cf. Remark 1.1). If such a subspace exists for the projection P_{l_q}, X is isomorphic to a subspace of l_q.

Suppose now that $P_{l_p}|Y$ and $P_{l_q}|Y$ are not isomorphisms for any subspaces $Y \subset X$, $\dim X/Y < \infty$. By $(0, e_i)$ and $(f_i, 0)$ we will denote the canonical unit vector bases in l_q and l_p, respectively. The standard "gliding hump" procedure (cf. [2]) gives us a sequence of natural numbers (n_k) and sequences of vectors (x_k) and (y_k) such that

$$\Big\|x_k - \sum_{i=n_k+1}^{n_{k+1}} a_i(0, e_i)\Big\| \leqslant \frac{1}{2^{2+k}} \quad \text{and} \quad \Big\|y_k - \sum_{i=n_k+1}^{n_{k+1}} \beta_i(f_i, 0)\Big\| \leqslant \frac{1}{2^{2+k}}$$

for some scalars (a_i) and (β_i). Since $\overline{\operatorname{sp}}\{\sum_{i=n_k+1}^{n_{k+1}} a_i(0, e_i), \sum_{i=n_k+1}^{n_{k+1}} \beta_i(f_i, 0)\}_{k=1}^{\infty}$ is linearly homeomorphic to $l_p + l_q$ and is a range of projection of norm one in $l_p + l_q$, by Theorem 2 of [2] we infer that $\overline{\operatorname{sp}}\{x_k, y_k\}_{k=1}^{\infty}$ is a complemented subspace of X isomorphic to $l_p + l_q$.

Theorem 1.1. *A complemented subspace of $l_p + l_q$, $1 < p, q \leqslant \infty$ is isomorphic to l_p, l_q, $l_p + l_q$ or is finite-dimensional.*

Proof. If X is an infinite dimensional, complemented subspace of the space $l_p + l_q$, then by Lemma 1.9. there are three possibilities. In the cases a) and b) of Lemma 1.9. the conclusion of Lemma 1.7. gives that X is isomorphic to l_p or X is isomorphic to l_q. In the case c) of Lemma 1.9. we use Lemma 1.8. to obtain that X is isomorphic to $l_p + l_q$. Thus the Theorem is proved.

Remark 1.4. By Whitley [17], Corollary 4.8, spaces $L_p(0, 1)$ are superprojective for $1 < p \leqslant 2$. In [15], Theorem A.2, Rosenthal stated when all operators from $L_p(\mu)$ into $L_p(\gamma)$ are compact. Using the results of Banach and Mazur [1], Paley [12] and Kadec [7] on linear dimension of spaces l_p and L_q and the results of Pełczyński [13], one easily finds when spaces l_p and $L_q(0, 1)$ are totally incomparable. Those facts and our Lemmas 1.2 and 1.3 yield the following results:

Proposition 1.1. *Let X be a complemented subspace of $L_q(0, 1) + l_p$ for $1 < q < p < 2$ and let X be isomorphic to a subspace of l_p. Then X is isomorphic to l_p.*

Proposition 1.2. *Let X be a complemented subspace of $L_q(0, 1) + l_p$ and let X be isomorphic to a subspace of $L_q(0, 1)$ where $1 \leqslant p < 2$ and $p < q$. Then X is isomorphic to a complemented subspace of $L_q(0, 1)$.*

2. Unconditional bases. In this section we are dealing with seminormalized, unconditional bases in $l_p + l_q$ where $1 < p \leqslant 2 \leqslant q < \infty$.

Lemma 2.1. *There exists a finite constant B_p, $1 \leqslant p < \infty$, such that for any finite set $\{x_j\}_{j=1}^n \subset l_p$ we have*

$$\min_{|\varepsilon_j|=1} \Big\|\sum_{j=1}^n \varepsilon_i x_j\Big\| \leqslant B_p \Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac{1}{2}} \quad \text{for } p > 2,$$

$$\min_{|\varepsilon_j|=1} \Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\| \leqslant B_p \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{\frac{1}{p}} \quad \text{for } 1 \leqslant p \leqslant 2.$$

The proof of this Lemma is a non-essential modification of the proof of Theorem 1 in Kadec [6]. It was observed by Edelstein [4], Proposition 2. An independent proof of the second inequality was given by D. Sarason and H. P. Rosenthal (cf. [15] Lemma A.3).

Remark 2.1. This Lemma is obviously not true for $p = \infty$.

The next Lemma is a well known theorem of Orlicz.

Lemma 2.2 (cf. [11]). *There exists a positive constant γ such that for any finite set $\{x_j\}_{j=1}^n \subset l_p$, $1 \leqslant p \leqslant 2$, we have*

$$\max_{|\varepsilon_j|=1} \Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\| \geqslant \gamma \Big(\sum_{j=1}^n \|x_j\|^2\Big)^{\frac{1}{2}}.$$

Lemma 2.3. *Let $z_n = (x_n, y_n)$ be a seminormalized, unconditional basis in $l_p + l_q$, $1 \leqslant p \leqslant 2 \leqslant q < \infty$. Then 0 is a limit point of the set $\{\|x_n\|\}_{n=1}^{\infty}$.*

Proof. Suppose that there exists a constant $\mu > 0$ such that $\|x_n\| \geqslant \mu$ for $n = 1, 2, 3, \ldots$ By the standard "gliding hump" and stability arguments (cf. [2]) we can assume that there exist a subbasis (e_k) of the unit vector basis in l_q and a sequence (n_k) of natural numbers such that

$$(0, e_k) = \sum_{i=n_k+1}^{n_{k+1}} a_i(x_i, y_i).$$

Since (z_n) is an unconditional basis, for any sequence of numbers $(\varepsilon_i)_{i=1}^{\infty}$, $|\varepsilon_i| = 1$ the sequence $\sum_{i=n_k+1}^{n_{k+1}} \varepsilon_i a_i(x_i, y_i)$ is a basic sequence equivalent to (e_k). Thus $P_{l_p}|\overline{\mathrm{sp}}\{\sum_{i=n_k+1}^{n_{k+1}} \varepsilon_i a_i z_i\}_{k=1}^{\infty}$ is a compact operator (by Lemma 1.6), and so $\lim_k \sup_{|\varepsilon_i|=1} \|\sum_{i=n_k+1}^{n_{k+1}} \varepsilon_i a_i x_i\| = 0$. But by Lemma 2.2,

$$\sup_{|\varepsilon_i|=1} \Big\|\sum_{i=n_k+1}^{n_{k+1}} \varepsilon_i a_i x_i\Big\| \geqslant \gamma\Big(\sum_{i=n_k+1}^{n_{k+1}} \|a_i x_i\|^2\Big)^{\frac{1}{2}} \geqslant \gamma\mu\Big(\sum_{i=n_k+1}^{n_{k+1}} |a_i|^2\Big)^{\frac{1}{2}},$$

and we conclude that $\lim_k \big(\sum_{i=n_k+1}^{n_{k+1}} |a_i|^2\big)^{\frac{1}{2}} = 0$.

On the other hand, by Lemma 2.1

$$\min_{|\varepsilon_i|=1} \Big\|\sum_{i=n_k+1}^{n_{k+1}} \varepsilon_i a_i y_i\Big\| \leqslant B_q\Big(\sum_{i=n_k+1}^{n_{k+1}} \|a_i y_i\|^2\Big)^{\frac{1}{2}} \leqslant CB_q\Big(\sum_{i=n_k+1}^{n_{k+1}} |a_i|^2\Big)^{\frac{1}{2}},$$

where C is a positive constant. So we can find a sequence $(\varepsilon_i)_{i=1}^{\infty}$ $|\varepsilon_i| = 1$ such that the sequence $\sum_{i=n_k+1}^{n_{k+1}} \varepsilon_i a_i z_i$ converges to zero. This contradicts the fact that this sequence is equivalent to the basic sequence (e_k). This contradiction proves the Lemma.

LEMMA 2.4. *Let $z_n = (x_n, y_n)$ be a seminormalized, unconditional basis in $l_p + l_q$, $1 < p \leqslant 2 \leqslant q < \infty$. Suppose that we have a subbasis (z_{n_k}) such that $x_{n_k} \to 0$. Then $\overline{\mathrm{sp}}\{z_{n_k}\}_{k=1}^{\infty} \sim l_q$.*

Proof. Let us consider the sequence $z_n^* = (x_n^*, y_n^*)$ of biorthogonal functionals. The sequence (z_n^*) is a seminormalized, unconditional basis in $l_{p^*} + l_{q^*}$, $1 < q^* \leqslant 2 \leqslant p^* < \infty$ $(1/p^* + 1/p = 1, 1/q^* + 1/q = 1)$. Since $x_{n_k} \to 0$, we may assume that $\inf_k \|y_{n_k}\| > 0$. Since

$$1 = |z_{n_k}^*(z_{n_k})| \leqslant |x_{n_k}^*(x_{n_k})| + |y_{n_k}^*(y_{n_k})| \quad \text{for } k = 1, 2, 3, \ldots \text{ and}$$

$\sup_k \|x_{n_k}^*\| < \infty$ and $\lim_k \|x_{n_k}\| = 0$, we infer that there exists a $\delta > 0$ such that $\|y_{n_k}^*\| > \delta$ for $k = 1, 2, 3, \ldots$ By Lemma 2.3, $\overline{\mathrm{sp}}\{z_{n_k}^*\}$ is not isomorphic to $l_{p^*} + l_{q^*}$, and so by Theorem 1.1 we have either $\overline{\mathrm{sp}}\{z_{n_k}^*\} \sim l_{p^*}$ or $\overline{\mathrm{sp}}\{z_{n_k}^*\} \sim l_{q^*}$. Suppose that $\overline{\mathrm{sp}}\{z_{n_k}^*\} \sim l_{p^*}$. In this case the operator $P_{l_{q^*}}|\overline{\mathrm{sp}}\{z_{n_k}^*\}$ is compact and $P_{l_{q^*}}(z_{n_k}^*) = y_{n_k}^*$, a contradiction. Thus we have obtained $\overline{\mathrm{sp}}\{z_{n_k}^*\} \sim l_{q^*}$ and $\overline{\mathrm{sp}}\{z_{n_k}\} \sim l_q$. This completes the proof.

LEMMA 2.5. *Let $z_n = (x_n, y_n)$ be a seminormalized, unconditional basis in $l_p + l_q$, $1 < p \leqslant 2 \leqslant q < \infty$ and let (z_{n_k}) be a subbasis such that $\inf_k \|x_{n_k}\| > 0$. Then $\overline{\mathrm{sp}}\{z_{n_k}\} \sim l_p$.*

Proof. Immediate from Theorem 1.1 and Lemmas 2.3 and 2.4.

Let $d(X, Y)$, X and Y being isomorphic Banach spaces, denote the greatest lower bound of numbers $\|T\|\|T^{-1}\|$, where T is an invertible operator from X onto Y. Let l_p^n denote the n-dimensional space equipped with the norm $\|(a_i)_{i=1}^n\| = \big(\sum_{i=1}^{n} |a_i|^p\big)^{\frac{1}{p}}$.

We will need one Lemma about spaces l_p^n.

LEMMA 2.6. *Let $1 < p < 2 \leqslant q < \infty$ and*

$$a(n) = \inf\{\|T\|\|T^{-1}\|,$$

where T is an isomorphic embedding from l_p^n into $l_q\}$.

Then $a(n) \xrightarrow[n\to\infty]{} \infty$.

Proof. Obviously $a(n) \leqslant a(n+1)$ for $n = 1, 2, 3, \ldots$ Suppose $a(n)$ is bounded. Then by [10] Corollary 2 of Proposition 7.1 and an easy approximation argument we find that $L_q(0, 1)$ contains a subspace isomorphic to l_p. But this contradicts the classical result of Paley [12].

LEMMA 2.7. *Let $z_n = (x_n, y_n)$ be a seminormalized, unconditional basis in $l_p + l_q$, $1 < p < 2 \leqslant q < \infty$. Then zero is an isolated limit point of the set $\{\|x_n\|\}_{n=1}^{\infty}$.*

Proof. Suppose that zero is not an isolated limit point of this set. Then there exist a decreasing sequence (a_i) tending to zero and a sequence N_i of infinite, pairwise disjoint subsets of natural numbers such that $a_{i+1} \leqslant \|x_n\| < a_i$ for $n \in N_i$. By Lemma 2.5 $\overline{\mathrm{sp}}\{z_n\}_{n\in N_i} \sim l_p$ and let $m_i = d(l_p, \overline{\mathrm{sp}}\{z_n\}_{n\in N_i})$. Let us choose a finite subset $\overline{N}_i \subset N_i$ such that $\mathrm{sp}\{z_n\}_{n\in\overline{N}_i}$ contains a subspace Y_i, contains a subspace Y_i, $\dim Y_i = k_i$ where $a(k_i) \geqslant i m_i$, $d(Y_i, l_p^{k_i}) \leqslant 2m_i$. Put $X = \overline{\mathrm{sp}}\{z_n\}_{n\in\bigcup_{i=1}^{\infty}\overline{N}_i}$. By Lemma 2.4 $X \sim l_q$. Obviously $Y_i \subset X$ for $i = 1, 2, 3, \ldots$ So for $i = 1, 2, 3, \ldots$ we have an embedding T_i: $l_p^{k_i} \to l_q$ such that $\|T_i\|\|T_i^{-1}\| \leqslant 2d(l_q, X)m_i$. But obviously $\|T_i\|\|T_i^{-1}\| \geqslant a(k_i) \geqslant i m_i$ and we obtain $2d(l_q, X) \geqslant i$ for $i = 1, 2, 3, \ldots$ This contradiction completes the proof of the Lemma.

THEOREM 2.1. *Let (z_n) be a seminormalized, unconditional basis in $l_p + l_q$, $1 < p \leqslant 2 \leqslant q < \infty$. Then one can divide a set of natural numbers into two disjoint sets N_1 and N_2 in such a way that $\overline{\mathrm{sp}}\{z_n\}_{n\in N_1} \sim l_p$ and $\overline{\mathrm{sp}}\{z_n\}_{n\in N_2} \sim l_q$.*

Proof. When $p = q = 2$, the statement is trivial. When $p \neq q$, by a simple duality argument we can assume that $1 < p < 2 \leqslant q < \infty$. Then by Lemma 2.7 we can divide the set of natural numbers into two disjoint subsets N_1 and N_2 in such a way that $\inf_{n\in N_1} \|x_n\| > 0$ and $\lim_{n\in N_2} \|x_n\| = 0$, where $x_n = P_{l_p} z_n$. To see that sets N_1 and N_2 have the desired property one has to apply Lemmas 2.4 and 2.5.

Added in proof. Theorem 1.1 can be generalized to the following form:

THEOREM 1.1a. *Let X be a complemented subspace of $\sum_{i=1}^{n} l_{p_i}$. Then X is finite dimensional or X is isomorphic to $\sum_{k=1}^{r} l_{p_{i_k}}$ for some subset $\{p_{i_k}\}_{k=1}^{r}$ of the set $\{p_i\}_{i=1}^{n}$.*

The details will appear elsewhere.

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INSTITUTE OF MATHEMATICS,
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Splitting quasinorms and metric approximation properties

by

S. SIMONS and T. J. LEIH* (Santa Barbara, Calif.)

Abstract. In this paper we consider quasinorms on the class of operators of finite rank between Banach spaces, the dual quasinorms that they define, and their connections with the Persson–Pietsch duality theory, maximal ideals and the metric approximation property.

INTRODUCTION

L stands for the class of all bounded operators between Banach spaces and L_0 stands for the subclass of L consisting of all operators of finite rank. In what follows, α is a quasinorm on L_0 (see Definition 3).

In Section 2 we consider three factorization conditions that can be imposed on α, namely that α be *left splitting*, *right splitting* or *splitting* (see Definition 7). (The second and third of these conditions were suggested by some comments of A. Pietsch. In particular, "splitting" was suggested by Pietsch's "upper semicontinuity".) We prove in Lemma 8 (c) that if α is left splitting then α' (see Notation 4) is right splitting and in Theorem 13 that if α is splitting then α' is splitting. We do not know whether if α is right splitting then α' is left splitting (see Problem 10).

In Section 3 we consider a general process by which splitting quasinorms on L_0 can be defined. In particular, we discuss the g_p and d_p norms of Saphar (see Remark 19).

Sections 4 and 5 are devoted to some technical results.

In Sections 6 and 7 we define a function $\alpha^D\colon L \to \overline{\mathscr{R}^+}$ and investigate some of its properties. In Section 8 we investigate the class D_α of operators for which $\alpha^D < \infty$. If α is *reasonable* (see Definition 38) then $(D_\alpha, \alpha^D|D_\alpha)$ is a normed ideal (see Lemma 42) even if α fails to be a norm on L_0. However, if α is a splitting norm on L_0 then we can prove a duality result (Theorem 44) which seems to be at the base of the Persson–Pietsch duality

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