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Semigroup algebras having regular multiplication

by

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Abstract. We obtain both algebraic and topological characterizations of those locally compact semi-topological semigroups whose measure algebras have regular multiplication (in the sense of Arens). The condition obtained also characterizes regularity of multiplication in the l_1-algebra of the semigroup.

A Banach algebra is said to have *regular multiplication* if the two Arens products on its bidual coincide. It is well known from the work of Day [2] that l_1-algebras of suitable semigroups afford examples of irregular multiplication; in fact it suffices to take any semigroup admitting two distinct invariant means. This is not the only way the multiplication can be irregular: Civin and Yood [1] find wide classes of groups whose algebras comprise further examples, and in [9] it is shown that the multiplication is irregular in the L^1-algebra of any infinite locally compact Hausdorff group. However, the Banach algebra $M(S)$ of finite regular Borel measures on an infinite locally compact Hausdorff semigroup S *can* have regular multiplication, as is shown by trivial examples: take $xy = y$ for all x, y, or take the multiplication in S to be constant. In the present note an algebraic characterization is given of those locally compact semigroups whose measure algebras have regular multiplication. Some related results can be found in a paper of N. Macri [5].

We are concerned throughout with *semi-topological* semigroups—that is, semigroups in which multiplication is assumed only to be separately continuous. Basic facts about measure algebras on semigroups can be found in [3]. We begin by giving the purely topological content of the characterization. βX denotes the Stone–Čech compactification of the completely regular space X.

1. An extension theorem. If X and Y are completely regular spaces and Z is compact Hausdorff, in order that a separately continuous mapping $f\colon X\times Y\to Z$ should admit a separately continuous extension $f'\colon \beta X\times\beta Y\to Z$ it is necessary and sufficient that, for all pairs $(x_n), (y_m)$ of sequences in X, Y respectively, the double sequence $(f(x_n, y_m))$ should have a double cluster point in Z ([7], Theorem 1). Here $w\in Z$ is called a double cluster point of the double sequence (z_{nm}) if every neighbourhood of w meets infinitely many rows and columns of the double sequence, each in an infinite number of terms. This concept was introduced by Grothendieck [4].

THEOREM 1. *Let X, Y and Z be completely regular spaces and let $f\colon X\times Y\to Z$ be separately continuous. f admits a separately continuous extension $f'\colon \beta X\times\beta Y\to\beta Z$ if and only if there is no pair of sequences $(x_n), (y_m)$ in X, Y respectively such that the sets*

$$A = \{f(x_n, y_m)\colon m > n\},\quad B = \{f(x_n, y_m)\colon m < n\}$$

are contained in disjoint zero-sets in Z.

Proof. To prove necessity consider any pair $(x_n), (y_m)$ of sequences in X, Y and suppose ξ, η are cluster points of these sequences in $\beta X, \beta Y$. It is then easily seen that $f'(\xi, \eta)$ is a double cluster point in βZ of the double sequence $(f(x_n, y_m))$, and hence is a cluster point in βZ of the two sets A and B above. However, if A, B are contained in disjoint zero sets they have disjoint closures in βZ.

Conversely, suppose that f admits no separately continuous extension. By the result cited there exist sequences $(x_n), (y_m)$ in X, Y such that $(f(x_n, y_m))$ has no double cluster point in βZ—that is, every $\xi\in\beta Z$ has a cozero-set neighbourhood U which *either* meets only finitely many rows infinitely often (say type 1) *or* meets only finitely many columns infinitely often (type 2). By compactness we can write βZ as a finite union $U_1\cup\ldots\cup U_N$ of such sets. Let P, Q be the unions of the U_i's of types 1, 2 respectively. P, Q are cozero-sets and are themselves of types 1, 2 respectively. By discarding finitely many x_n's, y_m's if necessary, we can arrange that P meets *no* row of the double sequence $(f(x_n, y_m))$ infinitely often and Q meets no column infinitely often. Set $H = \beta Z - P$, $K = \beta Z - Q$. Then H, K are disjoint zero-sets in βZ containing almost all terms of every row and column respectively of $(f(x_n, y_m))$. We now select subsequences of $(x_n), (y_m)$ inductively. Set $u_1 = x_1$ and $v_1 = y_1$. Let u_{k+1} be the first x_n such that $f(x_n, v_j)\in K$ for $1\leqslant j\leqslant k$; this is possible since $\{n\colon f(x_n, v_j)\notin K\}$ is finite for each j. Likewise we may choose v_{k+1} to be the first y_m such that $f(u_j, y_m)\in H$ for $1\leqslant j\leqslant k$. Then

$$f(u_n, v_m)\in\begin{cases} H & \text{if } m > n,\\ K & \text{if } m < n,\end{cases}$$

H and K being disjoint zero-sets. It is easily seen that if $X = Y = Z$ and f is an associative operation then its separately continuous extension f' to βX is also associative (when it exists). We have therefore:

COROLLARY. *If S is a completely regular semi-topological semigroup then βS admits the structure of a semi-topological semigroup, with S as a sub-semigroup, if and only if there is no pair of sequences $(x_n), (y_m)$ in S such that the sets $\{x_n y_m\colon m > n\}$ and $\{x_n y_m\colon m < n\}$ are contained in disjoint zero-sets in S.*

We note in passing that, on setting $Z = X\times Y$ in the theorem, we obtain a necessary and sufficient condition for $\beta X\times\beta Y$ to have a natural separately continuous embedding in $\beta(X\times Y)$.

2. Biduals of semigroup algebras. We now examine the question of how non-trivial the algebraic structure of a semigroup can become before its measure algebra loses its regularity of multiplication. S_d denotes S in its discrete topology.

THEOREM 2. *The following are equivalent for any locally compact Hausdorff semi-topological semigroup S:*

(a) *$M(S)$ has regular multiplication;*

(b) *$l_1(S)$ has regular multiplication;*

(c) *βS_d admits the structure of a semi-topological semigroup with S as a sub-semigroup;*

(d) *there is no pair of sequences $(x_n), (y_m)$ in S such that the sets $\{x_n y_m\colon m > n\}$ and $\{x_n y_m\colon m < n\}$ are disjoint;*

(e) *every bounded scalar-valued function on S is weakly almost periodic.*

Moreover, if these conditions are satisfied, the bidual of $l_1(S)$ is the measure algebra of the compact semigroup βS_d.

In the above statement $l_1(S)$ denotes $M(S_d)$, the scalar field can be taken to be either the real or the complex field throughout, and a bounded function on S is said to be *weakly almost periodic* if the set of all its translates is relatively weakly compact in the Banach space of all bounded scalar functions on S with supremum norm. The equivalences (b)$\Longleftrightarrow$(e) and (b)$\Longleftrightarrow$(c) can easily be deduced from the results of Pym [6].

Proof. The equivalence of (c) and (d) is the Corollary above. We prove (a)$\Rightarrow$(d). A bounded scalar valued function defined on the product of sets X, Y is said to *cluster* if, for all pairs of sequences (x_n) in X, (y_m) in Y, the two repeated limits of the double sequence $(f(x_n, y_m))$ are equal whenever they both exist (or equivalently, the double sequence $(f(x_n, y_m))$ has a double cluster point in the scalar field). A Banach algebra $\mathscr{A}$ has regular multiplication if and only if, for every continuous linear functional F on $\mathscr{A}$, the function $F(xy)$ clusters on $\mathscr{A}_1\times\mathscr{A}_1$, where $\mathscr{A}_1$ is the unit ball of $\mathscr{A}$ (see [6], Th. 4.2 or [8], Th. 9).

Suppose (d) fails, so that there are sequences $(x_n), (y_m)$ in S for which $\{x_n y_m\colon m > n\}\cap\{x_n y_m\colon m < n\} = \emptyset$. Let μ_n, ν_m be unit masses at x_n, y_m respectively and define h on S to be the characteristic function of $\{x_n y_m\colon m > n\}$. Then h is a bounded Borel function on S and so determines a continuous linear functional H on $M(S)$ by $H(\mu) = \int h\,d\mu$. We have $H(\mu_n * \nu_m) = h(x_n y_m)$, which is one or zero according as $m > n$ or $m < n$, so that H does not cluster on the unit ball of $M(S)$. Thus (a) implies (d).

If (e) is false then, by Grothendieck's criterion [4], there are sequences (x_n), (y_m) in S and a bounded function h on S such that

$$\lim_n \lim_m h(x_n y_m) = a,$$

$$\lim_m \lim_n h(x_n y_m) = b$$

with (say) $a < b$. Choose $a < c < d < b$, and set $A = \{s: h(s) < c\}$, $B = \{s: h(s) > d\}$. Then A and B are disjoint sets, and on discarding finitely many of the x_n and y_m if necessary, we have that, for fixed n, $x_n y_m \in A$ for almost all m, and, for fixed m, $x_n y_m \in B$ for almost all n. Proceeding by induction as in the proof of Theorem 1 we may extract subsequences such that $x_n y_m \in A$ for $m > n$ and $x_n y_m \in B$ for $m < n$. Thus (d) implies (e).

If (a) is false then by the criterion cited above there exist sequences (μ_n), (ν_m) in the unit ball of $M(S)$ and a continuous linear functional H on $M(S)$ such that the two repeated limits of $(H(\mu_n * \nu_m))$ exist and differ. Pick $\sigma \in M(S)$ such that the subalgebra of $M(S)$ generated by $\{\mu_n, \nu_n: n \geqslant 1\}$ is contained in $L^1(\sigma) \subseteq M(S)$. The restriction of H to $L^1(\sigma)$ agrees with integration of some bounded Borel function h on S, so that

$$H(\mu * \nu) = \int_S \int_S h(xy)\,\mu(dx)\,\nu(dy) \tag{1}$$

for $\mu, \nu \in L^1(\sigma)$. However, if (e) holds, then (again by Grothendieck's criterion), $h(xy)$ clusters on $S \times S$, and so by ([8], § 4.1, Example 2) the bilinear form (1) clusters on bounded sets of $L^1(\sigma) \times L^1(\sigma)$. Thus (e) implies (a).

We have so far shown (a), (c), (d) and (e) to be equivalent. The remaining equivalence is obtained by applying (a)$\Longleftrightarrow$(d) to S_d.

The dual of $l_1(S)$ is the space $m(S)$ of bounded scalar functions on S which is isometrically isomorphic to $C(\beta S_d)$, the space of continuous scalar functions on βS_d, as a Banach space in the natural way. The bidual of $l_1(S)$ can therefore be identified with $M(\beta S_d)$ as a Banach space. If (c) holds then $M(\beta S_d)$ is also a Banach algebra under convolution; its multiplication is separately $\sigma(M(\beta S_d), C(\beta S_d))$-continuous and extends that of $l_1(S)$. It therefore coincides with both Arens products on $M(\beta S_d)$.

This last step shows that the equivalence of (c) with the other conditions can be obtained without appealing to Theorem 1; however, if one were interested only in the topological portion of Theorem 2, it would be somewhat rococo to prove it using the theory of Banach algebras.

The class of semigroups satisfying the above conditions is quite restricted, as is best seen from (d). Suppose that S satisfies the condition

(f) for every pair of finite subsets $A, B \subseteq S$ there are $x, y \in S$ such that $xB \cap A = \emptyset = By \cap A$.

Then S does not satisfy (d). One may select sequences inductively as follows: pick x_1, y_1 arbitrarily, and choose x_{n+1}, y_{n+1} so that

$$x_{n+1}\{y_1, \ldots, y_n\} \cap \{x_r y_s: r, s \leqslant n\} = \emptyset$$

$$\{x_1, \ldots, x_{n+1}\} y_{n+1} \cap \{x_r y_s: r \leqslant n+1,\ s \leqslant n\} = \emptyset.$$

Then the points $x_n y_m$ are all distinct and so *a fortiori* have the property required.

It is easily seen that any infinite cancellation semigroup satisfies (f)—and in particular the additive semigroup of positive integers does. It follows that any semigroup satisfying (a)–(e) is periodic and has no infinite cancellation subsemigroup. One can also show (using (d)) that its semilattice of idempotents contains no infinite chains.

An example of a commuting semigroup of idempotents (thought of as a lower semilattice) which *does* satisfy (a)–(e) can be obtained by identifying the least elements in any collection of finite chains. One can make this example slightly more complicated (as was pointed out to me by Dr. J. Hickey) by adjoining a greatest element and taking the ordinal product of the resulting semilattice with itself. Another type of example, this time with a unique idempotent, is due to N. Macri: let S be a set with distinguished elements 0 and 1 and define $xy = 0$ unless $x = y \neq 1$, in which case $xy = 1$. In all these cases, though, the multiplication on βS is rather trivial: ξy is constant for $\xi \in \beta S \setminus S$, $y \in S$. It would be interesting to have an example in which this was not true. It would be especially interesting if S satisfied (d) but βS did not, as this would yield an example of a Banach algebra which had regular multiplication but whose bidual did not. I believe it is unknown whether such a thing is possible.

Finally, it may be worth pointing out that conditions (a)–(e) offer no guarantee about the ideal structure of the bidual. In any of the above examples $(\delta_\xi - \delta_\eta)M(\beta S) = \{0\}$ where δ denotes point mass and $\xi, \eta \in \beta S \setminus S$.

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