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Singular integrals in the spaces $\Lambda(B, X)$

by

ALBERTO TORCHINSKY* (Ithaca, N. Y.)

Abstract. This article is concerned with a generalization of the spaces $\Lambda(B, X)$ introduced by A. P. Calderón, and with singular integral and multiplier operators acting on them. It also contains examples for $B = L^p(R^n)$ which illustrate the theory developed and it shows how to obtain results concerning Lipschitz spaces of functions and distributions in R^n.

1. Introductory remarks. This paper may be viewed as an elaboration of paragraphs 14 and 34 of A. P. Calderón's article "*Intermediate spaces and interpolation, the complex method*". Indeed, we introduce classes of spaces, also denoted by $\Lambda(B, X)$, and we describe their main features. The insight we have gained in the process is then used to consider the action of suitable singular integral and multiplier operators on these spaces and continuity properties are established.

The paper is divided into 8 sections. Section 2 deals with a one-parameter group of transformations introduced by de Guzmán in [12]. We notice that the infinitesimal generator P of the group must satisfy the coercive condition $(Px, x) \geqslant (x, x)$ (Lemma 2.3).

Section 3 deals with lattices of locally summable functions on $(0, 1)$. The concept of φ-lattice is introduced and the main properties are established in Theorem 3.3. These lattices arise naturally in the theory developed here and the properties of the r-lattices were studied in [13].

Section 4 introduces the spaces $\Lambda(B, X)$. The concept of Banach-space valued function is used and the pertinent facts needed may be found in [15]. The spaces $\Lambda(B, X)$ are analyzed by means of the *representation* Theorem 4.6 and the *convergence* Theorem 4.8. The representation theorem is an abstraction of the simple fact that if a spherically symmetric function $\varphi(x)$ has $\|\varphi\|_2 = 1$, then

$$\frac{1}{C_n}\int_0^\infty \hat{\varphi}(tx)\,(t|x|)^n\,\overline{\hat{\varphi}}(tx)\,\frac{dt}{t} = 1,$$

for $x \in R^n - (0)$. The convergence theorem shows that the spaces $\Lambda(B, X)$

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are independent of the measure used to construct them. All these facts become more familiar when in Section 5 we characterize the spaces for $P = \operatorname{diag}(a_1, \ldots, a_n)$ with $a_i \in Z_+$ and in Section 6 we consider spaces of functions and distributions in R^n. It turns out that Lipschitz spaces such as those considered by Taibleson [27] and B. F. Jones, Jr. [17] are among the $\Lambda(B, X)$ for appropriate choices of B and X. The results obtained may now be interpreted in terms of fractional integration (see [13]; [23], Chapter V; and [27]), Theorem 4.8, equivalence of norms for elements of Lipschitz spaces if differences of high enough order are used to define them ([23], Chapter V; [27]), Corollary 4.11, and so on. We are also able to consider the Lipschitz classes in terms of the *boundary behaviour* of functions and distributions in R^n without worrying too much whether the functions are harmonic or temperatures for $t > 0$. For these two particular classes it suffices to take appropriate derivatives of μ_t and ν_t where $\hat{\mu}(tx) = e^{-t|x|}$ and $\hat{\nu}(tx) = e^{-t^2|x|^2}$ and apply Corollary 4.11. (See also [11] and [27] for this particular case.) We select two particular properties, the extension of Young's convolution theorem and Bernstein's theorem, to show the simplicity of the approach developed. The applications do not end here: indeed one may obtain related results to those of [3] and [24] by employing similar techniques.

Sections 7 and 8 are motivated by this fact: singular integrals of the form $Ku(x) = a(x)u(x) + \text{p.v.}\int k(x, x-y)u(y)\,dy$, $k(x, \cdot)$ a homogeneous kernel, are known to preserve some classes of Hölder continuous functions and the complex intermediate (Sobolev) spaces $[L^p, L^p_k]_s$, $1 < p < \infty$, $0 < s < 1$. (See [7] and [9].) In fact we show that even under the more general conditions of Hörmander [16] and Benedek–Calderón–Panzone [1] replacing homogeneity, the continuity of translation invariant singular integrals may be established for $\Lambda(B, X)$ spaces. The case described above is treated as a pseudo-differential operator in 8.6. We also show multiplier theorems and they provide in some instances a more convenient way to handle some of the operators which interest us. In this context see also Petree [19].

The notation used is standard. The letters M, N will be reserved for multi-indices of non-negative integers. Z_+ will denote the set of positive integers. $P = \operatorname{diag}(a_1, \ldots, a_n)$ will mean that the matrix P has entries $p_{ii} = a_i$ and $p_{ij} = 0$ for $i \neq j$, $1 \leqslant i, j \leqslant n$. C will denote a positive constant, not necessarily the same from one occurrence to the next. Other notations will be defined or will be clear from the context.

Part of this work was presented as a Thesis to The University of Chicago, written under Professor A. P. Calderón with whom I had the privilege of learning these and many other things. I would also like to thank Professor Max A. Jodeit, Jr. for his help throughout my graduate studies.

2. A group of transformations.

2.1. Let $\{A_t\}_{t>0}$ be a group of linear transformations of R^n such that $A_s A_t = A_{st}$ $(A_1 = I)$ and such that the mapping $t \to A_t$ is continuous with respect to the uniform operator topology. Denoting with (the $n \times n$ real matrix) P the infinitesimal generator of the group we obtain the representation

$$A_t = e^{P\ln t} = t^P, \quad t > 0$$

(see [15] for the justification). We shall further assume that the group satisfies

$$(2.2) \qquad |t^P x| \leqslant t|x|, \quad 0 < t \leqslant 1, \ x \in R^n,$$

and in this case we have:

1) $|t^P x|/t$ is a non-decreasing function of t.

2) For $x \in R^n - (0)$ there exists a unique value t_x such that $t_x^{-P} x \in S^{n-1}$.

3) The function $\varrho(x) = t_x$ for $x \neq 0$ and $\varrho(0) = 0$ satisfies $\varrho(t^P x) = t\varrho(x)$ and $\varrho(x+y) \leqslant \varrho(x) + \varrho(y)$.

Since for $s < t$ we have $|s^P x| = |(s/t)^P t^P x| \leqslant (s/t)|t^P x|$, 1) follows and 2) is an immediate consequence. In addition if $\varrho(t^P x) = s$, then $(t/s)^P x \in S^{n-1}$ and $s = t\varrho(x)$. To complete 3) let us show that if $t_1^P x_1' + t_2^P x_2' = t_3^P x_3'$, with $|x_i'| = 1$, $i = 1, 2, 3$, then $t_3 \leqslant t_1 + t_2$. Assume to the contrary that $t_3 > t_1 + t_2$, then

$$|x_3'| = 1 \leqslant |(t_1/t_3)^P x_1'| + |(t_2/t_3)^P x_2'| \leqslant (t_1/t_3)|x_1'| + (t_2/t_3)|x_2'| < 1,$$

which is a contradiction.

The following lemma characterizes the groups for which the sufficient condition to define $\varrho(x)$ holds.

2.3. Lemma. *$|t^P x| \leqslant t|x|$ for $0 < t \leqslant 1$, $x \in R^n$ if and only if $(Px, x) \geqslant (x, x)$.*

Proof. Let $f(t) \doteq |t^P x|^2 - t^2|x|^2$. By (2.2) and 1) we have $f(t) \leqslant 0$ for $0 < t < 1$, $f(1) = 0$ and $f(t) \geqslant 0$ for $t > 1$, thus we must have $\frac{d}{dt} f(t)|_{t=1} \geqslant 0$. Since $\frac{d}{dt}|t^P x|^2 = \frac{2}{t}(Pt^P x, t^P x)$ we obtain $0 \leqslant \frac{d}{dt} f(t)|_{t=1}$ $= \frac{2}{t}(Pt^P x, t^P x) - 2t(x, x)|_{t=1} = 2(Px, x) - 2(x, x)$.

Conversely, let $g(t) = |t^P x|^2$. Then $\frac{d}{dt} g(t) = \frac{2}{t}(Pt^P x, t^P x) \geqslant \frac{2}{t}|t^P x|^2$ $= \frac{2}{t} g(t)$. For $0 < s \leqslant 1$ this inequality obtains

$$\int_s^1 g'(t)/g(t)\,dt \geqslant 2\ln(1/s),$$

which implies $ln(g(1)/g(s)) = ln(|x|^2/|s^P x|^2) \geqslant ln(1/s^2)$, and so $|s^P x| \leqslant s|x|$ for $0 < s \leqslant 1$.

The following remarks are immediate consequences of the lemma just proved. If $(P^*x, x) = (x, Px) = (Px, x)$, then $(Px, x) \geqslant (x, x)$ implies that $\{t^{P^*}\}_{t>0}$ also satisfies

$$|t^{P^*}x| \leqslant t|x|, \quad 0 < t \leqslant 1, \ x \in R^n,$$

and it thus determines a function $\varrho^*(x)$ with similar properties to those of $\varrho(x)$. Also, since $(Px, x) = (Sx, x)$ where S is the real symmetric matrix $(P+P^*)/2$, (2.2) is equivalent to the fact that the smallest eigenvalue of S be $\geqslant 1$.

2.4. For a Borel measure ν with finite total mass $\|\nu\|$ on R^n we define a one-parameter family of dilations as follows: we set $\nu_t(E) = \nu(t^{-P}E)$ for every ν-measurable set E and $t > 0$. Observe that if $d\nu(x) = \varphi(x)dx$, $\varphi \in L^1(R^n)$ and dx the ordinary Lebesgue measure on R^n, then $d\nu_t(x) = t^{-trP}\varphi(t^{-P}x)dx = \varphi_t(x)dx$ since $|\det t^P| = t^{trP}$.

2.5. To $x \in R^n$, we assign the "polar coordinates" $x \to \varrho(x), \theta_1, \ldots, \theta_{n-1}$ where $\varrho(x)^{-P}x = x'$ and $x_1' = \cos\theta_1 \ldots \cos\theta_{n-1}$, $x_2' = \cos\theta_1 \ldots \sin\theta_{n-1}, \ldots$ $\ldots, x_n' = \sin\theta_1$.

Then a computation shows that $dx = \varrho^{trP-1}(Px', x')dx'd\varrho$, where dx' is the surface element on $S^{n-1} = \{|x| = 1\} = \{\varrho(x) = 1\}$. (See [21].)

3. Lattices of locally integrable functions.

3.1. A *lattice*, or Banach lattice, X of locally integrable functions $f(t)$ on $(0, 1)$ is a linear class of functions such that there is a norm defined on X with respect to which it is complete, and if $f \in X$ and $|g| \leqslant |f|$ then also $g \in X$ with $\|g\|_X \leqslant \|f\|_X$.

3.2. Given a lattice X and a positive, monotone increasing (in the wide sense) submultiplicative function $\varphi(t)$ defined on $(0, \infty)$ we say that X is a φ-lattice if the mappings $f \to \int_0^t f(s)\varphi(t/s)(t/s)^{-\varepsilon}\frac{ds}{s}$ and $f \to \int_t^1 f(s)\varphi(t/s)(t/s)^{\varepsilon}\frac{ds}{s}$ are continuous from X into itself for all $\varepsilon > 0$.

We now prove some properties which will be needed in the sequel.

3.3. THEOREM. *Let X be a φ-lattice, we then have:*

1) *If for some $\delta > 0$ $\psi(t)/t^\delta$ is increasing, $\varphi(t)\psi(t) \in X$. In particular $\varphi(t)t^\varepsilon \in X$ for $\varepsilon > 0$.*

2) *If for some $\delta > 0$ $\varphi(t)\psi(t)/t^\delta$ is increasing,* $\left|\int_0^1 g(t)\psi(t)\frac{dt}{t}\right| \leqslant C\|g\|_X$.

Proof. Let $f(t) \geqslant 0$ be a non-vanishing element of X and let $\chi(t)$ be the characteristic function of the interval (1/2,1).

For $0 < t \leqslant 1/2$ and δ as in 1) we have

$$a(t) = \int_t^1 f(s)\varphi(t/s)(t/s)^\delta\frac{ds}{s} \geqslant \varphi(t)t^\delta\int_t^1 f(s)\frac{1}{s^\delta}\frac{ds}{s}$$
$$\geqslant \varphi(t)t^\delta\psi(t)/t^\delta\int_{1/2}^1 f(s)/\psi(s)\frac{ds}{s} = C\varphi(t)\psi(t).$$

For $1/2 \leqslant t < 1$ we likewise obtain

$$b(t) = \int_0^t f(s)\varphi(t/s)(t/s)^{-\delta}\frac{ds}{s} \geqslant \chi(t)\varphi(1)2^\delta\int_0^{1/2} f(s)s^\delta\frac{ds}{s},$$

and

$$\varphi(t)\psi(t) \leqslant \varphi(t)\psi(t)/t^\delta \leqslant \varphi(1)\psi(1)\chi(t).$$

1) now follows from the observation that

$\varphi(t)\psi(t) \leqslant C\{\chi(t) + \chi(1-t)a(t)\} \in X$.

To show 2) we start by assuming, as we may, that $g(t) \geqslant 0$. Then

$$\varphi(s)s^\delta\int_s^1 g(t)\psi(t)\frac{dt}{t} \leqslant \int_s^1 g(t)\varphi(s/t)\frac{\varphi(t)\psi(t)}{t^\delta}\left(\frac{s}{t}\right)^\delta t^{2\delta}\frac{dt}{t}$$
$$\leqslant \varphi(1)\psi(1)\int_s^1 g(t)\varphi(s/t)(s/t)^\delta\frac{dt}{t} \in X.$$

Also,

$$\varphi(s)s^\delta\int_0^s g(t)\psi(t)\frac{dt}{t} \leqslant \int_0^s g(t)\varphi(s/t)\frac{\varphi(t)\psi(t)}{t^\delta}\left(\frac{s}{t}\right)^{-\delta}s^{2\delta}\frac{dt}{t}$$
$$\leqslant \varphi(1)\psi(1)\int_0^s g(t)\varphi(s/t)(s/t)^{-\delta}\frac{dt}{t} \in X.$$

Now the conclusion follows since by part 1), $\varphi(s)s^\delta \in X$ and the elements of X in the right-handside of the inequalities have norm $\leqslant C\|g\|_X$ by (3.2).

For $\varphi(t) = t^r$, $0 \leqslant r < \infty$, we simply choose to call a t^r-lattice an r-lattice.

3.4. Given a lattice X and a positive function $\gamma(t)$ defined on $(0, 1)$ we construct the lattice $\gamma X = \{f \in L^1_{loc}(0, 1): \gamma(t)^{-1}f(t) \in X\}$, $\|f\|_{\gamma X} = \|\gamma^{-1}f\|_X$.

4. The spaces $\Lambda(B, X)$.

4.1. Let $B = V^*$ for a complex Banach space V and for $y \in R^n$ let τ_y be a representation of R^n into a group of uniformly bounded linear operators of B into itself, i.e. $\|\tau_y u\|_B \leqslant C\|u\|_B$, C independent of $y \in R^n$ and $u \in B$ such that:

i) For each $u \in B$, $v \in V$, $(\tau_y u, v)$ is a continuous, and bounded, function of y.

ii) The τ_y are the adjoints of a family $\tilde{\tau}_y$ acting continuously on V, i.e., $(\tau_y u, v) = (u, \tilde{\tau}_y v)$ for $u \in B$, $v \in V$.

For a finitely valued Borel measure ν and $u \in B$ we define $\int \tau_y u d\nu(y)$ as the element $w \in B$ for which

$$(w, v) = \int (\tau_y u, v)\, d\nu(y) \tag{4.2}$$

for all $v \in V$. Similarly we define $U(x, t) \in B$ acting on $v \in V$ as

$$\big(U(x, t), v\big) = \int (\tau_{x+y} u, v)\, d\nu_t(y) = \int (\tau_{x+t} P_y u, v)\, d\nu(y). \tag{4.3}$$

It is readily seen that the above assumptions imply that $(\tau_t P_y u, v)$ is a jointly continuous function of the variables (y, t), and so the integral (4.3) is well defined.

4.4. Given B as above and a lattice X as in 3, we denote by $X(B) = \{F(t)\colon F(t)$ is B-valued weakly measurable and $\|F\|_B \in X\}$, $\|F\|_{X(B)} = \|(\|F\|_B)\|_X$. Finally we let

$$A_\nu(B, X) = \Big\{u \in B\colon \int \tau_y u\, d\nu_t(y) \in X(B)\Big\}. \tag{4.5}$$

Normed with $\|u\|_{A_\nu} = \|u\|_B + \big\|\int \tau_y u\, d\nu_t(y)\big\|_{X(B)}$, $A_\nu(B, X)$ becomes a Banach space in which B is continuously embedded.

The description of the spaces $A_\nu(B, X)$ and the independence, up to equivalence in norm, of the choice of the measure ν may be achieved by constructing a mapping S of the direct sum $X(B) \oplus B$ into $A_\nu(B, X)$; these will be done in the remaining of this section.

4.6. THEOREM. *Let B, τ_y be as above. Let ν satisfy*

i) $\int |x^M|\, |d\nu(x)| = C_M < \infty$ *for all multi-indices M of non-negative integers; and*

ii) $(\nu_t)^\wedge(x) = \hat{\nu}(t^{P^*} x) \not\equiv 0$ as a function of $t > 0$ for $x \in R^n - (0)$.

There exist, then, functions $\varphi, \psi \in \mathscr{S}(R^n)$ such that for all $v \in V$

$$(u, v) = \int (\tau_y u, v)\varphi(y)\, dy + \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \int \big(U(y, t), v\big)\psi_t(y)\, dy \frac{dt}{t}.$$

In fact these functions may be chosen so that $\hat{\psi} \in C_0^\infty(R^n)$ vanishes in a neighbourhood of the origin and $\hat{\varphi} \in C_0^\infty(R^n)$.

Proof. For $x \in R^n$ put

$$g_m(x) = \int_{1/m}^{m} |\hat{\nu}(t^{P^*} x)|^2 \frac{dt}{t}, \qquad m \in Z_+.$$

Then ii) and a compactness argument obtain that there exist $\varepsilon > 0$ and $m \in Z_+$ such that $g_m(x) > \varepsilon$ for $x \in S^{n-1}$.

Let now $\eta(x)$ be a positive, $C_0^\infty(R^n)$ function vanishing near the origin and infinity and so that $\eta(x) \equiv 1$ for $1/m \leqslant \varrho^*(x) \leqslant m$.

We now introduce the function

$$1/N(x) = \int_0^\infty |\hat{\nu}(t^{P^*} x)|^2 \eta(t^{P^*} x) \frac{dt}{t}.$$

The change of variables $t \to t/\varrho^*(x)$ shows that $N(x) = N(x')$, where $\varrho^*(x)^{-P^*} x = x' \in S^{n-1}$, i.e., $N(x)$ is homogeneous of degree zero with respect to the group t^{P^*}.

Now we are able to define the functions φ and ψ in terms of their Fourier transforms as follows:

$$\hat{\psi}(x) = \overline{\hat{\nu}}(x)\eta(x)N(x),$$

$$\hat{\varphi}(x) = \int_1^\infty \hat{\psi}(t^{P^*} x)\hat{\nu}(t^{P^*} x)\frac{dt}{t}, \qquad \hat{\varphi}(0) = 1.$$

We notice that for $\varepsilon > 0$,

$$(\varphi_\varepsilon)^\wedge(x) = \hat{\varphi}(\varepsilon^{P^*} x) = \hat{\varphi}(x) + \int_\varepsilon^1 \hat{\psi}(t^{P^*} x)\hat{\nu}(t^{P^*} x)\frac{dt}{t},$$

as the change of variables $\varepsilon t \to t$ readily shows.

We now define u_ε acting on $v \in V$ as

$$(u_\varepsilon, v) = \int (\tau_y u, v)\varphi(y)\, dy + \int_\varepsilon^1 \int \big(U(y, t), v\big)\psi_t(y)\, dy \frac{dt}{t} \ (*).$$

Since $(\tau_y u, v)$ is a continuous bounded function of $y \in R^n$ and $\hat{\nu}(y)$ is a C^∞ function with bounded derivatives of all orders, and $\varphi, \psi_t \in \mathscr{S}(R^n)$, the interpretation of $(*)$ in $\mathscr{S}'(R^n)$ by means of Parseval relation obtains

$$\begin{aligned}
(u_\varepsilon, v) &= \big((\tau_y u, v), \varphi(y)\big) + \int_\varepsilon^1 \Big(\big(U(y, t), v\big), \psi_t(y)\Big)\frac{dt}{t}\\
&= \big((\tau_y u, v)^\wedge, \hat{\varphi}(-y)\big) + \int_\varepsilon^1 \Big(\big(U(y, t), v\big)^\wedge, \hat{\psi}_t(-y)\Big)\frac{dt}{t}\\
&= \big((\tau_y u, v)^\wedge, \hat{\varphi}(-y)\big) + \int_\varepsilon^1 \big((\tau_y u, v)^\wedge, \hat{\nu}_t(-y)\hat{\psi}_t(-y)\big)\frac{dt}{t}\\
&= \big((\tau_y u, v)^\wedge, (\varphi_\varepsilon)^\wedge(-y)\big) = \big((\tau_y u, v), \varphi_\varepsilon(y)\big).
\end{aligned}$$

Whence $\lim_{\varepsilon\to 0}(u_\varepsilon, v) = \lim_{\varepsilon\to 0}\big((\tau_y u, v), \varphi_\varepsilon(y)\big) = (\tau_y u, v)]_{y=0} = (u, v)$ as it was to be shown.

Remark. i) may be replaced by i′) $\hat{\nu}(x) \in C^\infty(R^n - (0))$ with bounded derivatives of all orders for $x \neq 0$.

4.7. In order to construct the mapping S alluded to we introduce the notation A_k for the class of integrable functions $f(x)$ for which there exists a "polynomial" of degree $< k$, $\sum_{|M|<k} a_M(x) y^M$ such that

i) $a_M(x) \in L^1(R^n)$ and

ii) $\int |f(x-y) - \sum a_M(x) y^M|\, dx = O(|y|^k)$ for $y \in R^n$.

We may now state and prove the following theorem.

4.8. THEOREM. *Let $k \in Z_+$ and let μ be a finite Borel measure such that*

i) $\int x^M d\mu_t(x) = 0$ *for* $|M| < k$ *and*

ii) $\int |x|^k |d\mu(x)| < \infty$.

Further let X be a β-lattice and let $\gamma(t)$ be such that $\beta(t)\gamma(t)/t^\varepsilon$ increases for some $\varepsilon > 0$ and $\beta(t)\gamma(t)/t^{k-\delta}$ decreases for some δ, $0 < \delta < k$. Then for fixed functions $\varphi, \psi \in A_k$ and $(u, F(t)) \in B \oplus X(B)$ the integrals $S(u, F)$ $= \int \tau_y u \varphi(y)\, dy + \int_0^1 \int \tau_y F(t) \psi_t(y) \gamma(t)\, dy \frac{dt}{t}$ converge absolutely (in the B-norm) and

$$\|S(u, F)\|_{A_\mu(B,\gamma X)} \leqslant C\{\|u\|_B + \|F\|_{X(B)}\},$$

C independent of u and F.

Proof. We begin by proving a lemma.

4.9. LEMMA. *Let μ be as in the Theorem and let $f \in A_k$, then $\|f_t * \mu_s\|_1 = O(\min(1, (s/t)^k))$.*

Proof of the lemma. Put $I(x) = f_t * \mu_s(x)$, then

$$\begin{aligned} I(x) &= \int t^{-\operatorname{tr} P} f\big(t^{-P}x - (t/s)^{-P} y\big)\, d\mu(y) \\ &= \int t^{-\operatorname{tr} P}\Big\{f\big(t^{-P}x - (t/s)^{-P}y\big) - \sum_{|M|<k} a_k(t^{-P}x)\big((t/s)^{-P}y\big)^M\Big\} d\nu(y) \\ &= \int I(x, y, s, t)\, d\mu(y). \end{aligned}$$

Thus,

$$\int |I(x)|\, dx \leqslant \iint |I(x, y, s, t)|\, dx\, d\mu(y)| \leqslant C \int |(t/s)^{-P} y|^k |d\mu(y)| \leqslant C(s/t)^k$$

for $s/t \leqslant 1$. Since

$$\int |I(x)|\, dx \leqslant \|f_t\|_1 \|\mu\| = O(1),$$

the lemma now follows.

Let us return to the theorem. According to (4.5) we must show that $S(u, F) \in B$ and $\int \tau_y S(u, F)\, d\mu_t(y) \in (\gamma X)(B)$, with the desired norm inequalities. We first notice that the integrals in the definition of S are absolutely convergent in the B-norm: in the case of

$$\int \|\tau_y u \varphi(y)\|_B\, dy \leqslant C \int |\varphi(y)|\, dy\, \|u\|_B$$

this is immediate and for

$$\begin{aligned} \int_0^1 \int \|\tau_y F(t) \psi_t(y)\|_B \gamma(t)\, dy \frac{dt}{t} &\leqslant C \int |\psi(y)|\, dy \int_0^1 \|F(t)\|_B \gamma(t) \frac{dt}{t} \\ &\leqslant C \int |\psi(y)|\, dy\, \big\|(\|F\|_B)\big\|_X \end{aligned}$$

it is a consequence of Theorem 3.3. This shows that $S(u, F) \in B$ with the desired bounds on its norm.

Let us now consider

$$\begin{aligned} \int \big(\tau_z S(u, F), v\big)\, d\mu_s(z) &= \int \big(S(u, F), \tilde{\tau}_z v\big)\, d\mu_s(z) \\ = \iint (\tau_y u, \tilde{\tau}_z v) \varphi(y)\, dy\, d\mu_s(z) &+ \int \int_0^1 \int \big(\tau_y F(t), \tilde{\tau}_z v\big) \psi_t(y) \gamma(t)\, dy\, d\mu_s(z) \frac{dt}{t} \\ &= I_1 + I_2. \end{aligned}$$

We first examine I_2, the exchange of order of integration being justified by Fubini's theorem once the finiteness of the integral has been established.

$$\begin{aligned} I_2 &= \int \int_0^1 \int \big(\tau_{y+z} F(t), v\big) \psi_t(y) \gamma(t)\, dy \frac{dt}{t}\, d\mu_s(z) \\ &= \int \int_0^1 \int \big(\tau_y F(t), v\big) \int \psi_t(y - z)\, d\mu_s(z)\, dy\, \gamma(t) \frac{dt}{t} \end{aligned}$$

and so

$$\begin{aligned} |I_2| &\leqslant C\|v\|_V \int_0^1 \|F(t)\|_B \|\psi_t * \mu_s\|_1 \gamma(t) \frac{dt}{t} \\ &\leqslant C\|v\|_V \|F\|_{X(B)} < \infty, \end{aligned}$$

since $\|\psi_t * \mu_s\|_1 = O(1)$ and Theorem 3.3 applies. Also we have

$$\begin{aligned} |I_2| &\leqslant C\|v\|_V \int_0^s \|F(t)\|_B \gamma(t) \frac{dt}{t} + C\|v\|_V \int_s^1 \|F(t)\|_B \Big(\frac{s}{t}\Big)^k \gamma(t) \frac{dt}{t} \\ &= I_{2,1}(s) + I_{2,2}(s). \end{aligned}$$

It will be sufficient now to show that $\gamma(s)^{-1}I_{2,1}(s)$ and $\gamma(s)^{-1}I_{2,2}(s)\in X$ to complete the consideration of this term. For $s>t$ we have

$$\gamma(t)\gamma(s)^{-1} = \gamma(t)\beta(t)t^{-\varepsilon}\gamma(s)^{-1}\beta(s)^{-1}s^{\varepsilon}\beta(t)^{-1}t^{\varepsilon}\beta(s)s^{-\varepsilon}$$
$$\leqslant \beta(s)\beta(t)^{-1}\left(\frac{s}{t}\right)^{-\varepsilon} \leqslant \beta(s/t)(s/t)^{-\varepsilon}$$

and so

$$\gamma(s)^{-1}I_{2,1}(s)\leqslant C\|v\|_V\int_0^s \|F(t)\|_B\beta(s/t)(s/t)^{-\varepsilon}\frac{dt}{t}\in X$$

with

$$\|\gamma(s)^{-1}I_{2,1}(s)\|_X\leqslant C\|v\|_V\|F\|_{X(B)}.$$

Likewise for $t>s$ we have

$$\gamma(t)\gamma(s)^{-1}(s/t)^k = \gamma(t)\beta(t)t^{-k+\delta}\gamma(s)^{-1}\beta(s)^{-1}s^{k-\delta}\beta(s)\beta(t)^{-1}(s/t)^{\delta}$$
$$\leqslant \beta(s/t)(s/t)^{\delta}$$

whence

$$\gamma(s)^{-1}I_{2,2}(s)\leqslant C\|v\|_V\int_s^1 \|F(t)\|_B\beta(s/t)(s/t)^{\delta}\frac{dt}{t}\in X$$

with

$$\|\gamma(s)^{-1}I_{2,2}(s)\|_X\leqslant C\|v\|_V\|F\|_{X(B)}.$$

A similar but certainly less involved argument shows that

$$\left\|\gamma(s)^{-1}\int\int(\tau_y u, v)\varphi(y-z)\,d\mu_s(z)\,dy\right\|_X\leqslant C\|\beta(s)s^{\delta}\|_X\|v\|_V\|u\|_B$$

and this completes the proof of the theorem.

4.10. Corollary. *Let the measure μ of Theorem 4.8 satisfy the hypothesis of Theorem 4.6 and let $\varphi(x)$, $\psi(x)$ be the functions constructed in Theorem 4.6. If $\gamma(t)\equiv 1$, $S(u,F)$ maps $B\oplus X(B)$ onto $\Lambda_\mu(B,X)$.*

Proof. Let $F(t)=\int\tau_y u\,d\mu_t(y)$, then Theorems 4.6 and 4.8 assert that $S(u,F(t))=u$, for $u\in\Lambda_\mu(B,X)$.

4.11 Corollary. *Let the measures $\mu\neq\nu$ satisfy both the conditions of Theorem 4.6 and 4.8. Then the corresponding spaces $\Lambda_\mu(B,X)$ and $\Lambda_\nu(B,X)$ coincide algebraically and topologically (they have the same elements with equivalent norms).*

Proof. Since ν satisfies the hypothesis of Theorem 4.6 we may construct S so that $S(u,\int\tau_y u\,d\nu_t(y))=u$, and by Theorem 4.8

$$\|u\|_{\Lambda_\mu(B,X)}\leqslant C\left(\|u\|_B+\left\|\int\tau_y u\,d\nu_t(y)\right\|_{X(B)}\right)=C\|u\|\Lambda\nu(B,X).$$

The converse inequality follows by exchanging the roles of ν and μ.

4.12. Remark. From now on we shall denote the $\Lambda_\nu(B,X)$ plainly with $\Lambda(B,X)$ with the understanding that the measure ν satisfies both the conditions of Theorems 4.6 and 4.8.

4.13. Corollary. (Cf. [3], paragraph 6.1). *Let φ be a compactly supported, integrable function and let X be an r-lattice $0<r<1$. Then $\|u-\int\tau_y u\varphi_t(y)\,dy\|_B\in X$ implies $\|u-\tau_{t^P z}u\|_B\in X$ for any $z\in R^n-(0)$.*

Proof. Set $\nu(x)=\delta(x)-\varphi(x)dx$ and $\mu(x)=\delta(x)-\delta(z-x)$ where δ is the Dirac measure centered at the origin. Then ν satisfies the hypothesis of Theorem 4.6, indeed $\hat{\nu}(t^{P^*}z)=1-\hat{\varphi}(t^{P^*}z)\not\equiv 0$ since $\hat{\varphi}(y)\to 0$ as $\varrho^*(y)\to\infty$, and $\mu(x)$ those of Theorem 4.8, $\int d\mu(x)=0$. The conclusion now follows from 4.8.

5. The case $P=\operatorname{diag}(a_1,\ldots,a_n)$.

5.1. This section is devoted to the characterization of the spaces $\Lambda(B,X)$ for r-lattices X and diagonal matrices $P=\operatorname{diag}(a_1,\ldots,a_n)$ with $a=(a_1,\ldots,a_n)\in Z_+^n$.

We first introduce the measure ν which will be used to define (4.2). For $k\in Z_+$ and a fixed $z\in R^n-(0)$ we set

$$\nu_z(y)\equiv\nu(y)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j\delta(y-j^P z).$$

5.2. Remark. $\int(t^P y)^M d\nu(y)=0$ for $0\leqslant(a\cdot M)<k$. Notice that $t^P z=(t^{a_1}z_1,\ldots,t^{a_n}z_n)$, thus

$$\int(t^P y)^M d\nu(y)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j\big((jt)^P z\big)^M$$
$$=t^{(a\cdot M)}z^M\sum_{j=0}^{k}\binom{k}{j}(-1)^j j^{(a\cdot M)}=0$$

for $0\leqslant(a\cdot M)<k$ as an inductive argument shows.

5.3. With $\Delta_{t,z}^k u$ we shall denote the expression

$$\Delta_{t,z}^k u=\int\tau_y u\,d\nu_t(y)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j\tau_{(jt)^P z}u.$$

The theorem we now state and prove is illustrated in 6.2 and 6.3.

5.4. Theorem. *Let the matrix P be as in the introduction. Let X be an r-lattice, with $r=k-1+\varepsilon$, $0<\varepsilon<1$, $k\in Z_+$, and let ν and $\Delta_{t,z}^m$ be as defined above, for $1\leqslant m$.*

a) *For $u\in\Lambda(B,X)$ and $(a\cdot M)\leqslant k-1$ we have:*

i) $(\partial/\partial x)^M \tau_x u]_{x=0} \in \Lambda(B, t^{-(a\cdot M)}X)$.

ii) $(\partial/\partial x)^M \tau_x u]_{x=0} \in B$ *and* $\sup_{|z|=1} t^{(a\cdot N)} \Delta^m_{t,z} (\partial/\partial x)^N \tau_x u]_{x=0} \in X(B)$ *for* $(a\cdot N) = k-1$.

b) *Conversely, if* a) ii) *holds, then* $u \in \Lambda(B, X)$.

In both cases the statements are accompanied by the corresponding norm inequalities.

Proof. We first define $Q_M u = (\partial/\partial x)^M \tau_x u]_{x=0}$. Since for $u \in B$, $\tau_x u \in B$ and the integrals in (4.6) are absolutely convergent in the B-norm we have

$$\tau_x u = \int \tau_y(\tau_x u)\varphi(y)\,dy + \int_0^1 \int U(x+y, t)\psi_t(y)\,dy\,\frac{dt}{t}$$
$$= \int \tau_y u\varphi(y-x)\,dy + \int_0^1 U(y, t)\psi_t(y-x)\,dy\,\frac{dt}{t},$$

whence it follows that

$$(\partial/\partial x)^M \tau_x u$$
$$= \int \tau_y u (\partial/\partial y)^M \varphi(y-x)\,dy + \int_0^1 \int U(y, t) t^{-(a\cdot M)} \big((\partial/\partial y)^M \psi\big)_t (y-x)\,dy\,\frac{dt}{t},$$

this being justified since the resulting integrals are absolutely convergent in the B-norm. Setting $x = 0$ and applying (4.8) we obtain

$$\|Q_M u\|_{\Lambda(B, t^{-(a\cdot M)}X)} \leqslant C\|u\|_{\Lambda(B,X)}, \tag{5.5}$$

which proves a) i).

Also since B is continuously embedded in $\Lambda(B, t^{-(a\cdot M)}X)$, we infer that the first half of a) ii) is also true. Moreover $t^{-(a\cdot N)}X$ is of type ε for $(a\cdot N) = k-1$ and so (4.8) implies that $\Delta^m_{t,z} Q_N u \in (t^{-(a\cdot N)}X)(B)$ or $t^{(a\cdot N)}\Delta^m_{t,z} Q_N u \in X(B)$, and from (5.5) it follows that

$$\sup_{|z|=1} \|t^{(a\cdot N)} \Delta^m_{t,z} Q_N u\|_{x(B)} \leqslant C\|u\|_{\Lambda(B,X)}$$

with C *independent* of z: this last statement either follows from the derivation of (5.5) or from the application of an appropriate version of the uniform boundedness principle. Now that a) ii) has been proved we consider b). We first notice that $Q_M \tau_y = \tau_y Q_M$; this requires a small computation. Set

$$g(t) = t^{k-1} \sup_{|z|=1} \|\Delta^m_{t,z} Q_N u\|_B \cdot g(t) \in X$$

by assumption.

Moreover, since a short computation shows that

$$\Delta^m_{t,z} Q_N u = \Delta^m_{t\varrho(z), z'} Q_N u \quad (\text{where } \varrho(z)^{-P} z = z')$$

we have

$$\|t^{k-1}\Delta^m_{t,z} Q_N u\|_B = (t\varrho(z))^{k-1}\varrho(z)^{-k+1}\|\Delta^m_{t\varrho(z),z'} Q_N u\|_B$$
$$\leqslant \varrho(z)^{-k+1} g(t\varrho(z)).$$

We now construct a function η with the required properties for which $\int \tau_y u \eta_t(y)\,dy \in X(B)$, which together with $u \in B$ (setting $M = (0, \ldots, 0)$ in (5.5)) will complete the proof of b).

Let us begin by considering a $C_0^\infty(R^n)$ function $\theta(y)$ with support contained in $\{\varrho(y) \leqslant 1\}$ and with vanishing moments of order $< m$. Then

$$\Big\|t^{k-1}\int \Delta^m_{t,y} Q_N u \theta(y)\,dy\Big\|_B \leqslant C \int_{\{\varrho(y)\leqslant 1\}} \varrho(y)^{-k+1} g(t\varrho(y))\,dy$$
$$\leqslant C\int_0^t g(s)(t/s)^{k-1}\frac{ds}{s} \in X \tag{5.6}$$

by (3.2). Set

$$f(t) = \int \Delta^m_{t,z} Q_N u\theta(y)\,dy = \int \sum_{j=1}^m \binom{m}{j}(-1)^j \tau_{(jt)^P y} Q_N u\theta(y)\,dy$$
$$= \int \sum_{j=1}^m \binom{m}{j}(-1)^j \tau_y Q_N u \theta_{jt}(y)\,dy$$
$$= \int Q_N \tau_y u \sum_{j=1}^m \binom{m}{j}(-1)^j \theta_{jt}(y)\,dy.$$

This last integral is successively integrated by parts to obtain

$$f(t) = \int \tau_y u \sum_{j=1}^m \binom{m}{j}(-1)^j (tj)^{-k+1}\big((\partial/\partial y)^N\theta\big)_{tj}(y)\,dy$$
$$= t^{-k+1}\int \tau_y u \eta_t(y)\,dy,$$

with

$$\eta(y) = \sum_{j=1}^m \binom{m}{j}(-1)^j j^{-trP-k+1}(\partial/\partial y)^N \theta(j^{-P}y).$$

Since $\theta(y)$ does not vanish identically neither does $\eta(y)$. Moreover $\eta(y)$ satisfies the required conditions for Corollary 4.11 to apply and since $\int \tau_y u\eta_t(y)\,dy = t^{k-1}f(t) \in X(B)$ by (5.6), the proof is completed.

6. Lipschitz spaces of functions and distributions in Euclidean pace. We now consider the usual Lipschitz spaces, the Lipschitz spaces of parabolic type and we show that they are among the $\Lambda(B, X)$ spaces for appropriate choices of B and X. We then illustrate the simplicity of the method of proof for the $\Lambda(B, X)$ spaces by considering two instances which often arise in dealing with function spaces: the extension of Young's convolution theorem and the properties of the Fourier transform. Fractional integration (corresponding to Theorem 4.8 for r-lattices) and duality are dealt with in [13].

6.1. Let $B \subset \mathscr{S}'(R^n)$ be a Banach space such that $\mathscr{S}(R^n) \subset B$ continuously, $B = V^*$ for a complex Banach space V and $\|\tau_y u\|_B \leqslant C\|u\|_B$ where $(\tau_y u, \varphi) = (u, \varphi(\cdot + y))$ for $\varphi \in \mathscr{S}$. We may, for example, let $B = L^p(R^n)$, $1 < p \leqslant \infty$, so that the elements of B are functions $u(x)$ on R^n and $\tau_y u = u(x-y)$.

For the lattice X we choose the r-lattice $X = t^r L^q\left(0, 1; \frac{dt}{t}\right)$ (that X indeed is an r-lattice is shown in [13]), $1 \leqslant q \leqslant \infty$.

We now describe explicitly the spaces $\Lambda(B, X)$ for particular choices of $P = \operatorname{diag}(a_1, \ldots, a_n)$.

6.2. Let $P = \operatorname{diag}(1, \ldots, 1) = I$; $B = L^p(R^n)$, $1 < p \leqslant \infty$, $k \in Z_+$ and

$$X = t^{k-1+\varepsilon} L^q\left(0, 1; \frac{dt}{t}\right) \quad \text{for} \quad 0 < \varepsilon < 1,\ 1 \leqslant q \leqslant \infty.$$

Then $\Lambda(B, X) = \{u \in L^p(R^n)$: $(\partial/\partial x)^M u \in L^p(R^n)$ for $|M| \leqslant k-1$ and

$$\sup_{|z|=1} t^{-\varepsilon} \left\| \sum_{j=0}^{k} \binom{k}{j} (-1)^j u_\alpha(x - tjz) \right\|_p \in L^q\left(0, 1; \frac{dt}{t}\right)$$

where u_α is a derivative of u of order α, $|\alpha| = k-1\}$.

These are the spaces of Taibleson and others.

6.3. Let $P = \operatorname{diag}(1, \ldots, 1, 2)$; $B = L^p(R^n)$, $1 < p \leqslant \infty$; $k \in Z_+$ and $X = t^{k-1+\varepsilon} L^q\left(0, 1; \frac{dt}{t}\right) \cap t^{k-1+\delta} L^q\left(0, 1; \frac{dt}{t}\right)$ for $0 < \varepsilon, \delta < 1$ and $1 \leqslant q \leqslant \infty$. For simplicity put $k = 3$. Then

$$\Lambda(B, X) = \{u \in L^p(R^n): (\partial/\partial x)^M u \in L^p(R^n) \text{ for } M_1 + \ldots + M_{n-1} + 2M_n \leqslant 2 \text{ and } \left(t^{-\varepsilon} \left\| \sum_{j=0}^{2} \binom{2}{j} (-1)^j u_\alpha\left(x_1 - ((tj)^P z)_1, \ldots, x_{n-1} - ((tj)^P z)_{n-1}, x_n\right) \right\|_p \right.$$
$$\left. + t^{-\delta} \left\| \sum_{j=0}^{1} \binom{1}{j} (-1)^j u_\alpha(x_1, \ldots, x_{n-1}, x_n - (tj)^P \tilde{z}) \right\|_p \right) \in L^q\left(0, 1; \frac{dt}{t}\right) \text{ for } \alpha_1 + \ldots + \alpha_{n-1} + 2\alpha_n = 2,\ z = (z_1, \ldots, z_{n-1}, 0) \in S^{n-1},\ \tilde{z} = (0, \ldots, 0, 1) \in S^{n-1}\}.$$

These are the parabolic Lipschitz spaces of Jones and others.

6.4. For the case P not diagonal we have the following complementary result to (5.4). We are interested in operators Q such as the operators Q_M of Theorem 5.4 which satisfy:

i) $Q\tau_y = \tau_y Q$ and

ii) $Q\psi_t = t^{-\varepsilon}(Q\psi)_t$, $\varepsilon > 0$.

i) implies that $(Q\varphi)\hat{}\,(x) = q(x)\hat{\varphi}(x)$, $q \in L^\infty(R^n)$, and by taking Fourier transforms in ii) we see that $q(t^{P^*}x) = t^\varepsilon q(x)$. (We may set $q(x) = \varrho^* = \varrho^*(x)^\varepsilon \varphi(x)$, where $\varphi(x) = \varphi(x')$, with $\varrho^*(x)^{-P^*}x = x'$.) We may thus prove:

Lemma. *If ν is a measure such that $\int y^M d\nu_t(y) = 0$, $t > 0$, $|M| < k$ and X is an r-lattice, $r < k$, then* $\|Qu\|_B + \|t^r(Qu * \nu_t)\|_{X(B)} \leqslant C\|u\|_{\Lambda(B,X)}$.

6.5 We now adopt the notation $\Lambda[\alpha; p, q]$ for the spaces $\Lambda\left(L^p(R^n), t^{-\alpha}L^q\left(0, 1; \frac{dt}{t}\right)\right)$, $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. $\Lambda[\alpha; p, q] = \Lambda(\alpha; p, q)$ in the notation of Taibleson [27] only for $P = I$.

We now prove an extension of Young's convolution theorem. Notice that (4.2) and (4.3) are the usual convolutions.

6.6. Theorem. $\Lambda[\alpha_0; p_0, q_0] * \Lambda[\alpha_1; p_1, q_1] \subset \Lambda[\alpha_2; p_2, q_2]$, *where* $\alpha_2 = \alpha_0 + \alpha_1$, $\frac{1}{p_2} = \frac{1}{p_0} + \frac{1}{p_1} - 1$, $\frac{1}{q_2} = \frac{1}{q_0} + \frac{1}{q_1}$, *for* $1 < p_0, p_1, p_2 \leqslant \infty$ *and* $1 \leqslant q_0, q_1, q_2 \leqslant \infty$.

Proof. Let $u^i \in \Lambda[\alpha_i; p_i, q_i]$, $i = 0, 1$. Young's theorem for convolutions asserts that $u^0 * u^1 \in L^{p_2}(R^n)$. Let ψ be the function defined in (4.6), and set $\eta = \psi * \psi$. To complete the proof we now show that $t^{-\alpha_2}\|u^0 * u^1 * \eta_t\|_{p_2} \in L^{q_2}\left(0, 1; \frac{dt}{t}\right)$.

We have $\|u^0 * u^1 * \eta_t\|_{p_2} \leqslant \|u^0 * \psi_t\|_{p_0} \|u^1 * \psi_t\|_{p_1}$. The case $q_0 = \infty$ or $q_1 = \infty$ is readily seen to be true. For the remaining cases we have

$$\left(\int_0^1 (t^{-\alpha_0-\alpha_1}\|u^0 * u^1 * \eta_t\|_{p_2})^{q_2} \frac{dt}{t} \right)^{1/q_2}$$
$$\leqslant \left(\int_0^1 (t^{-\alpha_0}\|u^0 * \psi_t\|_{p_0})^{q_0} \frac{dt}{t} \right)^{1/q_0} \left(\int_0^1 (t^{-\alpha_1}\|u^1 * \psi_t\|_{p_1})^{q_1} \frac{dt}{t} \right)^{1/q_1},$$

by Hölder's inequality.

Notice that we have actually shown the norm inequality

$$\|u * v\|_{\Lambda[\alpha_2; p_2, q_2]} \leqslant C\|u\|_{\Lambda[\alpha_0; p_0, q_0]} \|v\|_{\Lambda[\alpha_1; p_1, q_1]}.$$

This theorem is still valid for p_0 or p_1 equal to 1, with the natural definition of the spaces $\Lambda[\alpha; p, q]$ and the interpretation of (4.2) and

(4.3) as (strongly convergent by Theorem 4.8) Riemann vector valued integrals. This justification is omitted here but made in order to give the full range of validity of Theorems 6.6 (see the definition given in [3] and [20]).

We now consider the Fourier transformation keeping in mind the interesting results of [14], [11] and [27] extending Bernstein's theorem on absolutely convergent Fourier transforms.

6.7. THEOREM. *For* $2 \leqslant p < \infty$ *and* $1 \leqslant q \leqslant 2$ *we have*

i) $\mathfrak{F}\colon \Lambda[tr\,P(1/q-1/p);\, p',\, q] \to L^q(R^n)$,

ii) $\mathfrak{F}\colon L^{q'}(R^n) \to \Lambda[tr\,P(1/q'-1/p');\, p,\, q']$.

Proof. Since by results of Benedek–Panzone on spaces of mixed norms [1] we have that $\Lambda[\alpha; p, q]^* = \Lambda[-\alpha, p', q']$ for $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, ii) will follow from i) by duality (see [13] and [27]).

We now prove i). Let $f \in \mathscr{S}(R^n)$, which is dense in $\Lambda[\alpha; p, q]$ (see [27]) and let $\varphi \in \mathscr{S}(R^n)$, with $\hat{\varphi}(x)$ vanishing near zero and $0 < C^{-1} \leqslant \int_0^\infty |\hat{\varphi}(t^{P^*}x)|^2 \frac{dt}{t} < \infty$ (see the introduction or Theorem 4.6). We then have

$$\int |\hat{f}(x)|^q dx \leqslant C \int_0^\infty \int |\hat{f}(x)|^q |\hat{\varphi}(t^{P^*}x)|^2 dx \frac{dt}{t}$$
$$\leqslant C \int_0^\infty \Big(\int |\hat{f}(x)\,\hat{\varphi}(t^{P^*}x)|^{qp/q} dx\Big)^{q/p} \Big(\int |\hat{\varphi}(t^{P^*}x)|^{(2-q)\frac{p}{p-q}} dx\Big)^{\frac{p-q}{p}} \frac{dt}{t}.$$

Moreover a simple computation shows that $\big(\int |\hat{\varphi}(t^{P^*}x)|^r dx\big)^{1/r} = O(t^{-trP/r})$. This fact together with the Haussdorf-Young inequality shows that

$$\Big(\int |\hat{f}(x)|^q dx\Big)^{1/q} \leqslant C \Big\{\int_0^\infty (\|f * \varphi_t\|_{p'} t^{-trP(1/q-1/p)})^q \frac{dt}{t}\Big\}^{1/q},$$

whence the proof would be completed if we could show that the expression on the right is dominated by the integral from 0 to 1.

That this is the case is the content of the next lemma, which also explains the equivalence for $P = I$ between the $\Lambda(B, X)$ introduced in [4], and their present version.

6.8 LEMMA. *Let* $\tilde{X}$ *be an r-lattice on* $(0, \infty)$, *i.e.*,

$$f(t) \to \int_0^t f(s)(t/s)^p \frac{ds}{s} \quad \textit{and} \quad f(t) \to \int_t^\infty f(s)(t/s)^q \frac{ds}{s}$$

are bounded linear mappings of $\tilde{X}$ *into itself for* $p < r < q$. *Let* X *denote the restriction of* $\tilde{X}$ *to* $(0, 1)$. *Then* $\Lambda(B, \tilde{X}) = \Lambda(B, X)$, *the spaces having the same elements with equivalent norms.*

Proof. As in [4] we may show that $f(t) = \min(t^p, t^q) \in \tilde{X}$, $p < r < q$. Moreover, if $\chi(t)$ denotes the characteristic function of the interval $(1, \infty)$, then since $\chi(t) \leqslant f(t)$ also $\chi(t) \in \tilde{X}$ and $\|\chi\|_{\tilde{X}} \leqslant \|f\|_{\tilde{X}}$. Clearly $\Lambda(B, \tilde{X}) \subset \Lambda(B, X)$. Conversely,

$$\Big\|\int \tau_y u\, d\nu_t(y)\Big\|_B = (1-\chi(t))\Big\|\int \tau_y u\, d\nu_t(y)\Big\|_B + \chi(t)\Big\|\int \tau_y u\, d\nu_t(y)\Big\|_B$$
$$= I_1(t) + I_2(t).$$

Thus if $u \in \Lambda(B, X)$, then $I_1(t) \in \tilde{X}$ and since $\|I_2\|_{\tilde{X}} \leqslant C\|u\|_B \|\chi\|_{\tilde{X}}$ the lemma follows.

To complete this section we show some results related to [14]. For $\varphi(x) \geqslant 0$ we set $L^q(\varphi, R^n) = \{u\colon (\int |u(x)|^q \varphi(x)^q dx)^{1/q} < \infty\}$.

6.9 THEOREM. *For* $1 < p \leqslant 2$, *we have*

i) $\mathfrak{F}\colon \Lambda[\alpha; p, p'] \to L^{p'}(\varrho^{*\alpha}, R^n)$.

ii) $\mathfrak{F}\colon L^p(\varrho^{*\alpha}, R^n) \to \Big(t^\alpha L^p\Big(0, 1; \frac{dt}{t}\Big)\Big)(L^{p'}(R^n))$.

iii) $\mathfrak{F}\colon L^p(R^n) \to \Lambda[0; p', p]$.

Proof. To define the spaces $\Lambda(B, X)$ we may choose $\varphi \in \mathscr{S}(R^n)$ radial, $\operatorname{supp}\hat{\varphi} \subset \{1/2 < \varrho^*(x) < 4\}$ and $|\hat{\varphi}(x)| > C > 0$ for $\{1 < \varrho^*(x) < 2\}$. We do so.

Let $u * \varphi_t \in (t^\alpha L^{p'}(0, 1; dt/t))(L^p(R^n)) = \Lambda$. Then

$$\|u * \varphi_t\|_\Lambda = \Big(\int_0^1 (t^{-\alpha}\|u * \varphi_t\|_p)^{p'} \frac{dt}{t}\Big)^{1/p'}$$
$$\geqslant \Big(\int_0^1 (t^{-\alpha}\|\hat{u}(x)\,\hat{\varphi}(t^{P^*}x)\|_{p'})^{p'} \frac{dt}{t}\Big)^{1/p'}$$
$$= \Big(\int |\hat{u}(x)|^{p'} \int_0^1 t^{-\alpha p'} |\hat{\varphi}(t^{P^*}x)|^{p'} \frac{dt}{t}\, dx\Big)^{1/p'}$$
$$\geqslant C\Big(\int |\hat{u}(x)|^{p'} \int_{1/\varrho^*(x)}^{2/\varrho^*(x)} t^{-\alpha p'} \frac{dt}{t}\, dx\Big)^{1/p'}$$
$$= C\Big(\int |\hat{u}(x)|^{p'} \varrho^*(x)^{\alpha p'} dx\Big)^{1/p'}.$$

This proves a slightly more general statement than i).

Likewise

$$\int |u(x)|^p \varrho^*(x)^{ap}\,dx \geqslant C\int |u(x)|^p \int\limits_{1/\varrho^*(x)}^{2/\varrho^*(x)} t^{-ap}\frac{dt}{t}\,dx$$

$$\geqslant C\int |u(x)|^p \int\limits_0^1 |\hat{\varphi}(t^{P^*}x)|^p t^{-ap}\frac{dt}{t}\,dx$$

$$= C\int\limits_0^1 \left(t^{-a}\|u(x)\hat{\varphi}(t^{P^*}x)\|_p\right)^p\frac{dt}{t}$$

$$\geqslant C\int\limits_0^1 (t^{-a}\|\hat{u}*\varphi_t\|_{p'})^p\frac{dt}{t}.$$

This proves ii) and iii) (setting $a = 0$).

7. Singular integrals. We shall now consider the continuity properties of a suitable class of singular integrals acting on the spaces $\Lambda(B, X)$. We begin by proving two lemmas.

7.1. LEMMA. *Assume all the hypothesis of Theorem 4.8 hold, with the exception of the choice of the functions φ and ψ_t used to define the mapping S which will now be replaced by $\varphi^t(x)$ and $\psi^t(x)$ with the following properties:*

$$\varphi^t(x) = \varphi * m^t(x),\ \varphi\in A_k \quad \textit{and}\ \|m^t\|_1 \leqslant C,$$

and

$$\psi^t(x) = \psi_t * n^t(x),\ \psi\in A_k \quad \textit{and}\ \|n^t\|_1 \leqslant C.$$

Then S defines a continuous mapping from $B\oplus X(B)$ into $\Lambda(B, \gamma X)$.

Proof. We notice that it suffices to show that the equivalent to Lemma 4.9 holds. Thus, we would like to prove

$$\|\psi^t * \mu_s\|_1 = O\big(\min(1, (s/t)^k)\big)$$

and

$$\|\varphi^t * \mu_s\|_1 = O(s^k).$$

But by 4.9

$$\|\psi^t * \mu_s\|_1 \leqslant \|n^t\|_1\|\psi_t * \mu_s\|_1 = O\big(\min(1, (s/t)^k)\big)$$

and

$$\|\varphi^t * \mu_s\|_1 \leqslant \|m^t\|_1\|\varphi * \mu_s\|_1 = O(s^k),$$

and the proof is completed.

7.2. LEMMA. *Let $\psi\in\mathscr{S}(R^n)$ be such that $\psi = \hat{\varphi}$ with $\varphi\in C_0^\infty(R^n)$ and* $\operatorname{supp}\varphi \subset \{1/k \leqslant |x| \leqslant k\}$ *for some $k\in Z_+$. Then $\psi = \eta*\zeta$, where $\eta\in\mathscr{S}(R^n)$, $\zeta\in C_0^\infty(R^n)$ and $\int\zeta(x)\,dx = 0$.*

Proof. Let $0 \not\equiv \theta(x)\in C_0^\infty(R^n)$ be a spherically symmetric function with the additional properties that $\theta(x)$ and $\hat{\theta}(x)$ are real valued and $\int\theta(x)\,dx = 0$. An argument similar to that of Theorem 4.6 shows that there exist $\delta > 0$ and $m\in Z_+$ such that $g_m(x) > \delta$ for $1/2k \leqslant |x| \leqslant 2k$, where

$$g_m(x) = \left(\int\limits_{1/m}^m \theta_t*\theta_t\frac{dt}{t}\right)^{\wedge}(x) = \int\limits_{1/m}^m |\hat{\theta}(tx)|^2\frac{dt}{t}.$$

Now $g_m\in M$ because $\hat{g}_m\in C_0^\infty$ and $\varphi = (\varphi/g_m)\cdot g_m$ (where $\varphi/g_m = 0$ whenever $\varphi = 0$).

Thus $\psi = \hat{\varphi} = (\varphi/g_m)^{\wedge}*\hat{g}_m = \eta*\zeta$, with $\eta\in\mathscr{S}$, $\zeta\in C_0^\infty$ and $\int\zeta = 0$.

Having completed the preliminaries we pass to the theory of singular integrals.

7.3. THEOREM. *Let $k\in\mathscr{S}'(R^n)$ be defined by $(k,\varphi) = $ p.v. $\int k(x)\varphi(x)\,dx$, where $k(x)$ coincides with a locally integrable function away from the origin and which satisfies:*

i) *For $0 < r < R$, $\Big|\int\limits_{r<\varrho(x)<R} k(x)\,dx\Big| \leqslant C$ and $\int\limits_{r<\varrho(x)<1} k(x)\,dx$ converges as $r\to 0$.*

ii) *For $R > 0$, $\int\limits_{\varrho(x)<R} \varrho(x)|k(x)|\,dx \leqslant C\cdot R$.*

iii) $\int\limits_{\varrho(x)\geqslant 4\varrho(y)} |k(x-y)-k(x)|\,dx \leqslant C$ *for $y\in R^n$.*

Then the convolution operator K: $f\to k*f$ defined for $f\in C_0^\infty(R^n)$ satisfies

a) $\|Kf\|_2 \leqslant C\|f\|_2$, *and thus K can be extended to a bounded operator on $L^2(R^n)$.*

b) *If $\operatorname{supp} f \subset \{\varrho(x) < \lambda\}$ and $\int f(x)\,dx = 0$, then Kf_t is a C^∞ function and $\|Kf_t\|_1 \leqslant C(\lambda)(\|f\|_1 + \|f\|_2)$, where $C(\lambda) = C(1+\lambda^{\operatorname{tr}P/2})$.*

Proof. a) is well known ([4], [12], [16], [21]).

b) Note that $\operatorname{supp} f_t \subset \{\varrho(x) < \lambda t\}$, $\|f_t\|_1 = \|f\|_1$ and $\|f_t\|_2 = t^{-\operatorname{tr}P/2}\|f\|_2$, for $t > 0$.

We then have

$$\|Kf_t\|_1 = \int\limits_{\varrho(x)\leqslant 4\lambda t} |Kf_t(x)|\,dx + \int\limits_{\varrho(x)>4\lambda t} |Kf_t(x)|\,dx = I_1 + I_2.$$

By Hölder's inequality and a) we see that

$$I_1 \leqslant \Big(\int\limits_{\varrho(x)\leqslant 4\lambda t} |Kf_t(x)|^2\,dx\Big)^{1/2}(4\lambda t)^{\operatorname{tr}P/2} \leqslant C\|f_t\|_2(4\lambda t)^{\operatorname{tr}P/2} = C\lambda^{\operatorname{tr}P/2}\|f\|_2.$$

Likewise

$$I_2 = \int\limits_{\varrho(x)>4\lambda t} \Big|\int k(x-y)f_t(y)\,dy\Big|\,dx$$

$$= \int\limits_{\varrho(x)>4\lambda t} \Big|\int (k(x-y)-k(x))f_t(y)\,dy\Big|\,dx$$

$$\leqslant \int\limits_{\varrho(y)<\lambda t} |f_t(y)| \int\limits_{\varrho(x)>4\lambda t} |k(x-y)-k(x)|\,dx\,dy \leqslant C\|f\|_1.$$

Whence

$$\|Kf_t\|_1 \leqslant C\lambda^{trP/2}\|f\|_2 + C\|f\|_1 \leqslant C(\lambda)(\|f\|_1 + \|f\|_2),$$

as we wished to show.

Singular integrals. For $k(x)$ as in Theorem 7.3 and $u \in \Lambda(B, X)$ we define the *singular integral* Ku as follows

$$\text{(7.4)} \qquad (Ku, v) = \text{p.v.} \int (\tau_y u, v) k(y)\, dy = \lim_{\varepsilon\to 0+} \int_{\varepsilon<\varrho(y)<1/\varepsilon} (\tau_y u, v) k(y)\, dy, \quad \text{for } v \in V.$$

The next theorem is better understood if we recall the following remark due to Taibleson ([27], page 928). The Riesz transforms R_i defined by $(R_i u)\hat{}\,(x) = (x_i/|x|)\hat{u}(x)$ are not bounded mappings of $\Lambda(\alpha; p, \infty)$ into itself for $p = 1, \infty$.

7.5 Theorem. *Let $A = U^*$, $B = V^*$ and $C = W^*$ be complex Banach spaces such that $V \cap W$ is dense in both V and W and if $\varphi(y)$ is the function defined in Theorem 4.6, then $\int \tau_y u \varphi(y)\, dy$: $B \to C$ continuously. Further assume that $\tilde{\tau}_z$ is defined on U and $V \cap W$ with bounded adjoint τ_z acting on A, B and C. If K: $A \to B$ continuously, then*

i) *$K\tau_z = \tau_z K$ and $(\tau_z Ku, v)$ is a bounded function of z for $u \in A$, $v \in V$.*

ii) *K: $A \cap \Lambda(C, X) \to B \cap \Lambda(C, X)$ continuously.*

Proof. Observe that

$$(K\tau_z u, v) = \text{p.v.} \int (\tau_y \tau_z u, v) k(y)\, dy = \text{p.v.} \int (\tau_y u, \tilde{\tau}_z v) k(y)\, dy = (Ku, \tilde{\tau}_z v) = (\tau_z Ku, v).$$

Since for $u \in A$, $\|Ku\|_B \leqslant C\|u\|_A$ we have

$$|(\tau_z Ku, v)| \leqslant \|\tau_z Ku\|_B \|v\|_V \leqslant C\|u\|_A \|v\|_V.$$

This proves i).

To show ii) we apply the representation Theorem 4.6 to $Ku \in B$, for $u \in A$ and $v \in V \cap W$, thus obtaining

$$(Ku, v) = \left(\int \tau_y Ku \varphi(y)\, dy, v\right) + \lim_{\varepsilon\to 0} \int_\varepsilon^1 \left(\int\int (\tau_{z+y} Ku, v)\, d\nu_t(z) \psi_t(y)\, dy\right) \frac{dt}{t}.$$

The inner integral of the second summand is absolutely convergent by i) and thus inverting the order of integration we obtain

$$\int\int (\tau_{z+y} Ku, v)\, d\nu_t(z) \psi_t(y)\, dy = \int\int (K\tau_y u, \tilde{\tau}_z v) \psi_t(y)\, dy\, d\nu_t(z).$$

Moreover

$$\int (K\tau_y u, \tilde{\tau}_z v) \psi_t(y)\, dy = \int \text{p.v.} \int (\tau_x u, \tilde{\tau}_z v) k(x-y)\, dx\, \psi_t(y)\, dy = \int (\tau_x u, \tilde{\tau}_z v) \text{p.v.} \int k(x-y) \psi_t(y)\, dy\, dx = \int (\tau_{z+x} u, v)(K\psi_t)(x)\, dx$$

where now $(K\psi_t)(x)$ is the operator defined in 7.3.

In view of Lemma 7.2, $\psi = \eta * \zeta$ where $\eta \in \mathscr{S}$, $\zeta \in C_0^\infty(R^n)$ and $\int \zeta(x)\, dx = 0$. Also by Theorem 7.3

$$\|K\zeta_t\|_1 \leqslant c(\|\zeta\|_1 + \|\zeta\|_2) = C.$$

Now since $K(f*g) = f*Kg$ whenever f, g and $Kg \in L^1$ we have that

$$K\psi_t = K(\eta_t * \zeta_t) = \eta_t * K\zeta_t.$$

Whence

$$(Ku, v) = \left(\int \tau_y Ku \varphi(y)\, dy, v\right) + \lim_{\varepsilon\to 0} \int_\varepsilon^1 \int (U(y, t), v)(\eta_t * K\zeta_t)(y)\, dy \frac{dt}{t}$$

and Lemma 7.1 obtains the desired conclusion.

In case the kernel k satisfies a homogeneity condition in the sense explained below the situation is more easily handled as shown by the next theorem.

7.6 Theorem. *Let $k(x)$ be a homogeneous kernel of degree $-trP$, i.e., $k(t^P x) = t^{-trP} k(x)$ for $x \in R^n - (0)$ and $t > 0$, such that*

i) *$k \in L^1_{\text{loc}}(R^n - (0))$, $k \in L^1(S^{n-1})$ and $\int_{S^{n-1}} k(x')(Px', x')\, dx' = 0$.*

ii) $$\int_{\varrho(x)>4\varrho(y)} |k(x-y) - k(x)|\, dx \leqslant C \quad \text{for } y \in R^n.$$

iii) *$\hat{k}(x)$ coincides in $R^n - (0)$ with a C^∞ function for which $\hat{k}(t^{P^*} x) = \hat{k}(x)$.*

By regarding $k(x)$ as the distribution p.v. $k(x)$ the convolution operator K: $f \to k*f$ defined for functions $f(x)$ with integrable Fourier transforms may be written

$$Kf(x) = \int e^{2\pi i(x,z)} \hat{k}(z) \hat{f}(z)\, dz.$$

With K thus defined for $f \in \mathscr{S}$ and Ku defined by (7.4) for $u \in \Lambda(C, X)$ we have

a) *If ψ is the function constructed in Theorem 4.6, then $K\psi_t = \eta_t$ with $\eta \in \mathscr{S}$.*

b) *For A, B, C as in Theorem 7.5, K maps $A \cap \Lambda(C, X) \to B \cap \Lambda(C, X)$ continuously.*

Proof. Since b) follows readily from a), we only show a) by means of this remark:

$$K\psi_t(x) = \int e^{2\pi i(x,z)} \hat{k}(z) \hat{\psi}(t^{P^*} z)\, dz = t^{-trP} \int e^{2\pi i(t^{-P}x, z)} \hat{k}(z) \hat{\psi}(z)\, dz = \eta_t(x).$$

Furthermore $\eta \in \mathscr{S}$ since $\hat{\eta} \in C_0^\infty$.

8. Multipliers.

8.1. The theory developed in Section 7 may be viewed as describing the behaviour of an important class of singular integral operators K which are translation invariant (i.e., $K\tau_y = \tau_y K$) if we select τ_y to be the translations in R^n defined in (6.1). In this section we shall develop the theory of arbitrary translation invariant operators M, or *multiplier operators*, acting on the $\Lambda(B, X)$ spaces of functions and distributions on Euclidean space. These results, together with those of Sections 5 and 6 provide numerous interesting applications of the theory developed in this paper.

In this section, then, we assume that the situation described in Section 5 holds: the spaces B are Banach spaces of functions or distributions in Euclidean space and the τ_y are the translations.

8.2. A function $m(x)$, continuous and bounded for $x \in R^n - (0)$, is said to be a *multiplier* of type (A, B), $\mathcal{S}(R^n)$ being dense in A, if the mapping M defined by

$$(Mu)\hat{}\,(x) = m(x)\hat{u}(x), \quad u \in \mathcal{S},$$

satisfies $\|Mu\|_B \leqslant C\|u\|_A$, C independent of u, and it may thus be extended to a bounded mapping M of A into B. Clearly $M\tau_y = \tau_y M$.

The following theorem, although simple to prove, is indeed of interest.

8.3. THEOREM. *Let $A = U^*$, $B = V^* \subset \mathcal{S}'(R^n)$. Let $M\colon A \to B$ be the adjoint of a continuous mapping $\tilde{M}\colon V \to U$. If τ_y maps continuously A into itself and B into itself, then $M\colon \Lambda(A, X) \to \Lambda(B, X)$ continuously.*

Proof. Since $Mu \in B$ whenever $u \in A$ it will suffice to show that $\int \tau_y Mu\psi_t(y)\,dy \in X(B)$ for ψ as in Theorem 4.6 and $u \in \Lambda(X, A)$. For $v \in V$ we have

$$\int (\tau_y Mu, v)\psi_t(y)\,dy = \int (M\tau_y u, v)\psi_t(y)\,dy$$
$$= \int (\tau_y u, \tilde{M}u)\psi_t(y)\,dy = \Big(M\int \tau_y u\psi_t(y)\,dy, v\Big).$$

Thus

$$\Big\|\int \tau_y Mu\,\psi_t(y)\,dy\Big\|_B = \Big\|M\int \tau_y u\psi_t(y)\,dy\Big\|_B \leqslant C\Big\|\int \tau_y u\psi_t(y)\,dy\Big\|_A \in X$$

and the proof is completed.

To prove the next Theorem recall that Lemma 7.2 allowed us to factor $\psi = \eta * \zeta$, with $\eta \in \mathcal{S}$ and $\operatorname{supp}\eta \subset \{1/k < \varrho^*(z) < k\}$. Also φ will be the function defined in Theorem 4.6.

8.4 THEOREM. *Let A, B be as in Theorem 8.3 and $C = W^* \subset \mathcal{S}'(R^n)$ be such that $\int \tau_y u\varphi(y)\,dy\colon B \to C$ continuously. Suppose further that $m(x)(\mathfrak{F}^{-1}\eta)(t^{P^*}x) = \mathfrak{F}^{-1}(f^t)(x)$, with $\|f^t\|_1 \leqslant C$ for $0 < t < 1$. If $\mathcal{S}$ is dense in $A \cap C$ and $V \cap W$ and $m(x)$ is a multiplier of type (A, B), then $m(x)$ is a multiplier of type $\big(A \cap \Lambda(C, X), B \cap \Lambda(C, X)\big)$.*

Proof. Since $Mu \in B$ for $u \in A$, the representation Theorem 4.6 obtains, for $u \in \mathcal{S}$ and $v \in \mathcal{S}$,

$$M(u, v) = \Big(\int \tau_y Mu\varphi(y)\,dy, v\Big) + \lim_{\varepsilon \to 0}\int_\varepsilon^1 \int\int (\tau_{y+z}Mu, v)\,d\nu_t(z)\,\psi_t(y)\,dy\,\frac{dt}{t}$$
$$= I_1 + I_2.$$

We have that

$$|I_1| \leqslant \Big\|\int \tau_y Mu\varphi(y)\,dy\Big\|_C \|v\|_W \leqslant c\|Mu\|_B\|v\|_W \leqslant c\|u\|_A\|v\|_W.$$

Likewise for I_2 we have,

$$\int\int (\tau_{y+z}Mu, v)\,d\nu_t(z)\,\psi_t(y)\,dy = \int (Mu)\hat{}\,(y)\check{v}(y)\check{\nu}(t^{P^*}y)\check{\psi}(t^{P^*}y)\,dy$$
$$= \int \hat{u}(y)\check{v}(y)\check{\nu}(t^{P^*}y)(f^t)\check{}\,(y)\check{\xi}(t^{P^*}y)\,dy$$
$$= \int\int (\tau_{y+z}u, v)\,d\nu_t(z) f^t * \zeta_t(y)\,dy$$

and the result follows from Lemma 7.1.

Remark. Observe that the proof given above actually shows that if

$$m(x)\check{\eta}(t^{P^*}x) = (f^t)\check{}\,(x) \quad \text{with } t^{-r}\|f^t\|_1 \leqslant C \text{ for } 0 < t < 1,$$

then $m(x)$ is a multiplier of type $\big(A \cap \Lambda(C, X), B \cap \Lambda(C, t^r X)\big)$.

8.5. COROLLARY. *Let A, B, C be as in Theorem 8.4 and let $m(x)$ be $\left[\frac{n}{2}\right]+1$ continuously differentiable with*

$$\sum_{|N| \leqslant \left[\frac{n}{2}\right]+1} \int_\Omega |(\partial/\partial x)^N m(t^{P^*}x)|^2\,dx \leqslant C, \quad t \geqslant 1,$$

where $\Omega = \operatorname{supp}\eta$.

Then $m(x)$ is a multiplier of type $\big(A \cap \Lambda(C, X), B \cap \Lambda(C, X)\big)$.

Proof. The known facts that L^1 norms are invariant under dilations $x \to t^{P^*}x$ performed in the space of the Fourier transforms and that a sufficient condition (a sharper, but more involved condition to describe is Theorem 6.7 with $q = 1$) for $g = \hat{f}$, $f \in L^1(R^n)$, is that $\sum_{|N| \leqslant \left[\frac{n}{2}\right]+1} \int |(\partial x)^N g(x)|^2\,dx \leqslant C$, applied to Theorem 8.4 obtain the conclusion.

To finish this section and this paper we give a simple but nevertheless interesting application of the results developed to the theory of pseudo-differential operators as developed in [5]: this we do to show how to deal with non-translation invariant singular integral and other important classes of operators which are known to preserve Lipschitz classes with $P = I$.

8.6 *A pseudo-differential operator* Q is an operator of the form

$$Qu(x) = \sum_{j=1}^{k} \int e^{2\pi i(x,z)} q_j(x, z)\hat{u}(z)\,dz$$

where $u \in \mathscr{S}$ and

i) The functions $q_j(x, z)$ are bounded.

ii) For $|z| > C$, $q_j(x, z)$ coincides for each x with a homogeneous function of degree $-d_j$, $0 \leqslant d_j \leqslant d_{j+1}$.

iii) $|(\partial/\partial x)^M (\partial/\partial_z)^N q_j(x, z)| \leqslant C_{M,N}$ for all multi-indices N and $|M| \leqslant 2([d_k]+1)-[d_j]$.

Since the general case will follow from addition (this is explained in ([5], page 95) we restrict ourselves to the case

$$q(x, z) = a(x)\,Y(z)\,\psi(z)\,|z|^{-d}$$

where

$$\psi(z) = |z|^d \int_0^1 \varphi(tz)\,t^d \frac{dt}{t}, \quad \text{with } \varphi \in C^\infty$$

a spherically symmetric function supported in $1/2 \leqslant |z| \leqslant 1$ and $Y(z) \in C^\infty(R^n - (0))$ a homogeneous function of degree zero. This means that $q(x, z) = a(x)\int_0^1 \eta(tz)\,t^d \frac{dt}{t}$, for $\eta(z) = Y(z)\varphi(z)$. (See [8], page 314).

Thus the operator Q in question has the form

$$Qu(x) = a(x)\int_0^1 t^d \int h_t(y-x)\,u(y)\,dy \frac{dt}{t}, \qquad h = \hat{\eta}.$$

Now if $d > 0$ this integral is absolutely convergent and setting $k(x) = \int_0^1 h_t(-x)\,t^d \frac{dt}{t}$ we finally obtain

$$Qu(x) = a(x)\int k(x-y)\,u(y)\,dy,$$

the representation still being valid for $d = 0$ provided that $\int_{S^{n-1}} Y(x')\,dx' = 0$ and that the integral be interpreted in the sense of Cauchy's principal value.

We have discussed the convolution operators which preserve the $\Lambda(B, X)$ spaces and since these spaces have also been fully described it is now possible to give sufficient conditions on $a(x)$ so that multiplication by it preserves the class in consideration. (See [25] for this last remark in the setting of Sobolev spaces.) The case $P = \operatorname{diag}(a_1, \ldots, a_n)$ may be treated similarly once the appropriate mixed homogeneity conditions are required as in [10]. The details of this closing remark are left for the reader to verify.

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Semigroup algebras having regular multiplication

by

N. J. YOUNG (Glasgow)

Abstract. We obtain both algebraic and topological characterizations of those locally compact semi-topological semigroups whose measure algebras have regular multiplication (in the sense of Arens). The condition obtained also characterizes regularity of multiplication in the l_1-algebra of the semigroup.

A Banach algebra is said to have *regular multiplication* if the two Arens products on its bidual coincide. It is well known from the work of Day [2] that l_1-algebras of suitable semigroups afford examples of irregular multiplication; in fact it suffices to take any semigroup admitting two distinct invariant means. This is not the only way the multiplication can be irregular: Civin and Yood [1] find wide classes of groups whose algebras comprise further examples, and in [9] it is shown that the multiplication is irregular in the L^1-algebra of any infinite locally compact Hausdorff group. However, the Banach algebra $M(S)$ of finite regular Borel measures on an infinite locally compact Hausdorff semigroup S *can* have regular multiplication, as is shown by trivial examples: take $xy = y$ for all x, y, or take the multiplication in S to be constant. In the present note an algebraic characterization is given of those locally compact semigroups whose measure algebras have regular multiplication. Some related results can be found in a paper of N. Macri [5].

We are concerned throughout with *semi-topological* semigroups—that is, semigroups in which multiplication is assumed only to be separately continuous. Basic facts about measure algebras on semigroups can be found in [3]. We begin by giving the purely topological content of the characterization. βX denotes the Stone–Čech compactification of the completely regular space X.

1. An extension theorem. If X and Y are completely regular spaces and Z is compact Hausdorff, in order that a separately continuous mapping $f\colon X\times Y\to Z$ should admit a separately continuous extension $f'\colon \beta X\times\beta Y\to Z$ it is necessary and sufficient that, for all pairs $(x_n), (y_m)$ of sequences in X, Y respectively, the double sequence $(f(x_n, y_m))$ should have a double cluster point in Z ([7], Theorem 1). Here $w\in Z$ is called