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A Guichard theorem on connected monothetic groups

by

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Abstract. A theorem of Halmos and Samelson on the Haar measure of the set of generators of a monothetic group is deduced from a new theorem of the Guichard type on the representation of functions as sums of finite differences.

The purpose of this note is to show that Theorem III of Halmos and Samelson [2], on the Haar measure of the set of generators of a monothetic group, is an immediate corollary of a new theorem of the Guichard type (see [1]) on the representation of functions as sums of finite differences. Our first theorem of this general Guichard type was announced in [3] and proved in [4] for C^∞ functions and distributions on $\boldsymbol{R}^n$. Additional theorems of a Guichard type were proved in [5] for L^2-functions on compact Abelian groups.

Let G denote a compact and connected Abelian group with (discrete Abelian) dual group Γ. Let ε denote a fixed positive real number less than one, and let $A_\varepsilon(G)$ denote the linear space of all continuous complex-valued functions on G whose Fourier series, $f(x) = \sum_{\xi} \hat{f}(\xi)\,\xi(x)$ ($\xi \in \Gamma$ and $\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}\,dx$), are "better than absolutely convergent" in the sense that $\sum_{\xi\in\Gamma} |\hat{f}(\xi)|^\varepsilon < \infty$.

THEOREM 1. *For each f in $A_\varepsilon(G)$ with $\hat{f}(1) = 0$ there is a set G_f of Haar measure 1 contained in G, such that for each a in G_f there is a function g in $A_\varepsilon(G)$ satisfying*

$$f(x) = g(x) - g(x-a) \quad \textit{for all } x \in G. \tag{1}$$

Proof. This theorem corresponds to Theorem 1 of [5] with the parameters p and m used there set equal to ε and 1, respectively. Since the restrictions $p < 1$ and $m = 1$ allow some technical simplifications in the proof, and since we wish to show as clearly as possible exactly

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what is involved in our deduction of Theorem III of Halmos and Samelson [2], we give here the complete proof of our present Theorem 1.

If $h\colon H\to K$ is a continuous homomorphism of a compact Abelian group H *onto* a compact Abelian group K, and if $u\in L^1(K)$, then

$$\int_H (u\circ h)(x)\,dx = \int_K u(y)\,dy. \tag{2}$$

This is true because the left-hand side of (2) defines a normalized positive linear translation-invariant functional on $L^1(K)$, which is the familiar characterization of the normalized Haar integral on K. Now each ξ in $\Gamma\sim\{1\}$ maps G *onto* the circle group T, because $\xi(G)$ is a connected and compact non-trivial subgroup of T. Furthermore, the function $u(y)\equiv|1-y|^{-\varepsilon}$, for $y\in T$, belongs to $L^1(T)$ because $\int_T u(y)\,dy = \int_0^1 |1-e^{2\pi i\theta}|^{-\varepsilon}d\theta \leqslant 2^{1-2\varepsilon}\int_0^{1/2}\theta^{-\varepsilon}d\theta$. Consequently we may apply (2) to the case that $H=G$, $K=T$, $h=\xi$, and $u(y)\equiv|1-y|^{-\varepsilon}$ to obtain

$$I\equiv\int_G |1-\xi(x)|^{-\varepsilon}dx = \int_0^1 |1-e^{2\pi i\theta}|^{-\varepsilon}d\theta<\infty,$$

which shows that the value of the integral I is finite and independent of ξ in $\Gamma\sim\{1\}$. It now follows from the monotone convergence theorem for the Haar integral on G that the function

$$F(x)\equiv\sum_{\xi\in\operatorname{supp}\hat f}|\hat f(\xi)|^{\varepsilon}\cdot|1-\xi(x)|^{-\varepsilon}$$

is integrable with finite integral,

$$\int_G F(x)\,dx = I\cdot\sum_{\xi}|\hat f(\xi)|^{\varepsilon},$$

and, in particular, the series defining $F(x)$ converges for almost all x in G, say for x in G_f.

Now for each a in G_f we define $\hat g(\xi)=\hat f(\xi)/[1-\overline{\xi(a)}]$ for $\xi\in\operatorname{supp}\hat f$ and $\hat g(\xi)=0$ for $\xi\notin\operatorname{supp}\hat f$. Then $g(x)\equiv\sum_{\xi\in\Gamma}\hat g(\xi)\xi(x)$ belongs to $A_\varepsilon(G)$ and satisfies

$$\hat f(\xi)=\hat g(\xi)\cdot\big(1-\overline{\xi(a)}\big)\quad\text{for all }\xi\in\Gamma. \tag{3}$$

But Equation (3) is equivalent to Equation (1), so Theorem 1 is proved.

For each a in G let $\mathscr{A}(\Gamma,a)$ denote $\{\xi\in\Gamma\colon \xi(a)=1\}$, the annihilator of a. Recall that $\mathscr{A}(\Gamma,a)$ is a subgroup of Γ and is topologically isomorphic to the dual group of the quotient group $G/[a]$, where $[a]$ denotes the closed subgroup of G which is generated by a.

COROLLARY 1. *Let G denote a compact connected Abelian group with dual group Γ, and let ε denote a fixed real number satisfying $0<\varepsilon<1$. Then for each function f in $A_\varepsilon(G)$ with $\hat f(1)=0$, there is a set G_f of Haar measure 1 contained in G, such that*

$$\mathscr{A}(\Gamma,a)\subset \complement\operatorname{supp}\hat f\quad\textit{for all } a\in G_f.$$

Proof. Let G_f be as in Theorem 1 so that for each a in G_f there is a g in $A_\varepsilon(G)$ satisfying (3). It is immediate that the relation $\xi\in\mathscr{A}(\Gamma,a)$ entails the relation $\hat f(\xi)=0$.

COROLLARY 2. (Theorem III of [2].) *Let G denote a compact connected Abelian group satisfying the second axiom of countability. Then almost every member of G generates a dense cyclic subgroup of G.* (Groups containing a dense cyclic subgroup are called "*monothetic*".)

Proof. Since G has a countable basis of open sets, the (discrete) dual group Γ is countable. Consequently there exist functions f in $A_\varepsilon(G)$ such that $\operatorname{supp}\hat f=\Gamma\sim\{1\}$. But then Corollary 1 applies to such an f to yield a set G_f of Haar measure 1 contained in G such that for every a in G_f

$$\{1\}\subset\mathscr{A}(\Gamma,a)\subset\complement\operatorname{supp}\hat f=\{1\}.$$

That is, for almost every element a of G, the dual group of $G/[a]$ is the trivial group $\{1\}$, so that $[a]=G$.

Let S denote $\{a\in G\colon a=[G]$. Then, since $G\sim S=\bigcup\{\mathscr{A}(G,\xi)\colon \xi\in\Gamma\sim\{1\}\}$ and since each $\mathscr{A}(G,\xi)$ is a proper closed nowhere-densed subgroup of G, it follows that the set S is residual. This proves a theorem stated on page 304 of [6].

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