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A characterization of ω by block extensions

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Abstract. The following characterization of the space ω is given: A nuclear Fréchet space is isomorphic to ω if and only if it has a basis such that every block basic sequence of every permutation of that basis has a block extension. In fact it is shown that it suffices to consider only blocks of length $\leqslant 2$.

In studying nuclear Fréchet spaces it is natural to try to investigate situations which are familiar in Banach space theory and to see if results can be carried over. It is particularly interesting to see a situation in which a simple imitation is *a priori* impossible. This is exactly what occurs if one considers extensions of block basic sequences (see below for definitions). In order to point this out we mention the following two well-known results.

THEOREM OF ZIPPIN [6]. *In a Banach space, every block basic sequence has a block extension.*

THEOREM OF LINDENSTRAUSS AND TZAFRIRI [3]. *A Banach space is isomorphic to $l_p (1 \leqslant p < \infty)$ or c_0 iff it has an unconditional basis such that every block basic sequence of every permutation of this basis generates a complemented subspace.*

Since every basic sequence which can be extended to an *unconditional* basis in a Fréchet space generates a complemented subspace, the second theorem provides many examples in which the first theorem cannot be improved to assert the existence of an unconditional extension when, say, the original basis is unconditional. Actually, the first specific example of this situation was given by Pełczyński [4].

On the other hand, since every basis in a nuclear Fréchet space is unconditional, it follows that at least one of the above two theorems must be false in this context. Indeed, the second author [5] has shown that it is Zippin's theorem which does not carry over. He also pointed out that it does hold for the space ω and asked if Zippin's theorem characterizes ω among the nuclear Fréchet spaces.

It is our purpose in this paper to answer that question in the affirmative. Specifically, we prove, THEOREM. *A nuclear Fréchet space is isomorphic to ω iff it has a basis such that every block basic sequence, with blocks of length $\leqslant 2$, of every permutation of that basis has a block extension.*

It should be observed that this theorem is only vaguely analogous to the Lindenstrauss–Tzafriri theorem. It is "better" than that result in that one obtains the block basic sequence with very small blocks, but it is "worse" in that block extensions (in nuclear Fréchet spaces) are very special compared to complemented subspaces. It would be nice to have an analog of this result for nuclear Fréchet spaces but the first difficulty is that it is unclear what spaces to choose for the role of l_p. It seems that finite type power series spaces will not work, but other than that, nothing seems to be known.

Definitions and preliminary results. Let N be the set of positive integers. The term *basis* will always mean *Schauder basis*. The term *sequence* will indicate an infinite sequence of scalars (real or complex numbers).

If (e_n) is a basis in a Fréchet space, E, $0 = p_0 < p_1 < \ldots$ is a sequence of indices and (t_n) is a sequence such that

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} t_i e_i \neq 0, \quad n \in N,$$

then it is easy to see that (y_n) is a basis for the subspace it generates. We call (y_n) a *block basic sequence* of the basis (e_n). The number $p_n - p_{n-}$ is the *length* of the nth block. If (y_n) is a subset of a basis (z_i) then we call (z_i) an *extension* of (y_n). If, in addition, $z_{p_n} = y_n$ and z_i is an element of the subspace generated by $e_{p_{n-1}+1}, \ldots, e_{p_n}$ for $p_{n-1} < i < p_{n'}$, $n \in N$, then we say that (z_i) is a *block extension* of (y_n).

Let ξ, η be sequences and A, B sets of sequences. We write

$$\xi \cdot \eta = (\xi_n \eta_n)_n,$$

$$\xi \cdot A = \{\xi \cdot \eta \colon \eta \in A\},$$

$$A \cdot B = \{\xi \cdot \eta \colon \xi \in A,\ \eta \in B\},$$

$$\frac{1}{\xi} = \left(\frac{1}{\xi_n}\right)_n, \text{ provided } \xi_n \neq 0 \ \forall\ n \in N,$$

$$A^\times = \Big\{\xi \colon \sum_{n \in N} |\xi_n \eta_n| < \infty \ \forall\ \eta \in A\Big\},$$

$$D(A, B) = \{\xi \colon \xi \cdot \eta \in B \ \forall\ \eta \in A\}.$$

The set $A^\times$ is called the *Köthe dual* of A. The set $D(A, B)$ is called the set of *diagonal transformations* or *multipliers* from A to B. We say that A is *perfect* if $A = A^{\times\times}$. If A, B are closed under coordinate addition and scalar multiplication and if they are perfect, then it is well known and easy to check that

$$D(A, B) = (A \cdot B^\times)^\times.$$

If N_1 is a subsequence of N and $t = (t_n)_{n \in N}$ is a sequence, then we write $t_{N_1} = (t_n)_{n \in N_1}$ for the restriction of t to N_1 and $A_{N_1} = \{t_{N_1} \colon t \in A\}$.

We denote by ω the nuclear Fréchet space of all sequences, equipped with the usual product topology.

If $(a^k) = (a_n^k)$ is an infinite matrix of scalars with $0 < a_n^1 < a_n^2 < \ldots$, then it determines an *echelon space of order* 1 given by $\lambda = \bigcap_k \frac{1}{a_k} \cdot l_1$. Under the seminorms $p_{k'}$ $k \in N$, where $p_k(\xi)$ is the l_1-norm of $a^k \cdot \xi$, the space λ is a Fréchet space. It is a nuclear Fréchet space iff $\forall\ k \in N\ \exists j \in N$ *such that* $a^k \in a^j \cdot l_1$. An echelon space is always perfect and we have

$$\lambda^\times = \bigcup_{k \in N} a^k \cdot l_\infty.$$

For other properties and generalizations of echelon spaces, see [2].

We now formulate the result from [5] which we shall use. All of the notation will be adhered to in the next section. Let λ be an echelon space of order 1. It is easy to see that the coordinate sequences e^n, $n \in N$ form a basis for λ. Let $I = (i_n)$, $J = (j_n)$ be two disjoint subsequences of N, let $\tilde{y}_n = t_n e^{i_n} + e^{j_n} \neq 0$, $n \in N$ and let $N_0 = \{n \colon t_n \neq 0\}$.

According to our definition, $(\tilde{y}_n)$ is not a block basic sequence of (e^n) but we can use it to construct a block basic sequence (y_k) of a permutation of (e^n) consisting of the vectors

$$\tilde{y}_n, \quad n \in N_0$$

$$e^{j_n}, \quad n \in N \sim N_0$$

$$e_{k}, \quad k \in N \sim (I \cup J).$$

It is then clear that these vectors can be reordered to form a block basic sequence of a permutation of (e^n) with blocks of length $\leqslant 2$. If necessary, we could be more specific about the ordering, but since we will consider only nuclear spaces and hence unconditional bases, this becomes unnecessary.

Now, using exactly the same methods as in [5], Section 3 (we need only use a slightly more general notation and keep track of the blocks of length 1), we may establish the following

EXISTENCE THEOREM. *The block basic sequence (y_k) has a block extension iff we can find a decomposition $N_0 = N_1 \cap N_2$ where $N_1 \cap N_2 = \emptyset$ and*

(1) $\forall k \in N \ \exists m \in N$ *and* $M > 0$

such that

(a) $\dfrac{1}{|t_n|}\dfrac{a^k_{j_n}}{a^m_{i_n}} \leqslant M, \quad n \in \boldsymbol{N}_1,$

(b) $|t_n|\dfrac{a^k_{i_n}}{a^m_{j_n}} \leqslant M, \quad n \in \boldsymbol{N}_2.$

Proof of theorem. We begin by translating the conclusion of the existence theorem into a slightly more convenient notation. Consider the echelon spaces μ, ν determined by the matrices (b^k_n), (c^k_n) respectively, where $b^k_n = a^k_{i_n}$, $c^k_n = a^k_{j_n}$. From elementary set theoretic properties of sequence spaces and their Köthe duals it follows that (1a) is equivalent to

$$\frac{1}{t_{\boldsymbol{N}_1}} \in \bigcap_k \bigcup_m \frac{b^m_{\boldsymbol{N}_1}}{c^k_{\boldsymbol{N}_1}}\cdot l_\infty = \bigcap_k \frac{1}{c^k_{\boldsymbol{N}_1}}\cdot \mu^\times_{\boldsymbol{N}_1} = \big(\bigcup_k c^k_{\boldsymbol{N}_1}\cdot \mu_{\boldsymbol{N}_1}\big)^\times$$
$$= \big(\mu_{\boldsymbol{N}_1}\cdot \bigcup_k c^k_{\boldsymbol{N}_1}\cdot l_\infty\big)^\times = \big(\mu_{\boldsymbol{N}_1}\cdot \nu^\times_{\boldsymbol{N}_1}\big)^\times = \big(D(\mu,\nu)\big)_{\boldsymbol{N}_1}.$$

Using a similar argument for (1b) we conclude that (1) is equivalent to

$$(1)' \qquad \frac{1}{t_{\boldsymbol{N}_1}} \in \big(D(\mu,\nu)\big)_{\boldsymbol{N}_1} \quad \text{and} \quad t_{\boldsymbol{N}_2} \in \big(D(\nu,\mu)\big)_{\boldsymbol{N}_2}.$$

The first step in the proof of our theorem is to show that $\boldsymbol{I}$ and $\boldsymbol{J}$ can always be chosen so that the above spaces of diagonal transformations have a particularly convenient form.

LEMMA 1. *Suppose that the matrix* (a^k_n) *satisfies the conditions that* $a^1_n = n$, $n \in \boldsymbol{N}$ *and for each* $k \in \boldsymbol{N}$,

$$\sup_{n\in \boldsymbol{N}} n\frac{a^k_n}{a^{k+1}_n} < \infty.$$

Then $\boldsymbol{I}$ *and* $\boldsymbol{J}$ *can be chosen such that* $D(\mu,\nu) = \nu$ *and* $D(\nu,\mu) = \nu^\times$.

Proof. Take $i_n = 2n-1$, $n \in \boldsymbol{N}$ and let j_n be any even integer such that $j_n > j_{n-1}$ and $a^1_{j_n} \geqslant a^n_{i_n}$, $n \in \boldsymbol{N}$. Then for any $k \in \boldsymbol{N}$,

$$\sup_{n\in \boldsymbol{N}} \frac{a^k_{i_n}}{a^1_{j_n}} < \infty.$$

Hence for any $m, k \in \boldsymbol{N}$ it follows from our hypotheses that

$$\sup_{n\in \boldsymbol{N}} \frac{a^m_{j_n}a^k_{i_n}}{a^{m+1}_{j_n}} \leqslant \sup_{n\in \boldsymbol{N}} \frac{a^k_{i_n}}{a^1_{j_n}}\sup_{n\in \boldsymbol{N}} \frac{a^m_{j_n}a^1_{j_n}}{a^{m+1}_{j_n}} < \infty.$$

We may then compute,

$$D(\nu,\mu) = (\nu\cdot\mu^\times)^\times = \left(\left(\bigcap_m \frac{1}{c^m}\cdot l_1\right)\cdot \bigcup_k b^k\cdot l_\infty\right)^\times = \left(\bigcup_k b^k\cdot\left(\bigcap_m \frac{1}{c^m}\cdot l_1\right)\right)^\times$$
$$= \bigcap_k \frac{1}{b^k}\cdot\big(\bigcup_m c^m\cdot l_\infty\big) = \bigcap_k\bigcup_m \frac{c^m}{b^k}\cdot l_\infty = \bigcap_k\bigcup_m \left(\frac{a^m_{j_n}}{a^k_{i_n}}\right)_n\cdot l_\infty$$
$$= \bigcap_k\bigcup_m \left(\frac{a^{m+1}_{j_n}}{a^k_{i_n}}\right)_n\cdot l_\infty \supset \bigcap_k\bigcup_m (a^m_{j_n})_n\cdot l_\infty = \nu^\times.$$

On the other hand, since $a^k_{i_n} \geqslant a^1_{i_n} = i_n \geqslant 1$, we have,

$$D(\nu,\mu) \subset \bigcap_k\bigcup_m (a^m_{j_n})_n\cdot l_\infty = \nu^\times.$$

For the second equation,

$$D(\mu,\nu) = (\mu\cdot\nu^\times)^\times = \bigcap_k\bigcup_m \frac{b^m}{c^k}\cdot l_\infty$$
$$= \bigcap_k\bigcup_m \left(\frac{a^m_{i_n}}{a^k_{j_n}}\right)_n\cdot l_\infty \supset \bigcap_k\bigcup_m \left(\frac{1}{a^k_{j_n}}\right)_n\cdot l_\infty$$
$$\supset \bigcap_k \frac{1}{c^k}\cdot l_1 = \nu,$$

and

$$\mu\cdot\nu^\times = \left(\bigcap_k \frac{1}{b^k}\cdot l_1\right)\cdot\big(\bigcup_m c^m\cdot l_\infty\big)$$
$$= \bigcup_m c^m\cdot\left(\bigcap_k \frac{1}{b^k}\cdot l_1\right) = \bigcup_m\left(\frac{1}{c^m}\cdot\bigcup_k b^k\cdot l_\infty\right)^\times$$
$$= \bigcup_m\left(\bigcup_k \frac{b^k}{c^m}\cdot l_\infty\right)^\times = \bigcup_m\bigcap_k \frac{c^m}{b^k}\cdot l_1$$
$$= \bigcup_m\bigcap_k \left(\frac{a^m_{j_n}}{a^k_{i_n}}\right)_n\cdot l_1 \supset \bigcup_m\bigcap_k (a^{m-1}_{j_n})_n\cdot l_1$$
$$\supset \bigcup_m (a^{m-2}_{j_n})_n\cdot l_\infty = \nu^\times,$$

so $D(\mu,\nu) = (\mu\cdot\nu^\times)^\times \subset \nu^{\times\times} = \nu$.

Next we show that when the spaces of diagonal transformations have this special form, then the required decomposition is not always possible.

LEMMA 2. *If we assume that* $c^1_n > 1$, $n \in \boldsymbol{N}$ and $\lim\limits_n \dfrac{c^k_n}{c^{k+1}_n} = 0$, $k \in \boldsymbol{N}$, *then there exists a sequence* $\xi = (\xi_n)_{n\in \boldsymbol{N}_0}$ *of positive numbers such that for any decomposition* $\boldsymbol{N}_0 = \boldsymbol{N}_1 \cup \boldsymbol{N}_2$, $\boldsymbol{N}_1 \cap \boldsymbol{N}_2 = \emptyset$ *it is not the case that both* $\dfrac{1}{\xi_{\boldsymbol{N}_1}} \in \nu_{\boldsymbol{N}_1}$ *and* $\xi_{\boldsymbol{N}_2} \in \nu^\times_{\boldsymbol{N}_2}$.

Proof. For each $n \in \boldsymbol{N}_0$ we construct a strictly increasing function $f_n\colon [0,\infty) \to [0,\infty)$ such that

(i) $f_n(0) = 0$,

(ii) $f_n(t) = \dfrac{1}{f_n\left(\frac{1}{t}\right)}, \quad t > 0,$

(iii) $f_n(c^k_n) = k+1, \quad k \in \boldsymbol{N}.$

Indeed we can use (iii) to define f_n at the points $1 < c_n^1 < c_n^2 < \ldots$ and then connect these points with line segments and finally complete the function by (i), (ii). It is clear that f_n is a bijection.

Now suppose that $\xi \in \nu^\times$ and $\xi_n > 0$, $n \in \boldsymbol{N}$. Then for some $k \in \boldsymbol{N}$ and $M > 0$ we have, for n sufficiently large,

$$\xi_n \leqslant M c_n^k = M \frac{c_n^k}{c_n^{k+1}} c_n^{k+1} \leqslant c_n^{k+1}$$

so $f_n(\xi_n) \leqslant k+2$. This implies that $\big(f_n(\xi_n)\big)_n \in l_\infty$.

On the other hand suppose that $\xi \in \nu$ and $\xi_n \geqslant 0$, $n \in \boldsymbol{N}_0$. Suppose that we had $k \in \boldsymbol{N}$ such that $f_n(\xi_n) \geqslant \frac{1}{k+1}$ for infinitely many $n \in \boldsymbol{N}$. Then for these n we would have

$$f_n\left(\frac{1}{\xi_n}\right) = \frac{1}{f_n(\xi_n)} \leqslant k+1$$

so by applying f_n^{-1} we would obtain

$$\frac{1}{\xi_n} \leqslant f_n^{-1}(k+1) = c_n^k,$$

so that $\xi_n c_n^k \geqslant 1$ for infinitely many n. But this contradicts the fact that $\xi \in \nu = \bigcap_k \frac{1}{c^k} l_1$. Hence we may conclude that $\big(f_n(\xi_n)\big)_n \in c_0$.

Finally, let $\eta = (\eta_n)_{n \in \boldsymbol{N}_0}$ be some enumeration of the positive rationals and set $\xi_n = f_n^{-1}(\eta_n)$ $n \in \boldsymbol{N}_0$. If we had a decomposition as required, then from the foregoing it would follow that $\frac{1}{\eta_{\boldsymbol{N}_1}} \in c_0$ and $\eta_{\boldsymbol{N}_2} \in l_\infty$. That is, η would have been decomposed into two subsequences, one of which converges to infinity and the other is bounded. This is clearly impossible.

We are now able to prove the most important case of our theorem.

PROPOSITION. *Let E be a nuclear Fréchet space on which there exists a continuous norm and let (e_n) be a basis. Then there is a permutation of this basis which has a block basic sequence with blocks of length $\leqslant 2$ that has no block extension.*

Proof. From the nuclearity and the existence of a norm we can represent E as an echelon space of order 1. Using a result of Bessaga and Pełczyński ([1], p. 310) we can rearrange the basis and choose a fundamental system of norms such that the second condition of Lemma 1 is satisfied. The first condition is then easily achieved by an appropriate diagonal transformation. The requirements of Lemma 2 are then a consequence of the fact that $c_n^k = a_{j_n}^k$. We obtain the desired result by putting together the existence theorem and Lemmas 1, 2.

Finally we prove the main theorem. If E is isomorphic to ω then every block basic sequence has a block extension (see [5] for a detailed proof). Conversely, if E has a basis such that every block basic sequence with blocks of length $\leqslant 2$ of every permutation of the basis has a block extension, then we must show that E is isomorphic to ω. If not, then applying [2], Theorem 7, we can write E as a direct sum $E_1 \oplus E_2$ where a subset of the original basis is a basis for E_1 and its complement is a basis for E_2 and E_1 admits a continuous norm. We then apply the proposition to E_1 to obtain a block basic sequence in E_1 with blocks of length $\leqslant 2$ and having no block extension in E_1. Next we alternate these blocks with blocks having length 1 and consisting of the elements of the basis for E_2. It is then clear that this gives a block basic sequence of a permutation of the basis for E with blocks of length $\leqslant 2$ and having no block extension in E. This is a contradiction so the theorem is proved.

Remark. We do not know if the same characterization can be made without using a permutation of the basis.

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