References

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In general, Bernoulli convolutions have independent powers

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Abstract. For a sequence $b = (b_n)$ of non-negative real numbers such that \[
\sum_{n=1}^{\infty} b_n < 1,
\]
let $\nu(b)$ be the measure on the circle $T$ represented by the infinite convolution $\sum_{n=1}^{\infty} \frac{1}{2} \delta(-b_n) + \delta(b_n)$. It is shown that for a residual set of such $b_n$, the closure in the $\text{e}(L^\infty(\nu(b)), L^1(\nu(b)))$ topology of the characters of $T$ contains all constant functions with values in $[0,1]$. It follows that for those measures $\nu(b)$ it is singular to $\nu$ unless $\nu = \delta_0$.

1. Introduction. Let $T$ be the circle group $R/Z$ and let $M(T)$ denote the convolution algebra of bounded regular Borel measures on $T$. Further, let $B$ denote the set of sequences $(b_n)$ of real numbers satisfying $b_n \geq 0$ for $n = 1, 2, 3, \ldots$ and \[
\sum_{n=1}^{\infty} b_n ^2 < 1.
\] For each sequence $b = (b_n)$ in $B$, we write $\nu(b)$ for the infinite convolution product

\[
\nu(b) = \sum_{n=1}^{\infty} \frac{1}{2} \delta(-b_n) + \delta(b_n)
\]

where $\delta(x)$ is the positive measure of mass 1 concentrated at $x \in T$. The infinite convolution product converges in the weak* topology by [10], p. 121, so that we have defined a map $\nu: B \to M(T)$. Let $\mathcal{B}$ denote the image of $\nu$ (thus $\mathcal{B}$ is a set of symmetric Bernoulli convolutions). We regard $B$ as a subspace of the compact space $[0,1]^\mathbb{N}$. As such $B$ is a compact Hausdorff space. We shall say that a subset $\mathcal{A}$ of $\mathcal{B}$ is virtually all of $\mathcal{B}$ if $\nu^{-1}(\mathcal{A})$ is residual in $B$ (i.e., $B \setminus \nu^{-1}(\mathcal{A})$ is of first category).

Let $\mathcal{M}(M(T))$ denote the maximal ideal space of $M(T)$ which we regard as a topological subspace of $\prod \{L^\infty(\mu) : \mu \in M(T)\}$ where $L^\infty(\mu)$ has the $\text{e}(L^\infty(\mu), L^\infty(\mu))$-topology (see [9]). For $\chi \in \mathcal{M}(M(T))$ and $\mu \in M(T)$, $\chi_\mu$ indicates the $\mu$-coordinate of $\chi$. The dual group $T'(\cong Z)$ of $T$ is embedded in $\mathcal{M}(M(T))$ in an obvious and natural way. Using this embedding, we define a subset $\mathcal{A}$ of $\mathcal{B}$ to consist of all measures $\mu \in \mathcal{B}$ having the following property:
for each $x$ in $[-1, 1]$, there exists $\chi$ in the closure of $T'$ in $\mathcal{M}(\mathbb{T})$ such that $x_0(t) = x$ (a.e. $\mu$).

We can now state the main results of this paper.

**Theorem 1.** $\mathcal{J}$ is virtually all of $\mathcal{S}$.

**Corollary 1.** For virtually all $\mu$ in $\mathcal{S}$, $\delta(d) * \mu^*$ is singular to $\mu^*$ for all $d \in \mathbb{T}$ unless $r = s$.

We paraphrase this property of a measure $\mu$ by saying that it has *strongly independent powers*. We say that $\mu$ has independent powers if $\mu^*$ is singular to $\mu^*$ unless $r = s$.

**Corollary 2.** For virtually all $\mu$ in $\mathcal{S}$, $\mu$ has independent powers.

**Corollary 3.** For virtually all $\mu$ in $\mathcal{S}$, $\mu$ and all of its convolution powers are singular to Lebesgue measure.

**Remark 1.** The problem of deciding the singularity or absolute continuity of symmetric Bernoulli convolutions has been studied by several authors. A survey of some of their results, together with proofs of the sharpest of these results are to be found in a paper of Garsia [4].

**Remark 2.** We single out for special mention a result of Erdős [3] which says that, in a special case, the "random" symmetric Bernoulli convolution is absolutely continuous with respect to Lebesgue measure. Precisely, if $\mu_n = \delta_n$ (where $\alpha_n$ is a positive integer with $\sum_{n=1}^{\infty} \alpha_n^{-1}$ converging), then $\mu$ belongs to $\mathcal{J}$. The present authors have shown in [2] (Theorem 3.2) that in fact, $\mu$ belongs to $\mathcal{J}$ if and only if $\sum_{n=1}^{\infty} \alpha_n^{-1} = \infty$. Kaufman [7] generalised the results of Hewitt and Kakutani in a different direction—in particular he removed the strong arithmetical constraint that the $\alpha_n$ be integers of the special kind just mentioned. His most significant result for our present purposes is that, given any $\mu, f \in \mathcal{J}$, there is a dense set $\mathcal{I}$ of $\mathcal{J}$ such that for all $x$ in $\mathcal{I}$, $\mu * \delta_{(x, e_0)}$ belongs to $\mathcal{J}$. We shall make use of Kaufman's methods in proving our result. An alternative approach to deciding when a constant function $x$ belongs to the $\sigma(L^\infty(\mu), L^1(\mu))$ closure of $T'$ can be found in a paper of Johnson [6]. However, Johnson proves the result only in the special case which he requires. As some steps in the proof do not obviously carry over to our situation, we prefer to give a proof of this result.

**Lemma 1.** Let $\mu$ be a member of $\mathcal{S}$ and $\mu = \nu(b)$. Then $\exists$ a sequence $(n(k))$ of positive integers such that

$$
\lim_{k \to \infty} \int \frac{\exp \left(2\pi i n(k) \sum_{n=1}^{\infty} (\sum_{n=1}^{\infty} \alpha_n^{-1}) \right)}{\chi(\nu) \, d\lambda(\nu)} = \nu(b) \int \chi(x) \, d\lambda(x).
$$

**Proof.** Let $b_n$ be the sequence $(b_n, b_{n+1}, \ldots)$ and let $D$ be the subgroup of $T$ generated by the members of the sequence $b$. A consequence of the Three Series Theorem is that there is an almost everywhere (with
respect to \( \lambda \) defined map \( \varphi : D \to T \) given by

\[
\varphi(x) = \sum_{n=1}^{m} (-1)^m b_n,
\]

and \( \psi(b) \) is the measure induced on \( T \) by \( \lambda \) and this map according to the formula

\[
\int f \psi(b) = \int f \circ \varphi \, d\lambda
\]

for all \( f \in C(T) \). If \( g \in L^1(\varphi(b)) \), \( g \circ \varphi \in L^1(\lambda) \) and so can be approximated (in \( L^1(\lambda) \)) by linear combinations of characteristic functions of open and closed sets of the form

\[
(x; e_i = n_i; i = 1, 2, \ldots, N).
\]

Correspondingly \( g \) can be approximated in \( L^1(\psi(b)) \) by linear combinations of Radon–Nikodym derivatives of measures of the form \( \delta(d^* \varphi(b_0)) \) where \( d \delta \lambda \) is such that \( \delta(d^* \varphi(b_0)) \) is absolutely continuous with respect to \( \varphi(b) \). Therefore, showing that (i) and (ii) together imply that \( x \) is in the \( \sigma(L^\infty(\mu), L^1(\mu)) \) closure of \( T \) amounts to finding a sequence of integers \( (n(k)) \) such that \( \delta(d^* \varphi(b_0))^{\vee} (n(k)) \to x \) for each \( d \delta \lambda \) and positive integer \( N \). Moreover, since \( \varphi(b_0)^\vee (n) = \varphi(b_0)^\vee (-n) \), we may assume that the integers \( n(k) \) are positive. Since \( \delta(d^* \varphi(b_0)) \to x \) for each \( d \delta \lambda \) according to (ii), it suffices to show that

\[
\psi(b_0)^\vee (n(k)) \to x.
\]

To do this, we remark that

\[
\rho(b) = \sum_{k=1}^{2^N} \delta(d_k)^\vee \psi(b_0)
\]

for some \( d_k \delta \lambda \) (\( i = 1, 2, \ldots, 2^N \)). Since \( \rho(b)^\vee (n(k)) \to x \) and \( \delta(d_k)^\vee (n(k)) \to x \), a glance at (i) shows that \( \rho(b_0)(n(k)) \to x \) and the required result follows.

This result (or at least weaker forms of it) has been used by the present authors in [1] to prove that certain measures \( \mu \) satisfy Corollary 1. In fact, we were able, in many cases, to describe the elements \( d \delta \lambda \) in \( T \) having the property that, for some \( n, \delta(d) \mu^n \) is not singular to \( \mu^n \).

It will be clear to the reader that there exists a reformulation of Lemma 1 along the lines of the Hewitt–Kakutani result quoted above.

3. Proofs of results. Recalling Kaufman’s results [7] for a moment, we note that he used two facts:

(i) the set of independent power measures in \( M(T) \) is a \( \mathcal{G} \), in the

\[
\sigma(M(T), C(T)) \text{ topology};
\]

(ii) the map \( \varepsilon \to \sigma \{ (b_n, \varepsilon_n) \} \) is continuous from \( D \) to \( M(T) \) with the

\[
\sigma(M(T), C(T)) \text{ topology}.
\]

Thus the obvious frontal attack on the proof of Corollary 2 would use (ii) together with a proof that \( b \to \varphi(b) \) is continuous. However, as will be made clear, this map is not continuous so that the natural approach breaks down and we have to resort to an indirect method. To do this, we introduce an auxiliary space \( E \) which is the countable infinite product of copies of \( N(= (1, 2, 3, \ldots)) \) with the product metric. Of course \( E \) is a complete separable metric space homeomorphic to the irrationals. We define a map \( \psi : B \times E \to B \) by

\[
\psi(b, m) = (b_m, m^{-1}).
\]

It is clear that \( \psi \) is a continuous surjection. Moreover, we can obtain an analogue of (i) above for the map \( m \mapsto \psi(b, m) \), so that although \( \psi \) is not continuous \( \sigma \circ \varphi \) is continuous in the second variable. On the other hand, \( \psi \) is Borel measurable. These two facts are basic to our approach and we prove them together with some earlier assertions in the next lemma.

Lemma 2. (i) \( B \) is a compact subset of \( [0, 1]^b \).

(ii) For each \( f \in C(T) \), \( b \to \varphi(b)(f) \) is Borel measurable on \( B \), but not continuous.

(iii) For each \( k \in \mathbb{Z} \), and each \( b \in B \), \( m \to \psi(b, m)^k \) is continuous on \( B \).

Proof. (i) Let \( b^{(k)} \) belong to \( B \) (\( k = 1, 2, 3, \ldots \)) and \( b^{(k)} \to b \). Then, for each \( N \),

\[
\sum_{k=1}^{N} (b^{(k)})^k \leq 1,
\]

so that letting \( k \) tend to infinity, \( \sum_{k=1}^{N} b_{k}^k \leq 1 \). Since this is true, for all \( N \), it follows that \( b \in B \), so that \( B \) is closed and hence compact.

(ii) Let

\[
\varphi(b) = \sum_{n=1}^{N} \frac{1}{2} \delta(b_{-n} - b_{n}).
\]

Then, for each \( N \), \( b \to \varphi(b) \) is continuous on \( B \) and for each \( b \in B \) and \( f \in C(T), \varphi(b)(f) \to \varphi(b)(f) \). Thus \( b \to \varphi(b)(f) \) is a limit of a sequence of continuous functions and so is Borel measurable. On the other hand, let

\[
b \in B \text{ and } \sum_{n=1}^{N} b_{k}^k \leq \frac{1}{4}. \]

Let \( b^{(k)} \) be the member of \( B \) defined by

\[
b_{n}^{(k)} = b_n \quad 1 \leq n \leq k,
\]

\[
b_{n}^{(k)} = b_{k-n} \quad k+1 \leq n.
\]
Then $e^{i\theta} \to b$ but $\nu(e^{i\theta}) = \nu_\mathcal{F}(b) = \nu_\mathcal{F}(b') = \nu_\mathcal{F}(b)$ in the weak$^*$ topology. Since, in general, $\nu(b)^k \neq \nu(b)$, $b \to \nu(b)(f)$ is not continuous.

(iii) Note that

$$
\nu(b, m)^{-1}(k) = \int_{k=1}^m \cos 2\pi n h_k m^{-1} k.
$$

Thus if $m$ and $m'$ agree on the first $N$ coordinates

$$
\left| \nu(b, m)^{-1}(k) - \nu(b, m')^{-1}(k) \right| \leq \left| \sum_{n=N+1}^m \cos 2\pi n h_k m_n^{-1} k - \sum_{n=N+1}^{m'} \cos 2\pi n h_k m_n'^{-1} k \right|.
$$

However, for sufficiently large $N$,

$$
0 \leq 1 - \sum_{n=N+1}^m \cos 2\pi n h_k m_n^{-1} k \leq \sum_{n=N+1}^m \log \cos 2\pi n h_k m_n^{-1} k
$$

and

$$
- \sum_{n=N+1}^m \log \cos 2\pi n h_k m_n^{-1} k \leq \sum_{n=N+1}^m (2\pi n h_k)^2.
$$

Since the right hand side of (4) does not depend on $m$, and tends to zero as $N \to \infty$, these statements combined with (2) give the continuity we require.

Now let $\{V_r : r = 1, 2, 3, \ldots\}$ be a countable base for the topology of the real interval $[-1, 1]$, and define

$$
B(h, r) = \bigcup_{k=1}^r \{b \in E : \nu(b)^{-1}(k) \in V_r \}
$$

and

$$
W(b; h, r) = \{m \in E : \nu(b, m) \in B(h, r)\}.
$$

Let $A = \bigcap_{h=1}^\infty B(h, r)$. Then there exists a Borel set contained in $F$.

Lemma 3. (i) $A$ is a Borel set contained in $\nu^{-1}(\mathcal{F})$;

(ii) $\bigcap_{h=1}^\infty W(b; h, r)$ is a $\mathcal{G}$ subset of $E$ contained in $\nu^{-1}(\mathcal{F})$ for each $b \in E$.

Proof. (i) If $A$ is a Borel set since $b \to \nu(b)^{-1}(k)$ is Borel measurable and the inclusion is an immediate consequence of Lemma 1.

(ii) The inclusion follows as in (i), and the $\mathcal{G}$ property is a consequence of the continuity of $m \to \nu(b, m)^{-1}(k)$.

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$$
X = \bigcap_{h=1}^\infty W(b; h, r) \text{ is dense in } E.
$$

The next lemma, although not difficult to prove, is the crucial step in our indirect approach. Here we show how the original problem can first be replaced by one of proving that a certain subset of $B \times E$ is residual. This in turn can be reduced to proving that $X$ is residual in $B$. Then, in Lemmas 5, 6 and 7, we will give a direct proof that this is indeed the case.

Lemma 4. If $X$ is residual, then $A$ (and hence $\nu^{-1}(\mathcal{F})$) is residual.

Proof. Since $A$ is a Borel set, $\nu^{-1}(A)$ is also a Borel set and so has the property of Baire ([59], p. 56). Moreover, for each $b \in X$, the section of $\nu^{-1}(A)$ over $b$,

$$
\bigcap_{h=1}^\infty W(b; h, r)
$$

is residual in $E$. Under the stated hypothesis we can apply the Kuratowski-Ulam theorem ([8], pp. 222-224) to show that $\nu^{-1}(A)$ is residual.

Thus $\nu^{-1}(B(A))$ is of first category. Suppose, to obtain a contradiction, that $B(A)$ is of second category. Using the fact that $B(A)$ is a Borel set, we can express it as a symmetric difference $\bigtriangleup$ where $U$ is a non-empty open set and $D$ is a first category set (see [8], p. 54).

Therefore $\nu^{-1}(B(A)) = \nu^{-1}(U) \bigtriangleup \nu^{-1}(D)$ where $\nu^{-1}(U)$ is a non-empty open set. Thus a contradiction will be obtained if we can show that $\nu^{-1}(D)$ is of first category. This will be achieved by showing that $\nu^{-1}(C)$ is nowhere dense for every closed nowhere dense set $C$ in $B$.

Suppose that $\nu^{-1}(C)$ is not nowhere dense. Then there is included in $\nu^{-1}(C)$ a basic open set $B \times E$ of the form

$$
V = \{b, m : b_1 \in U_1, b_2 \in U_2, \ldots, b_N \in U_N; m_1 = M_1, m_2 = M_2, \ldots, m_N = M_N \}
$$

where $U_1, \ldots, U_N$ are open subsets of $[0, 1]$ and $M_1, \ldots, M_N$ are positive integers. But now,

$$
C = \nu(V) = \{b, m : b_1 \in M_1 U_1, b_2 \in M_2 U_2, \ldots, b_N \in M_N U_N \}
$$

which, as each $M_i U_i = [M_i x ; x \in U_i]$ is open, is open in $B$. This contradicts the fact that $C$ is nowhere dense, and the proof is complete.

Remark 5. Note that it would not be sufficient in our present context to use $D$ in place of $E$, as Kaufman did. For let $D \neq B \times D$ defined by $0 : b \in D$, and let $X$ be the subset of $E$ consisting of all $b$ with $b_0 = 0$. Clearly $X$ is closed and nowhere dense, however, $\nu^{-1}(X)$ contains $\{b, e : e_0 = 0\}$ which is an open and closed subset of $B \times D$. 
Our next lemma is a little unusual as category arguments go — before proving that the phenomenon under discussion holds virtually everywhere, we check first, by direct contruction, that it happens somewhere.

**Lemma 5.** $σ^{-1}(s')$ is non-empty.

**Proof.** For any $μ$ in $A$ write $S(μ)$ for the set of (necessarily real) numbers $s$, such that the constant function $x$ in the $σ[L^∞(μ), L^1(μ)]$-closure of $T$ in $A[M(T)]$. $S(μ)$ is clearly a semigroup under the usual multiplication.

Now take a sequence $(m_n)$ of positive integers not less than 2, write $c_n = (m_1, m_2, \ldots, m_n)^{-1}$, for $n = 1, 2, \ldots$, and demand that $2d = \sum_{n=1}^{∞} c_n < 1$.

By Theorem 3.2 of [2], $ω(c) \in \mathbb{F}$, where $c = (c_n)$. In other words for each $x ∈ C, |x| ≤ 1$, there is a sequence $(w(k))$ of integers such that $exp(2πim(k)x) → x$ in the $σ[L^∞(ω(c)), L^1(ω(c))]$-topology. Now write $α = (1/α_n)$ so that $σ(α) = δ(−d) ∗ α(c)$.

(Recall that we work modulo one.)

By passing to a subsequence if necessary, we can assume that $exp(2πim(k)(−d)) → x_1$, with $|x_1| = 1$, as $k → ∞$.

Now $m_1 ∈ S(σ(α))$; hence $x_1 = ± |x_1|$, and because $S(σ(α))$ is a semigroup this shows that

$$[0, 1] \subseteq S(σ(α)).$$

The required conclusion is that $[−1, 1]$ is contained in $S(σ(α))$ and we will achieve this by making a special choice of $(m_n)$ to ensure that $−1 ∈ S(σ(α))$. In fact take $m_n = 2^m, n = 1, 2, \ldots$, and write $n(b) = 2^{m+k}, k = 1, 2, \ldots$. It follows from Lemma 1 of [1], or from the version of Theorem 3.1 of [5] quoted in §2, that for these choices,

$$exp(2πim(k)x) → x \quad σ[L^∞(ω(c)), L^1(ω(c))].$$

A simple direct verification shows that

$$2^{−3} − 2^{−5} − 4 \quad (modulo \quad one) \quad as \quad k → ∞$$

i.e. that $exp(2πim(k)(−d)) → −1$. This completes the proof.

For $y ∈ R, |y|$ will denote the distance from $y$ to the nearest integer.

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and

$$U(k, p, q, n) = \{x ∈ R : \|kx − pn\| < q^{-1}\}$$

where $α$ is some fixed member of $σ^{-1}(s')$. Further, let

$$A(σ, r, s; k, m, p, q) = \bigcap_{n=1}^{∞} \bigcup_{k=1}^{∞} \bigcap_{p=1}^{∞} \bigcup_{q=1}^{∞} V(m, k, q)$$

and finally,

$$G = \bigcap_{α ∈ A} \bigcup_{r, s, k, m, p, q} \bigcup_{n=1}^{∞} A(σ, r, s; k, m, p, q).$$

Since $V(m, k, q)$ and $U(k, p, q, n)$ are open, $A(σ, r, s; k, m, p, q)$ is also open, and hence $G$ is a $\mathbb{F}$.

**Lemma 6.** $G$ is contained in $X$.

**Proof.** Fix $b ∈ G$. In view of Lemma 3 (ii), we have to prove that for each $h, r ∈ N$, $W(b; h, r)$ is dense in $E$. Accordingly, we choose $h, r$ and $f ∈ N$ with $t ≫ h$ and $m_n ∈ N$ for $1 ≤ n ≤ t$. It is enough to find $k$ in $N$ and $m_n$ ($n > t$) such that if $m = (m_n)$,

$$|1 − exp(2πim(b))| < h^{-1} \quad (1 ≤ n ≤ t)$$

and

$$\|exp(2πim(b))\| < \frac{1}{δ/4}.$$

In particular, we can (and do) assume that $h = t$. Now choose $σ ∈ V$, and $δ > 0$ so that the open interval of length $2δ$ about $x$ is in $V$. Also fix $p, r, s ∈ N$ such that $r = t$ and

$$|1 − exp(2πim(b))| < h^{-1} \quad (1 ≤ n ≤ t)$$

This is possible since $α$ is in $σ^{-1}(s')$. Finally, fix $m$ and $q$ such that $m = m_n, \ldots, m_1$ and

$$q > 8πδ^{-1} max(m, r).$$

According to the definition of $G$, we can find $h$ in $N$ and $σ$ in $σ(σ, r, s; k, m, p, q)$ corresponding to the choice of $m, p, q, r, s$. For such $h, b ∈ V(m, k, q)$ when $1 ≤ n ≤ h ≫ t$, so that

$$1 − exp(2πim(b))| < h^{-1}$$

and

$$\|exp(2πim(b))\| < \frac{1}{δ/4}.$$
which establishes (5). Similarly,
\begin{equation}
1 - \prod_{n=1}^{l} \cos 2\pi b_n m_n^{-1} k \leq \sum_{n=1}^{l} |1 - \cos 2\pi b_n m_n^{-1} k| \\
\leq \sum_{n=1}^{l} |1 - \exp(2i\pi b_n m_n^{-1} k)| < 2\pi m q^{-1} < \delta/4.
\end{equation}

At this point, we define \( m_n \) for \( n > t \) as follows:
\begin{itemize}
  \item \( m_n = 1 \) if \( n \) belongs to the image of \( \sigma \);
  \item \( m_n = 2^\left(\left\lfloor (4\pi b d^{-1})^{-1} \right\rfloor \right) \) otherwise
\end{itemize}
(\( \lfloor \ \rfloor \) denotes integer part.)

Further, let
\begin{align*}
a &= \prod_{n=1}^{l} \cos 2\pi m_n^{-1} b_n k, \\
b &= \prod_{n=t+1}^{\infty} \cos 2\pi m_n^{-1} b_n k, \\
g &= \prod_{n=t+1}^{\infty} \cos 2\pi m_n^{-1} b_n k,
\end{align*}
so that \( v(v(b, m))^* (k) = a^g \). By the definition of \( m_n \),
\begin{equation}
|b - \prod_{n=1}^{l} \cos 2\pi p_n a_n| \leq \sum_{n=1}^{l} |1 - \cos 2\pi b_n m_n^{-1} k - \cos 2\pi p_n a_n| \\
\leq \sum_{n=1}^{l} 2\pi \| b_n m_n^{-1} - p_n a_n \| < \delta/4,
\end{equation}
and
\begin{equation}
|g - 1| \leq \sum_{n=t+1}^{\infty} |1 - \cos 2\pi b_n m_n^{-1} k| \leq \sum_{n=t+1}^{\infty} 2^{-n-1} \delta < \delta/4.
\end{equation}

Combining (7), (8), (9) and (10), we obtain
\begin{equation}
|v(v(b, m))^* (k) - a| \leq |a-1| + |b-a| |g-1| < \delta
\end{equation}
which implies (6). This completes the proof.

To complete the proof of Theorem 1, we need to show that \( G \) is dense in \( B \). This is accomplished by the next lemma.

**Lemma 7.** If \( b \in B \) is such that \( b_n \) is a strictly positive rational number for each \( n \), then \( b \in G \).

**Proof.** Let \( b_n = p_n q_n^{-1} \) where \( p_n \) and \( q_n \) are positive integers with no common factor. Fix \( m, p, q, r, s \) in \( N \) and let \( \sigma(1) \) be any integer greater than \( r \), such that \( q_n^{-1} q_{n+1} \) is strictly greater than \( m \cdot q_n \cdot q_{n+1} \cdot \ldots \).
(Note that \( \sigma(1) \) exists since \( q_{n+1} \to \infty \) as \( n \to \infty \).) Having defined \( \sigma(1), \sigma(2), \ldots, \sigma(j) \) (for \( f < j \)), we choose \( \sigma(j+1) > \sigma(j) \) such that \( q^{-1} q_{n+1} \) is strictly greater than \( m \cdot q_n \cdot q_{n+1} \cdot \ldots \).
This defines \( \sigma \in \Sigma(r, s) \) inductively. Because of the construction of \( \sigma \), there exist positive integers \( p_i^l (1 \leq i \leq s) \) such that
\begin{equation}
p_i^l b_m p_l^s = 0 \pmod{m}
\end{equation}
for \( 1 \leq i \leq r \) and \( 1 < j < s \), and the denominator of \( p_i^l b_{n0} \) (in its lowest terms) is strictly greater than \( q \). Now we choose \( t_1, t_2, \ldots, t_s \) in \( N \) inductively so that
\begin{equation}
t_1^1 p_1^0 + t_2^1 p_2^0 + \ldots + t_s^l p_s^0 = 0 \pmod{q^{-1}}
\end{equation}
for \( 1 \leq i \leq s \). Define \( k = t_1^i p_1^0 + t_2^i p_2^0 + \ldots + t_s^i p_s^0 \).

Now it follows from (11) and (12) that
\begin{equation}
b^* A(\sigma, r; s; k, m, p, q, q)
\end{equation}
so that \( b \in G \).

The proof of Lemma 7 is based on Kaufman’s “Technical Lemma” in [7].

Proof of Corollaries. Clearly Corollary 1 implies Corollary 2. By Theorem 1, there is a positive generalised character \( \chi \) such that \( \chi_n = \alpha \) \( (\mu \text{ a.e.}) \) where \( 0 < \alpha < 1 \). For \( d \in T \) and any positive integer \( n \)
\begin{equation}
\chi_n \in \chi^* \{d \} = \alpha \quad (\delta(d) \cdot \mu \text{ a.e.}).
\end{equation}
To prove Corollary 1, we note that if \( \chi \) is absolutely continuous with respect both \( \delta(d) \cdot \mu \) and \( \mu \), then using (13), \( \alpha^* = \chi^* \{d \} = \alpha^* \) \( (\mu \text{ a.e.}) \) so that \( n = m \). For Corollary 3, we use the fact that if \( 0 = \lambda L(T) \) and \( \lambda \) is absolutely continuous with respect to \( \mu \), then \( \chi_l \) is either equal to a character of \( T \) or is 0 \( (\mu \text{ a.e.}) \). Neither of these possibilities can occur if \( \chi_n \) is as described in (13). This contradiction proves the result.

**References**
A characterization of $\omega$ by block extensions

by

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Abstract. The following characterization of the space $\omega$ is given: A nuclear Fréchet space is isomorphic to $\omega$ if and only if it has a basis such that every block basic sequence of every permutation of that basis has a block extension. In fact it is shown that it suffices to consider only blocks of length $\leq 2$.

In studying nuclear Fréchet spaces it is natural to try to investigate situations which are familiar in Banach space theory and to see if results can be carried over. It is particularly interesting to see a situation in which a simple imitation is a priori impossible. This is exactly what occurs if one considers extensions of block basic sequences (see below for definitions). In order to point this out we mention the following two well-known results.

Theorem of Zippin [6]. In a Banach space, every block basic sequence has a block extension.

Theorem of Lindenstrauss and Tzafriri [3]. A Banach space is isomorphic to $l_p(1 \leq p < \infty)$ or $c_0$ if and only if it has an unconditional basis such that every block basic sequence of every permutation of this basis generates a complemented subspace.

Since every basic sequence which can be extended to an unconditional basis in a Fréchet space generates a complemented subspace, the second theorem provides many examples in which the first theorem cannot be improved to assert the existence of an unconditional extension when, say, the original basis is unconditional. Actually, the first specific example of this situation was given by Pelczynski [4].

On the other hand, since every basis in a nuclear Fréchet space is unconditional, it follows that at least one of the above two theorems must be false in this context. Indeed, the second author [5] has shown that it is Zippin's theorem which does not carry over. He also pointed out that it does hold for the space $\omega$ and asked if Zippin's theorem characterizes $\omega$ among the nuclear Fréchet spaces.

It is our purpose in this paper to answer that question in the affirmative. Specifically, we prove,