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*Received April 20, 1972* (513)

STUDIA MATHEMATICA, T. XLVII. (1973)

# In general, Bernoulli convolutions have independent powers

by

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**Abstract.** For a sequence $\boldsymbol{b} = (b_n)$ of non-negative real numbers such that $\sum_{n=1}^{\infty} b_n^2 \leqslant 1$, let $\nu(\boldsymbol{b})$ be the measure on the circle $\boldsymbol{T}$ represented by the infinite convolution $\mathop{\ast}_{n=1}^{\infty} \frac{1}{2}(\delta(-b_n)+\delta(b_n))$. It is shown that for a residual set of such $\boldsymbol{b}$'s, the closure in the $\sigma(L^\infty(\nu(\boldsymbol{b})), L^1(\nu(\boldsymbol{b}))$ topology of the characters of $\boldsymbol{T}$ contains all constant functions with values in $[-1, 1]$. It follows that for these measures $\delta(x) \ast \nu(\boldsymbol{b})^r$ is singular to $\nu(\boldsymbol{b})^s$ unless $r = s$.

**1. Introduction.** Let $\boldsymbol{T}$ be the circle group $\boldsymbol{R}/\boldsymbol{Z}$ and let $M(\boldsymbol{T})$ denote the convolution algebra of bounded regular Borel measures on $\boldsymbol{T}$. Further, let $B$ denote the set of sequences $(b_n)$ of real numbers satisfying $b_n \geqslant 0$ $(n = 1, 2, 3, \ldots)$ and $\sum_{n=1}^{\infty} b_n^2 \leqslant 1$. For each sequence $\boldsymbol{b} = (b_n)$ in $B$, we write $\nu(\boldsymbol{b})$ for the infinite convolution product

$$\nu(\boldsymbol{b}) = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}\big(\delta(-b_n)+\delta(b_n)\big)$$

where $\delta(x)$ is the positive measure of mass 1 concentrated at $x \in \boldsymbol{T}$. The infinite convolution product converges in the weak* topology by ([10], p. 127), so that we have defined a map $\nu\colon B \to M(\boldsymbol{T})$. Let $\mathscr{B}$ denote the image of $\nu$ (thus $\mathscr{B}$ is a set of symmetric Bernoulli convolutions). We regard $B$ as a subspace of the compact space $[0, 1]^{\aleph_0}$. As such $B$ is a compact Hausdorff space. We shall say that a subset $\mathscr{C}$ of $\mathscr{B}$ is *virtually all* of $\mathscr{B}$ if $\nu^{-1}(\mathscr{C})$ is residual in $B$ (i.e. $B \setminus \nu^{-1}(\mathscr{C})$ is of first category).

Let $\Delta\big(M(\boldsymbol{T})\big)$ denote the maximal ideal space of $M(\boldsymbol{T})$ which we regard as a topological subspace of $\prod\{L^\infty(\mu)\colon \mu \in M(\boldsymbol{T})\}$ where $L^\infty(\mu)$ has the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$-topology (see [9]). For $\chi \in \Delta\big(M(\boldsymbol{T})\big)$ and $\mu \in M(\boldsymbol{T})$, $\chi_\mu$ indicates the $\mu$-coordinate of $\chi$. The dual group $\boldsymbol{T}\hat{}\ (\cong \boldsymbol{Z})$ of $\boldsymbol{T}$ is embedded in $\Delta\big(M(\boldsymbol{T})\big)$ in an obvious and natural way. Using this embedding, we define a subset $\mathscr{A}$ of $\mathscr{B}$ to consist of all measures $\mu \in \mathscr{B}$ having the following property:

for each $x$ in $[-1,1]$, there exists $\chi$ in the closure of $T^\wedge$ in $\Delta M(T)$) such that $\chi_\mu(t) = x$ ($\mu$ a.e.).

We can now state the main results of this paper.

THEOREM 1. *$\mathscr{A}$ is virtually all of $\mathscr{B}$.*

COROLLARY 1. *For virtually all $\mu$ in $\mathscr{B}$, $\delta(d)*\mu^r$ is singular to $\mu^s$ for all $d\in T$ unless $r = s$.*

We paraphrase this property of a measure $\mu$ by saying that it has *strongly independent powers*. We say that $\mu$ has *independent powers* if $\mu^r$ is singular to $\mu^s$ unless $r = s$.

COROLLARY 2. *For virtually all $\mu$ in $\mathscr{B}$, $\mu$ has independent powers.*

COROLLARY 3. *For virtually all $\mu$ in $\mathscr{B}$, $\mu$ and all of its convolution powers are singular to Lebesgue measure.*

Remark 1. The problem of deciding the singularity or absolute continuity of symmetric Bernoulli convolutions has been studied by several authors. A survey of some of their results, together with proofs of the sharpest of these results are to be found in a paper of Garsia [4].

Remark 2. We single out for special mention a result of Erdös [3] which says that, in a special case, the "random" symmetric Bernoulli convolution is absolutely continuous with respect to Lebesgue measure. Precisely, if $b_n = a^n$ $(n = 1, 2, 3, \ldots)$, then for almost all $a$ in $[\gamma, 1]$ $(0<\gamma<1)$ (with respect to Lebesgue measure), $\nu(\boldsymbol{b})$ is absolutely continuous. We explain the relationship of our work to that of more recent authors in this field in the next section.

Remark 3. Even the fact (Corollary 3) that "most" Bernoulli convolutions are singular seems to be new (cf. [4], p. 412).

Remark 4. The methods used here apply equally well if $T$ is replaced by $R$. Moreover, if $B$ is replaced by

$$F = \{(b_n)\colon \sum_{n=1}^{\infty} b_n \leqslant 1,\quad b_n \geqslant 0\ (n = 1, 2, 3, \ldots)\}$$

and $\nu$ by $\omega$ where

$$\omega(\boldsymbol{b}) = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}\big(\delta(0)+\delta(b_n)\big)$$

we can prove that for virtually all $\mu$ in $\omega(F)$, $\mu$ belongs to the set $\mathscr{I}$ of all the measures with the property that every *complex* constant of absolute value not greater than 1 is in the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$ closure of $T^\wedge$. From this, the analogous results to Corollaries 1, 2 and 3 follow quickly. In particular, for virtually all $\mu$ in $\omega(F)$, $\mu^r\tilde{\mu}^s \perp \mu^p\tilde{\mu}^q$ unless $r = p$ and $s = q$. ($\tilde{\mu}$ is the involute of $\mu$, i.e. $\tilde{\mu}(B) = \overline{\mu}(-B)$ for every Borel set $B$ of $T$.)



**2. Preliminaries.** Our first task is to find a criterion for a Bernoulli convolution to belong to $\mathscr{A}$, or equivalently, for a real constant function $x$ ($x\in[-1,1]$) to belong to the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$ closure of the characters of $T$. (Note that this is equivalent to $x = \chi_\mu$ for some $\chi$ in the closure of $T^\wedge$ in $\Delta\big(M(T)\big)$, because $\Delta\big(M(T)\big)$ is compact). One possible choice for this criterion is provided by a result of Hewitt and Kakutani ([5] Theorem 3.1). Let $D$ be the countable infinite product of copies of $Z(2)$ with the product topology, so that a member $\varepsilon$ of $D$ is an infinite sequence $(\varepsilon_n)$ where $\varepsilon_n$ is 0 or 1 for each $n$. Let $\lambda$ be the Haar measure of $D$, and let $\mu = \nu(\boldsymbol{b})$ where $\sum_{n=1}^{\infty} b_n \leqslant 1$ and $\boldsymbol{b}\in B$. Then Hewitt and Kakutani's result shows that $x$ is in the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$ closure of $T^\wedge$ if and only if there is a sequence $\big(n(k)\big)$ of integers such that

$$\lim_{k\to\infty} \int \exp\Big(2\pi i n(k) \sum_{m=1}^{\infty} (-1)^{\varepsilon_m} b_m\Big)\,\chi(\varepsilon)\,d\lambda(\varepsilon) = x\int \chi(\varepsilon)\,d\lambda(\varepsilon).$$

for each character $\chi$ of $D$. Using this result, Hewitt and Kakutani were able to show that if $b_n = m_1^{-1}m_2^{-1}\ldots m_n^{-1}$ where each $m_n$ is a positive integer and $\sum_{n=1}^{\infty} m_n^{-1}$ converges, then $\omega(\boldsymbol{b})$ belongs to $\mathscr{I}$. The present authors have shown in [2] (Theorem 3.2) that in fact, $\omega(\boldsymbol{b})$ belongs to $\mathscr{I}$ if and only if $\sup_n m_n = \infty$. Kaufman in [7] generalised the results of Hewitt and Kakutani in a different direction — in particular he removed the strong arithmetical constraint that the $b_n$ be reciprocal integers of the special kind just mentioned. His most significant result for our present purposes is that, given any $\boldsymbol{b}\in F$, then there is a dense $\mathscr{G}_\delta$ set $C$ in $D$ such that for all $\varepsilon$ in $C$, $\omega\big((b_n\varepsilon_n)\big)$ belongs to $\mathscr{I}$. We shall make use of Kaufman's methods in proving our result. An alternative approach to deciding when a constant function $x$ belongs to the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$ closure of $T^\wedge$ can be found in a paper of Johnson [6]. However, Johnson proves the result only in the special case which he requires. As some steps in the proof do not obviously carry over to our situation, we prefer to give a proof of this result.

LEMMA 1. *Let $\boldsymbol{b}$ be a member of $B$ and $\mu = \nu(\boldsymbol{b})$. Then $x$ belongs to the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$ closure of $T^\wedge$ if there exists a sequence $\big(n(k)\big)$ of positive integers such that*

(i) $\nu(\boldsymbol{b})^\wedge\big(n(k)\big)\to x$ *as* $k\to\infty$;

(ii) $\exp\big(2\pi i b_m n(k)\big)\to 1$ *as* $k\to\infty$ *for all* $m$.

Proof. Let $\boldsymbol{b}_N$ be the sequence $(b_{N+1}, b_{N+2}, \ldots)$ and let $D$ be the subgroup of $T$ generated by the members of the sequence $\boldsymbol{b}$. A consequence of the Three Series Theorem is that there is an almost everywhere (with respect to $\lambda$) defined map $\varphi\colon \boldsymbol{D}\to\boldsymbol{T}$ given by

$$\varphi(\varepsilon) = \sum_{m=1}^{\infty}(-1)^{\varepsilon_m}b_m,$$

and $\nu(\boldsymbol{b})$ is the measure induced on $\boldsymbol{T}$ by $\lambda$ and this map according to the formula

$$\int f d\nu(\boldsymbol{b}) = \int f\circ\varphi d\lambda$$

for all $f\in C(\boldsymbol{T})$. If $g\in L^1(\nu(\boldsymbol{b}))$, $g\circ\varphi$ belongs to $L^1(\lambda)$ and so can be approximated (in $L^1(\lambda)$) by linear combinations of characteristic functions of open and closed sets of the form

$$\{\varepsilon\colon \varepsilon_i = \eta_i;\ i = 1, 2, \ldots, N\}.$$

Correspondingly $g$ can be approximated in $L^1(\nu(\boldsymbol{b}))$ by linear combinations of Radon–Nikodym derivatives of measures of the form $\delta(d)*\nu(\boldsymbol{b}_N)$ where $d\in D$ is such that $\delta(d)*\nu(\boldsymbol{b}_N)$ is absolutely continuous with respect to $\nu(\boldsymbol{b})$. Therefore, showing that (i) and (ii) together imply that $\boldsymbol{x}$ is in the $\sigma(L^\infty(\mu), L^1(\mu))$ closure of $\boldsymbol{T}^\wedge$ amounts to finding a sequence of integers $(n(k))$ such that $(\delta(d)*\nu(\boldsymbol{b}_N))^\wedge(n(k))\to x$ for each $d\in D$ and positive integer $N$. Moreover, since $\nu(\boldsymbol{b}_N)^\wedge(n) = \nu(\boldsymbol{b}_N)^\wedge(-n)$, we may assume that the integers $n(k)$ are positive. Since $\delta(d)^\wedge(n(k))\to 1$ for each $d\in D$ according to (ii), it suffices to show that

$$\nu(\boldsymbol{b}_N)^\wedge(n(k)) \to x.$$

To do this, we remark that

$$\nu(\boldsymbol{b}) = \sum_{i=1}^{2^N}\delta(d_i)*\nu(\boldsymbol{b}_N) \tag{1}$$

for some $d_i\in D$ $(i = 1, 2, \ldots, 2^N)$. Since $\nu(\boldsymbol{b})(n(k))\to x$ and $\delta(d_i)(n(k))\to 1$ a glance at (1) shows that $\nu(\boldsymbol{b}_N)(n(k))\to x$ and the required result follows.

This result (or at least weaker forms of it) has been used by the present authors in [1] to prove that certain measures $\mu$ satisfy Corollary 1. In fact, we were able, in many cases, to describe the elements $d\in\boldsymbol{T}$ having the property that, for some $n$, $\delta(d)*\mu^n$ is not singular to $\mu^n$.

It will be clear to the reader that there exists a reformulation of Lemma 1 along the lines of the Hewitt–Kakutani result quoted above.

**3. Proofs of results.** Recalling Kaufman's results [7] for a moment, we note that he used two facts:

(i) the set of independent power measures in $M(\boldsymbol{T})$ is a $\mathscr{G}_\delta$ in the $\sigma(M(\boldsymbol{T}), C(\boldsymbol{T}))$ topology;



(ii) the map $\boldsymbol{\varepsilon}\to\omega((b_n\varepsilon_n))$ is continuous from $\boldsymbol{D}$ to $M(\boldsymbol{T})$ with the $\sigma(M(\boldsymbol{T}), C(\boldsymbol{T}))$ topology.

Thus the obvious frontal attack on the proof of Corollary 2 would use (ii) together with a proof that $\boldsymbol{b}\to\nu(\boldsymbol{b})$ is continuous. However, as will be made clear, this map is not continuous so that the natural approach breaks down and we have to resort to an indirect method. To do this, we introduce an auxiliary space $E$ which is the countable infinite product of copies of $\boldsymbol{N}(=\{1, 2, 3, \ldots\})$ with the product metric. Of course $E$ is a complete separable metric space homeomorphic to the irrationals. We define a map $\psi\colon B\times E\to B$ by

$$\psi(\boldsymbol{b}, \boldsymbol{m}) = (b_n m_n^{-1}).$$

It is clear that $\psi$ is a continuous surjection. Moreover, we can obtain an analogue of (i) above for the map $\boldsymbol{m}\to\nu(\psi(\boldsymbol{b}, \boldsymbol{m}))$, so that although $\nu$ is not continuous $\nu\circ\psi$ is continuous in the second variable. On the other hand, $\nu$ is Borel measurable. These two facts are basic to our approach and we prove them together with some earlier assertions in the next lemma.

LEMMA 2. (i) *$B$ is a compact subset of $[0, 1]^{\aleph_0}$.*

(ii) *For each $f\in C(\boldsymbol{T})$, $\boldsymbol{b}\to\nu(\boldsymbol{b})(f)$ is Borel measurable on $B$, but not continuous.*

(iii) *For each $k\in\boldsymbol{Z}$, and each $\boldsymbol{b}\in B$, $\boldsymbol{m}\to\nu(\psi(\boldsymbol{b}, \boldsymbol{m}))^\wedge(k)$ is continuous on $E$.*

Proof. (i) Let $\boldsymbol{b}^{(k)}$ belong to $B$ $(k = 1, 2, 3, \ldots)$ and $\boldsymbol{b}^{(k)}\to\boldsymbol{b}$. Then, for each $N$,

$$\sum_{n=1}^{N}(b_n^{(k)})^2 \leqslant 1,$$

so that letting $k$ tend to infinity, $\sum_{n=1}^{N} b_n^2 \leqslant 1$. Since this is true, for all $N$, t follows that $\boldsymbol{b}\in B$, so that $B$ is closed and hence compact.

(ii) Let

$$\nu^N(\boldsymbol{b}) = \mathop{\ast}_{n=1}^{N}\tfrac{1}{2}(\delta(-b_n)+\delta(b_n)).$$

Then, for each $N$, $\boldsymbol{b}\to\nu_N(\boldsymbol{b})$ is continuous on $B$ and for each $\boldsymbol{b}\in B$ and $f\in C(\boldsymbol{T})$, $\nu_N(\boldsymbol{b})(f)\to\nu(\boldsymbol{b})(f)$. Thus $\boldsymbol{b}\to\nu(\boldsymbol{b})(f)$ is a limit of a sequence of continuous functions and so is Borel measurable. On the other hand, let $\boldsymbol{b}\in B$ and $\sum_{n=1}^{\infty} b_n^2 \leqslant \frac{1}{2}$. Let $\boldsymbol{c}^{(k)}$ be the member of $B$ defined by

$$c_n^{(k)} = b_n \qquad 1\leqslant n\leqslant k,$$

$$c_n^{(k)} = b_{n-k} \qquad k+1\leqslant n.$$

Then $c^{(k)} \to b$ but $\nu(c^{(k)}) = \nu_k(b) * \nu(b) \to \nu(b)^2$ in the weak* topology. Since, in general, $\nu(b)^2 \neq \nu(b)$, $b \to \nu(b)(f)$ is not continuous.

(iii) Note that

$$\nu(\psi(b, m))\hat{}\,(k) = \prod_{n=1}^{\infty} \cos 2\pi b_n m_n^{-1} k.$$

Thus if $m$ and $m'$ agree on the first $N$ coordinates

$$(2)\qquad |\nu(\psi(b, m))\hat{}\,(k) - \nu(\psi(b, m'))\hat{}\,(k)| \leqslant \Big| \prod_{n=N+1}^{\infty} \cos 2\pi b_n m_n^{-1} k - \prod_{n=N+1}^{\infty} \cos 2\pi b_n (m_n')^{-1} k \Big|.$$

However, for sufficiently large $N$,

$$(3)\qquad 0 \leqslant 1 - \prod_{n=N+1}^{\infty} \cos 2\pi b_n m_n^{-1} k \leqslant - \sum_{n=N+1}^{\infty} \log\cos 2\pi b_n m_n^{-1} k$$

and

$$(4)\qquad - \sum_{n=N+1}^{\infty} \log\cos 2\pi b_n m_n^{-1} k \leqslant \sum_{n=N+1}^{\infty} (2\pi k b_n)^2.$$

Since the right hand side of (4) does not depend on $m$, and tends to zero as $N \to \infty$, these statements combined with (2) give the continuity we require.

Now let $\{V_r : r = 1, 2, 3, \ldots\}$ be a countable base for the topology of the real interval $[-1, 1]$, and define

$$B(h, r) = \bigcup_{k=1}^{\infty} \{b \in B : \nu(b)\hat{}\,(k) \in V_r \text{ and } |1 - \exp 2\pi i k b j| < h^{-1},\ 1 \leqslant j \leqslant h\}$$

and

$$W(b; h, r) = \{m \in E : \psi(b, m) \in B(h, r)\}.$$

Let $A = \bigcap_{h,r=1}^{\infty} B(h, r)$.

LEMMA 3. (i) *$A$ is a Borel set contained in $\nu^{-1}(\mathscr{A})$;*

(ii) *$\bigcap_{h,r=1}^{\infty} W(b; h, r)$ is a $\mathscr{G}_\delta$ subset of $E$ contained in $\{m \in E : \nu(\psi(b, m)) \in \mathscr{A}\}$, for each $b \in B$.*

Proof. (i) $A$ is a Borel set since $b \to \nu(b)\hat{}\,(k)$ is Borel measurable and the inclusion is an immediate consequence of Lemma 1.

(ii) The inclusion follows as in (i), and the $\mathscr{G}_\delta$ property is a consequence of the continuity of $m \to \nu(\psi(b, m))\hat{}\,(k)$.



Let

$$X = \{b : \bigcap_{h,r=1}^{\infty} W(b, h, r) \text{ is dense in } E\}.$$

The next lemma, although not difficult to prove, is the crucial step in our indirect approach. Here we show how the original problem can first be replaced by one of proving that a certain subset of $B \times E$ is residual. This in turn can be reduced to proving that $X$ is residual in $B$. Then, in Lemmas 5, 6 and 7, we will give a direct proof that this is indeed the case.

LEMMA 4. *If $X$ is residual, then $A$ (and hence $\nu^{-1}(\mathscr{A})$) is residual.*

Proof. Since $A$ is a Borel set, $\psi^{-1}(A)$ is also a Borel set and so has the property of Baire ([8], p. 56). Moreover, for each $b \in X$, the section of $\psi^{-1}(A)$ over $b$,

$$\bigcap_{h,r=1}^{\infty} W(b; h, r)$$

is residual in $E$. Under the stated hypothesis we can apply the Kuratowski–Ulam theorem ([8], pp. 222–224) to show that $\psi^{-1}(A)$ is residual.

Thus $\psi^{-1}(B \setminus A)$ is of first category. Suppose, to obtain a contradiction, that $B \setminus A$ is of second category. Using the fact that $B \setminus A$ is a Borel set, we can express it as a symmetric difference $U \triangle D$ where $U$ is a non-empty open set and $D$ is a first category set (see [8], p. 54). Therefore $\psi^{-1}(B \setminus A) = \psi^{-1}(U) \triangle \psi^{-1}(D)$ where $\psi^{-1}(U)$ is a non-empty open set. Thus a contradiction will be obtained if we can show that $\psi^{-1}(D)$ is of first category. This will be effected by showing that $\psi^{-1}(C)$ is nowhere dense for every closed nowhere dense set $C$ in $B$.

Suppose that $\psi^{-1}(C)$ is not nowhere dense. Then there is included in $\psi^{-1}(C)$ a basic open set of $B \times E$ of the form

$$V = \{(b, m) : b_1 \in U_1, b_2 \in U_2, \ldots, b_N \in U_N;\ m_1 = M_1, m_2 = M_2, \ldots, m_N = M_N\}$$

where $U_1, \ldots, U_N$ are open subsets of $[0, 1]$ and $M_1, \ldots, M_N$ are positive integers. But now,

$$C \supset \psi(V) = \{b : b_1 \in M_1^{-1} U_1, b_2 \in M_2^{-1} U_2, \ldots, b_N \in M_N^{-1} U_N\}$$

which, as each $M_i^{-1} U_i = \{M_i^{-1} x : x \in U_i\}$ is open, is open in $\boldsymbol{B}$. This contradicts the fact that $C$ is nowhere dense, and the proof is complete.

Remark 5. Note that it would not be sufficient in our present context to use $\boldsymbol{D}$ in place of $E$, as Kaufman did. For let $\theta : B \times \boldsymbol{D} \to B$ defined by $\theta(b, \varepsilon) = (b_n \varepsilon_n)$, and let $K$ be the subset of $B$ consisting of all $b$ with $b_1 = 0$. Clearly $K$ is closed and nowhere dense, however, $\theta^{-1}(K)$ contains $\{(b, \varepsilon) : \varepsilon_1 = 0\}$ which is an open and closed subset of $B \times \boldsymbol{D}$.

Our next lemma is a little unusual as category arguments go — before proving that the phenomenon under discussion holds virtually everywhere, we check first, by direct contruction, that it happens somewhere.

LEMMA 5. *$\nu^{-1}(\mathscr{A})$ is non-empty.*

Proof. For any $\mu$ in $\mathscr{B}$ write $S(\mu)$ for the set of (necessarily real) numbers $x$, such that the constant function $x$ is in the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$-closure of $\boldsymbol{T}$ in $\Delta\big(M(\boldsymbol{T})\big)$. $S(\mu)$ is clearly a semigroup under the usual multiplication.

Now take a sequence $(m_n)$ of positive integers not less than 2, write $c_n = (m_1\cdot m_2\cdot \ldots \cdot m_n)^{-1}$, for $n = 1, 2, \ldots$, and demand that $2d = \sum_{n=1}^{\infty} c_n < 1$. By Theorem 3.2 of [2], $\omega(\boldsymbol{c})\in\mathscr{I}$, where $\boldsymbol{c} = (c_n)$. In other words for each $z\in\boldsymbol{C}$, $|z|\leqslant 1$, there is a sequence $\big(n(k)\big)$ of integers such that $\exp\big(2\pi i n(k)t\big) \to z$ in the $\sigma\Big(L^\infty\big(\omega(\boldsymbol{c})\big), L^1\big(\omega(\boldsymbol{c})\big)\Big)$- topology. Now write $\boldsymbol{a} = (\frac{1}{2}c_n)$ so that

$$\nu(\boldsymbol{a}) = \delta(-d) * \omega(\boldsymbol{c}).$$

(Recall that we work modulo one.)

By passing to a subsequence if necessary, we can assume that

$$\exp(2\pi i n(k)(-d)) \to z_1, \quad \text{with } |z_1| = 1, \text{ as } k\to\infty.$$

Now $zz_1\in S\big(\nu(\boldsymbol{a})\big)$; hence $zz_1 = \pm|z|$, and because $S\big(\nu(\boldsymbol{a})\big)$ is a semigroup this shows that

$$[0, 1] \subseteq S\big(\nu(\boldsymbol{a})\big).$$

The required conclusion is that $[-1, 1]$ is contained in $S\big(\nu(\boldsymbol{a})\big)$ and we will achieve this by making a special choice of $(m_n)$ to ensure that $-1\in S\big(\nu(\boldsymbol{a})\big)$. In fact take $m_n = 3^{2n-1}$, $n = 1, 2, \ldots$, and write $n(k) = 3^{(2k+1)^2}$, $k = 1, 2, \ldots$ It follows from Lemma 1 of [1], or from the version of Theorem 3.1 of [5] quoted in § 2, that, for these choices,

$$\exp\big(2\pi i n(k)t\big)\to 1 \qquad \sigma\Big(L^\infty\big(\omega(\boldsymbol{c})\big), L^1\big(\omega(\boldsymbol{c})\big)\Big).$$

A simple direct verification shows that

$$n(k)\sum_{n=1}^{\infty} 2^{-1}3^{-n^2} \to \tfrac{1}{2} \text{ (modulo one)} \qquad \text{as } k\to\infty$$

i.e. that $\exp\big(2\pi i n(k)(-d)\big)\to -1$. This completes the proof.

For $y\in\boldsymbol{R}$, $\|y\|$ will denote the distance from $y$ to the nearest integer. For each positive integer $n$, let $\pi_n$ be the projection from $B$ to $\boldsymbol{R}$ defined by $\pi_n(\boldsymbol{b}) = b_n$, so that $\pi_n(B) = [0, 1]$. Now fix $k, m, p, q, r, s$ in $\boldsymbol{N}$ and define $\Sigma(r, s)$ to be the set of strictly increasing maps $\sigma$: $\{1, 2, \ldots, s\} \to \boldsymbol{N}\setminus\{1, 2, \ldots, r\}$.

$$V(m, k, q) = \{x\in\boldsymbol{R}: \|m^{-1}kx\| < q^{-1}\}$$



and

$$U(k, p, q, n) = \{x\in\boldsymbol{R}: \|kx - pa_n\| < q^{-1}\}$$

where $\boldsymbol{a}$ is some fixed member of $\nu^{-1}(\mathscr{A})$. Further, let

$$\Lambda(\sigma, r, s; k, m, p, q) = \bigcap_{n=1}^{s} \pi_{\sigma(n)}^{-1}\big(U(k, p, q, n)\big) \cap \bigcap_{n=1}^{r} \pi_n^{-1}\big(V(m, k, q)\big)$$

and finally,

$$G = \bigcap_{m,p=1}^{\infty}\ \bigcap_{q,r=1}^{\infty}\ \bigcap_{s=1}^{\infty}\ \bigcup_{k=1}^{\infty}\ \bigcup_{\sigma\in\Sigma(r,s)} \Lambda(\sigma, r, s; k, m, p, q).$$

Since $V(m, k, q)$ and $U(k, p, q, n)$ are open, $\Lambda(\sigma, r, s; k, m, p, q)$ is also open, and hence $G$ is a $\mathscr{G}_\delta$.

LEMMA 6. *$G$ is contained in $X$.*

Proof. Fix $\boldsymbol{b}$ in $G$. In view of Lemma 3 (ii), we have to prove that for each $h, r$ in $\boldsymbol{N}$, $W(\boldsymbol{b}; h, r)$ is dense in $E$. Accordingly, we choose $h, r$ and $t$ in $\boldsymbol{N}$ with $t\geqslant h$, and $m_n\in\boldsymbol{N}$ for $1\leqslant n\leqslant t$. It is enough to find $k$ in $\boldsymbol{N}$ and $m_n$ $(n > t)$ such that if $\boldsymbol{m} = (m_n)$,

$$(5) \qquad |1-\exp(2\pi i m_n^{-1} b_n k)| < h^{-1} \qquad (1\leqslant n\leqslant t)$$

and

$$(6) \qquad \nu\big(\psi(\boldsymbol{b}, \boldsymbol{m})\big)^\wedge(k)\in V_r.$$

In particular, we can (and do) assume that $h = t$. Now choose $x\in V_r$ and $\delta > 0$ so that the open interval of length $2\delta$ about $x$ is in $V_r$. Also fix $p, r, s\in\boldsymbol{N}$ such that $r = t$ and

$$(7) \qquad \Big|\prod_{n=1}^{s} \cos 2\pi p a_n - x\Big| < \delta/4.$$

This is possible since $\boldsymbol{a}$ is in $\nu^{-1}(\mathscr{A})$. Finally, fix $m$ and $q$ such that $m = m_1\cdot m_2\cdot \ldots \cdot m_t$ and

$$q > 8\pi\delta^{-1}\max(mr, s).$$

According to the definition of $G$, we can find $k$ in $\boldsymbol{N}$ and $\sigma$ in $\Sigma(r, s)$ corresponding to the choice of $m, p, q, r, s$. For such $k$, $b_n\in V(m, k, q)$ when $1\leqslant n\leqslant h = t$, so that

$$|1-\exp(2\pi i m_n^{-1} b_n k)| = \big|1-\big(\exp(2\pi i m^{-1} b_n k)\big)^{mm_n^{-1}}\big| \leqslant 2\pi m m_n^{-1} q^{-1} < h^{-1},$$

which establishes (5). Similarly,

$$(8)\qquad \Big|1-\prod_{n=1}^{t}\cos 2\pi b_n m_n^{-1}k\Big| \leqslant \sum_{n=1}^{t}\Big|1-\cos 2\pi b_n m_n^{-1}k\Big| \leqslant \sum_{n=1}^{t}\Big|1-\exp(2\pi i b_n m_n^{-1}k)\Big| < 2\pi m t q^{-1} < \delta/4.$$

At this point, we define $m_n$ for $n > t$ as follows:

$$m_n = 1 \quad \text{if } n \text{ belongs to the image of } \sigma;$$
$$m_n = 2^n([4\pi k\delta^{-1}]+1) \quad \text{otherwise}$$

([ ] denotes integer part.)

Further, let

$$\alpha = \prod_{n=1}^{t}\cos 2\pi m_n^{-1}b_n k,$$
$$\beta = \prod_{n\in \operatorname{im}\sigma}\cos 2\pi m_n^{-1}b_n k,$$
$$\gamma = \prod_{\substack{n\notin \operatorname{im}\sigma\\ n>t}}\cos 2\pi m_n^{-1}b_n k,$$

so that $\nu(\psi(\boldsymbol{b}, \boldsymbol{m}))^{\wedge}(k) = \alpha\beta\gamma$. By the definition of $\boldsymbol{m}$,

$$(9)\qquad \Big|\beta-\prod_{n=1}^{s}\cos 2\pi p a_n\Big| \leqslant \sum_{n=1}^{s}\Big|\cos 2\pi b_{\sigma(n)}k-\cos 2\pi p a_n\Big| \leqslant \sum_{n=1}^{s}2\pi\Big\|kb_{\sigma(n)}-pa_n\Big\| < \delta/4,$$

and

$$(10)\qquad |\gamma-1| \leqslant \sum_{\substack{n\notin \operatorname{im}\sigma\\ n>r}}\Big|1-\cos 2\pi b_n m_n^{-1}k\Big| < \sum_{n=r}2^{-n-1}\delta \leqslant \delta/4.$$

Combining (7), (8), (9) and (10), we obtain

$$\big|\nu(\psi(\boldsymbol{b}, \boldsymbol{m}))^{\wedge}(k)-x\big| \leqslant |\alpha-1|+|\beta-x|+|\gamma-1| < \delta$$

which implies (6). This completes the proof.

To complete the proof of Theorem 1, we need to show that $G$ is dense in $B$. This is accomplished by the next lemma.

LEMMA 7. *If $\boldsymbol{b}\in B$ is such that $b_n$ is a strictly positive rational number for each $n$, then $\boldsymbol{b}\in G$.*



Proof. Let $b_n = p_n q_n^{-1}$ where $p_n$ and $q_n$ are positive integers with no common factor. Fix $m, p, q, r, s$ in $\boldsymbol{N}$ and let $\sigma(1)$ be any integer greater than $r$, such that $q^{-1}\cdot q_{\sigma(1)}$ is strictly greater than $m\cdot q_1\cdot q_2\cdot\ldots\cdot q_r$. (Note that $\sigma(1)$ exists since $q_n\to\infty$ as $n\to\infty$.) Having defined $\sigma(1), \sigma(2), \ldots, \sigma(j)$ (for $j < s$), we choose $\sigma(j+1) > \sigma(j)$ such that $q^{-1}\cdot q_{\sigma(j+1)}$ is strictly greater than $m\cdot q_1\cdot q_2\cdot\ldots\cdot q_r\cdot q_{\sigma(1)}\cdot q_{\sigma(2)}\cdot\ldots\cdot q_{\sigma(j)}$. This defines $\sigma\in\Sigma(r, s)$ inductively. Because of the construction of $\sigma$, there exist positive integers $p_i'$ $(1\leqslant i\leqslant s)$ such that

$$(11)\qquad p_i' b_n \equiv p_i' b_{\sigma(j)} \equiv 0 \pmod m$$

for $1\leqslant n\leqslant r$ and $(1\leqslant j < i)$, and the denominator of $p_i' b_{\sigma(i)}$ (in its lowest terms) is strictly greater than $q$. Now we choose $t_1, t_2, \ldots, t_s$ in $\boldsymbol{N}$ inductively so that

$$(12)\qquad \|t_1 p_1' + t_2 p_2' + \ldots + t_i p_i' b_{\sigma(i)} - p a_i\| < q^{-1}$$

for $1\leqslant i\leqslant s$. Define $k = t_1 p_1' + t_2 p_2' + \ldots + t_s p_s'$.

Now it follows from (11) and (12) that

$$\boldsymbol{b}\in\Lambda(\sigma, r, s; k, m, p, q)$$

so that $\boldsymbol{b}\in G$.

The proof of Lemma 7 is based on Kaufman's "Technical Lemma" in [7].

Proof of Corollaries. Clearly Corollary 1 implies Corollary 2. By Theorem 1, there is a positive generalised character $\chi$ such that $\chi_\mu = \boldsymbol{a}$ ($\mu$ a.e.) where $0 < \boldsymbol{a} < 1$. For $d\in\boldsymbol{T}$ and any positive integer $n$

$$(13)\qquad \chi_{\delta(d)*\mu^n}(t) = a^n \quad \big(\delta(d)*\mu^n \text{ a.e.}\big).$$

To prove Corollary 1, we note that if $\lambda$ is absolutely continuous with respect both $\delta(d)*\mu^n$ and $\mu^m$, then using (13), $a^n = \chi_\lambda(t) = a^m$ ($\lambda$ a.e.) so that $n = m$. For Corollary 3, we use the fact that if $0\neq\lambda\in L^1(\boldsymbol{T})$ and $\lambda$ is absolutely continuous with respect to $\mu^n$, then $\chi_\lambda$ is either equal to a character of $\boldsymbol{T}$ or is 0 ($\lambda$ a.e.). Neither of these possibilities can occur if $\chi_{\mu^n}$ is as described in (13). This contradiction proves the result.

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*Received April 27, 1972* (517)

STUDIA MATHEMATICA, T. XLVII. (1973)

## A characterization of $\omega$ by block extensions

by

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**Abstract.** The following characterization of the space $\omega$ is given: A nuclear Fréchet space is isomorphic to $\omega$ if and only if it has a basis such that every block basic sequence of every permutation of that basis has a block extension. In fact it is shown that it suffices to consider only blocks of length $\leq 2$.

In studying nuclear Fréchet spaces it is natural to try to investigate situations which are familiar in Banach space theory and to see if results can be carried over. It is particularly interesting to see a situation in which a simple imitation is *a priori* impossible. This is exactly what occurs if one considers extensions of block basic sequences (see below for definitions). In order to point this out we mention the following two well-known results.

THEOREM OF ZIPPIN [6]. *In a Banach space, every block basic sequence has a block extension.*

THEOREM OF LINDENSTRAUSS AND TZAFRIRI [3]. *A Banach space is isomorphic to $l_p (1 \leq p < \infty)$ or $c_0$ iff it has an unconditional basis such that every block basic sequence of every permutation of this basis generates a complemented subspace.*

Since every basic sequence which can be extended to an *unconditional* basis in a Fréchet space generates a complemented subspace, the second theorem provides many examples in which the first theorem cannot be improved to assert the existence of an unconditional extension when, say, the original basis is unconditional. Actually, the first specific example of this situation was given by Pełczyński [4].

On the other hand, since every basis in a nuclear Fréchet space is unconditional, it follows that at least one of the above two theorems must be false in this context. Indeed, the second author [5] has shown that it is Zippin's theorem which does not carry over. He also pointed out that it does hold for the space $\omega$ and asked if Zippin's theorem characterizes $\omega$ among the nuclear Fréchet spaces.

It is our purpose in this paper to answer that question in the affirmative. Specifically, we prove,