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A fixed point theorem for transformations whose iterates have uniform Lipschitz constant

by

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Abstract. For every uniformly convex Banach space X there exists a constant $\gamma > 1$ which has the following property: If $K \subset X$ is nonempty, bounded, closed and convex, and if $T\colon K \to K$ has the property that each of its iterates T^i is lipschitzian with Lipschitz constant $k < \gamma$, then T has a fixed point in K. Some applications of this result are also discussed.

Let X be a Banach space and K a nonempty, bounded, closed and convex subset of X. A mapping $T\colon K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$. In 1965, F. E. Browder [1], D. Göhde [6], and W. A. Kirk [7] proved independently that if X is uniformly convex then T always has a fixed point in K. (Also see Goebel [3].) It was observed at that time (cf. [7]) that if one only assumes T to be lipschitzian with Lipschitz constant $k > 1$ then T need not have a fixed point, even if X is a Hilbert space and k is arbitrarily near 1. However, there are classes of transformations which lie between the nonexpansive transformations and those with Lipschitz constant $k > 1$ for which fixed point theorems do exist; in particular the asymptotically nonexpansive mappings of Goebel–Kirk [5] form such a class. These are mappings $T\colon K \to K$ having the property that T^i has Lipschitz constant k_i with $k_i \to 1$ as $i \to \infty$. The principal result of [5] states that nonexpansiveness of T in the theorem of Browder–Göhde–Kirk cited above may be replaced by asymptotic nonexpansiveness.

Our purpose here is to generalize the theorem of [5] by a sharpening of the original argument, thus obtaining a fixed point theorem for mappings which are uniformly lipschitzian (see below) with Lipschitz constant k sufficiently near 1 (but greater than 1). We direct several final remarks to applications of this result.

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The *modulus of convexity* of a Banach space X is the function δ: $[0,2]\to[0,1]$ defined as follows:

$$\delta(\varepsilon) = \inf\left[1-\left\|\frac{x+y}{2}\right\|: x, y\in X, \|x\|\leqslant 1, \|y\|\leqslant 1, \|x-y\|\geqslant\varepsilon\right].$$

In the theorem below we assume that X is *uniformly convex* [2], i.e., that $\delta(\varepsilon)>0$ if $\varepsilon>0$. In such a space it is easily verified that δ is strictly increasing and continuous, and moreover (cf. Opial [9]) the inequalities $\|x\|\leqslant d$, $\|y\|\leqslant d$, $\|x-y\|\geqslant\varepsilon$, imply

$$\left\|\frac{x+y}{2}\right\|\leqslant\left(1-\delta\left(\frac{\varepsilon}{d}\right)\right)d.$$

We say that a transformation $T: K\to K, K\subset X$, is *uniformly k-lipschitzian* if for each $x, y\in K$,

$$\|T^i x-T^i y\|\leqslant k\|x-y\|, \quad i=1,2,\ldots.$$

THEOREM 1. *Let X be a uniformly convex Banach space. Then there exists a constant $\gamma>1$ such that if K is a nonempty, bounded, closed and convex subset of X, and if $T: K\to K$ is uniformly k-lipschitzian for $k<\gamma$, then T has a fixed point in K.*

Proof. We take γ to be the solution of the equation $\gamma(1-\delta(1/\gamma))=1$ and assume $1\leqslant k<\gamma$, i.e., assume k satisfies the inequality

$$(*)\qquad k(1-\delta(1/k))<1.$$

For $x\in K$ let $d(x)=\limsup_{i\to\infty}\|x-T^i x\|$, and let $S(x;r)$ denote the closed spherical ball centered at x with radius $r>0$. Fix $x\in K$ and let R consist of those real numbers $\varrho>0$ for which there exists an integer n such that

$$K\cap\left(\bigcap_{i=n}^{\infty}S(T^i x;\varrho)\right)\neq\emptyset.$$

Then $R\neq\emptyset$ (because R contains the diameter of K) so we can define $\varrho_0=\varrho_0(x)=$ g.l.b. R. For each $\varepsilon>0$ define

$$C_\varepsilon=\bigcup_{n=1}^{\infty}\left(\bigcap_{i=n}^{\infty}S(T^i x;\varrho_0+\varepsilon)\right).$$

Then for each $\varepsilon>0$ the sets C_ε are nonempty and convex, so reflexivity of X implies that

$$C=\bigcap_{\varepsilon>0}(\bar{C}_\varepsilon\cap K)\neq\emptyset.$$

Let $z=z(x)\in C$. Notice that z and ϱ_0 have the properties:

(i) for each $\varepsilon>0$, $S(z;\varrho_0+\varepsilon)$ contains almost all terms of the sequence $\{T^i x\}$;

(ii) given $m\in K$ and $\varrho<\varrho_0$, the set $\{i: \|m-T^i x\|>\varrho\}$ is infinite.

Now if $\varrho_0=0$ or if $d(z)=0$, then $\lim_{i\to\infty}T^i x=z$ yielding $Tz=z$. We may therefore assume $\varrho_0>0$, $d(z)>0$. Let $\varepsilon>0$, $\varepsilon\leqslant d(z)$, and select j so that

$$\|z-T^j z\|\geqslant d(z)-\varepsilon.$$

By (i) there exists an integer N such that if $i\geqslant N$, then

$$\|z-T^i x\|\leqslant\varrho_0+\varepsilon\leqslant k(\varrho_0+\varepsilon).$$

Thus if $i-j\geqslant N$,

$$\|T^j z-T^i x\| = \|T^j z-T^j T^{i-j}x\|\leqslant k\|z-T^{i-j}x\|\leqslant k(\varrho_0+\varepsilon).$$

Letting $m=\dfrac{z+T^j z}{2}$, in view of the property of δ we have

$$\|m-T^i x\|\leqslant\left(1-\delta\left(\frac{d(z)-\varepsilon}{k(\varrho_0+\varepsilon)}\right)\right)k(\varrho_0+\varepsilon),$$

for $i\geqslant N+j$.

This implies (by (ii))

$$\varrho_0\leqslant\left(1-\delta\left(\frac{d(z)-\varepsilon}{k(\varrho_0+\varepsilon)}\right)\right)k(\varrho_0+\varepsilon);$$

hence by continuity of δ,

$$\left(1-\delta\left(\frac{d(z)}{k\varrho_0}\right)\right)k\geqslant 1.$$

This implies $d(z)\leqslant k\delta^{-1}(1-1/k)\varrho_0$. Furthermore, because of (ii), we have $\varrho_0\leqslant d(x)$ yielding

$$d(z)\leqslant k\delta^{-1}(1-1/k)d(x).$$

Therefore $d(z)\leqslant a d(x)$ where $a=k\delta^{-1}(1-1/k)$, and $a<1$ because k satisfies $(*)$. Also $\|z-x\|\leqslant d(x)+\varrho_0(x)\leqslant 2d(x)$.

To complete the proof, fix $x_0\in K$ and define the sequence $\{x_n\}$ by $x_{n+1}=z(x_n)$, $n=0,1,\ldots$, where $z(x_n)$ is selected in the same manner as $z(x)$. If for any n we have $\varrho_0(x_n)=0$ then $Tx_{n+1}=x_{n+1}$. Otherwise we have

$$\|x_{n+1}-x_n\|\leqslant 2d(x_n)\leqslant 2a^n d(x_0)$$

which implies $\{x_n\}$ is a Cauchy sequence. Therefore $x_n\to y\in K$ as $n\to\infty$. Also

$$\|y-T^i y\|\leqslant\|y-x_n\|+\|x_n-T^i x_n\|+\|T^i x_n-T^i y\|$$
$$\leqslant(1+k)\|y-x_n\|+\|x_n-T^i x_n\|.$$

This implies $d(y) \leqslant (1+k)\|y-x_n\| + d(x_n) \to 0$ as $n\to\infty$ yielding $Ty = y$.

We now list several consequences of Theorem 1.

1. Let K_1 be the unit ball in Hilbert space and S_1 its boundary. Theorem 1 has a connection with the following problem which was raised recently by K. Goebel [4]: If $T: K_1 \to K_1$ satisfies a Lipschitz condition $\|Tx-Ty\| \leqslant k\|x-y\|$, $x, y \in K_1$, then is $\inf_{x\in K_1} \|x-Tx\| = 0$? The answer to this problem is unknown, but it has the following equivalent formulation:

THEOREM (Goebel [4]). *The existence of a lipschitzian transformation* $T: K_1 \to K_1$ *such that* $\inf_{x\in K_1}\|x-Tx\| > 0$ *is equivalent to the existence of a lipschitzian retraction of* K_1 *onto* S_1.

Theorem 1 implies that there exists a constant $\gamma > 1$ such that for any uniformly k-lipschitzian transformation $T: K_1 \to K_1$ with $k < \gamma$, we have $\inf_{x\in K_1}\|x-Tx\| = 0$. Letting γ^* denote the least upper bound of all numbers γ for which this is true, it is not known if $\gamma^* < \infty$. This problem is also equivalent to the retraction problem:

THEOREM 2. $\gamma^* < \infty$ *if and only if there exists a lipschitzian retraction of* K_1 *onto* S_1.

Proof. If there exists a uniformly lipschitzian transformation $T: K_1 \to K_1$ with the property $\inf_{x\in K_1}\|x-Tx\| > 0$ then according to the result of Goebel cited above there is a lipschitzian retraction of K_1 onto S_1. On the other hand, if R is such a retraction then $T = -R$ is easily seen to be uniformly lipschitzian. Moreover, $\inf_{x\in K_1}\|x-Tx\| > 0$ (see [4]) implying $\gamma^* < \infty$.

2. Kirk has shown [8] that if T is a transformation of a bounded closed convex set in a Banach space X such that $\|T^i x - T^i y\| \leqslant k\|x-y\|$, $i = 1, \ldots, n-1$, and if $T^n = \mathfrak{I}$ (the identity transformation on X), then there exists $\gamma > 1$ (independent of T but depending on n) such that if $k < \gamma$, then T has a fixed point. Theorem 1 implies that in a uniformly convex space such a constant γ may be chosen independently of n.

3. If $T: K\to K$ is uniformly k-lipschitzian then T is nonexpansive with respect to the metric

$$r(x, y) = \sup[\|T^i x - T^i y\|: i = 0, 1, 2, \ldots], \quad x, y \in K,$$

and this metric is equivalent to the norm metric since

$$\|x-y\| \leqslant r(x, y) \leqslant k\|x-y\|.$$

On the other hand, if there is a metric $s(x, y)$ on K such that

$$(**) \qquad \alpha\|x-y\| \leqslant s(x, y) \leqslant \beta\|x-y\|, \quad x, y \in K,$$

and a transformation $T: K\to K$ nonexpansive with respect to $s(x, y)$, then for $i = 1, 2, \ldots$,

$$\|T^i x - T^i y\| \leqslant \frac{1}{\alpha} s(T^i x, T^i y) \leqslant \frac{1}{\alpha} s(x, y) \leqslant \frac{\beta}{\alpha}\|x-y\|.$$

Therefore T is uniformly $\frac{\beta}{\alpha}$-lipschitzian and Theorem 1 implies:

THEOREM 3. *Let X be uniformly convex and $K \subset X$ closed bounded and convex. If $T: K\to K$ is nonexpansive with respect to a metric $s(x, y)$ on K satisfying (**) where $\frac{\beta}{\alpha} < \gamma$ for γ as in Theorem 1, then T has a fixed point in K.*

4. Suppose X is uniformly convex and $K \subset X$ is a lipschitzian retract of some closed bounded convex subset H of X. Theorem 1 assures that if the Lipschitz constant k of the retraction is sufficiently near 1 (but larger than 1) then every nonexpansive mapping of K into itself has a fixed point. Thus, fixed point theorems hold for nonexpansive mappings defined on non-convex domains provided these domains are sufficiently "nice" retracts of convex sets.

5. Theorem 1 actually has a slightly more general formulation. If $T: K\to K$ is continuous then one only need assume in Theorem 1 that T is "eventually" uniformly k-lipschitzian, i.e., that there exist an integer N such that if $x, y \in K$ and $i \geqslant N$, then

$$\|T^i x - T^i y\| \leqslant k\|x-y\|.$$

The proof carries over immediately.

6. We remark that Theorem 1 is an explicit generalization of the theorem of Browder–Göhde–Kirk described in the opening paragraph. In fact, if T is nonexpansive then T is uniformly 1-lipschitzian, and the inequality (*) in the proof of Theorem 1 obviously holds for $k = 1$ when X is uniformly convex. Moreover, one can show (see [5]) that for any $x \in K$ the point $z(x)$ as constructed in the proof of Theorem 1 is fixed under T.

7. The fundamental open question raised by Theorem 1 is, of course, whether or not the solution of $\gamma(1-\delta(1/\gamma)) = 1$ is the largest number γ for which the theorem holds. We note that this number is only slightly larger than 1; for example, in Hilbert space $\gamma = \frac{\sqrt{5}}{2}$.

References

[1] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA 54 (1965), pp. 1041–1044.
[2] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40 (1936), pp. 396–414.
[3] K. Goebel, *An elementary proof of the fixed point theorem of Browder and Kirk*, Michigan Math. J. 16 (1969), pp. 381–383.
[4] — *On the minimal displacement of points under lipschitzian mappings*, Pacific J. Math. (to appear).
[5] — and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35(1972), pp. 171–174.
[6] D. Göhde, *Zum prinzip der kontraktiven Abbildung*, Math. Nachr. 30 (1965), pp. 251–258.
[7] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly 72 (1965), pp. 1004–1006.
[8] — *A fixed point theorem for mappings with a non-expansive iterate*, Proc. Amer. Math. Soc. 29 (1971), pp. 294–298.
[9] Z. Opial, *Lecture notes on nonexpansive and monotone mappings in Banach spaces*, Center for Dynamical Systems, Brown University, Providence, R. I., USA (1967).

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In general, Bernoulli convolutions have independent powers

by

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Abstract. For a sequence $\boldsymbol{b} = (b_n)$ of non-negative real numbers such that $\sum_{n=1}^{\infty} b_n^2 \leqslant 1$, let $\nu(\boldsymbol{b})$ be the measure on the circle $\boldsymbol{T}$ represented by the infinite convolution $\mathop{\ast}_{n=1}^{\infty} \frac{1}{2}(\delta(-b_n)+\delta(b_n))$. It is shown that for a residual set of such $\boldsymbol{b}$'s, the closure in the $\sigma(L^\infty(\nu(\boldsymbol{b})), L^1(\nu(\boldsymbol{b}))$ topology of the characters of $\boldsymbol{T}$ contains all constant functions with values in $[-1, 1]$. It follows that for these measures $\delta(x) \ast \nu(\boldsymbol{b})^r$ is singular to $\nu(\boldsymbol{b})^s$ unless $r = s$.

1. Introduction. Let $\boldsymbol{T}$ be the circle group $\boldsymbol{R}/\boldsymbol{Z}$ and let $M(\boldsymbol{T})$ denote the convolution algebra of bounded regular Borel measures on $\boldsymbol{T}$. Further, let B denote the set of sequences (b_n) of real numbers satisfying $b_n \geqslant 0$ $(n = 1, 2, 3, \ldots)$ and $\sum_{n=1}^{\infty} b_n^2 \leqslant 1$. For each sequence $\boldsymbol{b} = (b_n)$ in B, we write $\nu(\boldsymbol{b})$ for the infinite convolution product

$$\nu(\boldsymbol{b}) = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}\big(\delta(-b_n)+\delta(b_n)\big)$$

where $\delta(x)$ is the positive measure of mass 1 concentrated at $x \in \boldsymbol{T}$. The infinite convolution product converges in the weak* topology by ([10], p. 127), so that we have defined a map $\nu: B \to M(\boldsymbol{T})$. Let $\mathscr{B}$ denote the image of ν (thus $\mathscr{B}$ is a set of symmetric Bernoulli convolutions). We regard B as a subspace of the compact space $[0, 1]^{\aleph_0}$. As such B is a compact Hausdorff space. We shall say that a subset $\mathscr{C}$ of $\mathscr{B}$ is *virtually all* of $\mathscr{B}$ if $\nu^{-1}(\mathscr{C})$ is residual in B (i.e. $B \setminus \nu^{-1}(\mathscr{C})$ is of first category).

Let $\Delta\big(M(\boldsymbol{T})\big)$ denote the maximal ideal space of $M(\boldsymbol{T})$ which we regard as a topological subspace of $\prod\{L^\infty(\mu): \mu \in M(\boldsymbol{T})\}$ where $L^\infty(\mu)$ has the $\sigma\big(L^\infty(\mu), L^1(\mu)\big)$-topology (see [9]). For $\chi \in \Delta\big(M(\boldsymbol{T})\big)$ and $\mu \in M(\boldsymbol{T})$, χ_μ indicates the μ-coordinate of χ. The dual group $\boldsymbol{T}^\wedge (\cong \boldsymbol{Z})$ of $\boldsymbol{T}$ is embedded in $\Delta\big(M(\boldsymbol{T})\big)$ in an obvious and natural way. Using this embedding, we define a subset $\mathscr{A}$ of $\mathscr{B}$ to consist of all measures $\mu \in \mathscr{B}$ having the following property: