A fixed point theorem for transformations whose iterates have uniform
Lipschitz constant

by

K. GOEBEL (Lublin) and W. A. KIRK (Iowa City, la.)

Abstract. For every uniformly convex Banach space $X$ there exists a constant $\gamma > 1$ which has the following property: If $K \subset X$ is nonempty, bounded, closed and convex, and if $T : K \to K$ has the property that each of its iterates $T^i$ is Lipschitzian with Lipschitz constant $k < \gamma$, then $T$ has a fixed point in $K$. Some applications of this result are also discussed.

Let $X$ be a Banach space and $K$ a nonempty, bounded, closed and convex subset of $X$. A mapping $T : K \to K$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. In 1965, F. E. Browder [1], D. Göhde [6], and W. A. Kirk [7] proved independently that if $X$ is uniformly convex then $T$ always has a fixed point in $K$. (Also see Goebel [3]). It was observed at that time (cf. [7]) that if one only assumes $T$ to be Lipschitzian with Lipschitz constant $k > 1$ then $T$ need not have a fixed point, even if $X$ is a Hilbert space and $K$ is arbitrarily near $1$. However, there are classes of transformations which lie between the nonexpansive transformations and those with Lipschitz constant $k > 1$ for which fixed point theorems do exist; in particular the asymptotically nonexpansive mappings of Goebel–Kirk [5] form such a class. These are mappings $T : K \to K$ having the property that $T^i$ has Lipschitz constant $k_i$ with $k_i \to 1$ as $i \to \infty$. The principal result of [5] states that nonexpansiveness of $T$ in the theorem of Browder–Göhde–Kirk cited above may be replaced by asymptotic nonexpansiveness.

Our purpose here is to generalize the theorem of [5] by a sharpening of the original argument, thus obtaining a fixed point theorem for mappings which are uniformly Lipschitzian (see below) with Lipschitz constant $k$ sufficiently near 1 (but greater than 1). We direct several final remarks to applications of this result.

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The modulus of convexity of a Banach space $X$ is the function $\delta: [0, 2] \rightarrow [0, 1]$ defined as follows:

$$\delta(\varepsilon) = \inf \{1 - \frac{\|x + y\|}{2} : x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}.$$

In the theorem below we assume that $X$ is uniformly convex [3], i.e., that $\delta(\varepsilon) > 0$ if $\varepsilon > 0$. In such a space it is easily verified that $\delta$ is strictly increasing and continuous, and moreover (cf. Opial [9]) the inequalities $\|x\| \leq \delta, \|y\| \leq \delta, \|x - y\| \geq \varepsilon$ imply

$$\frac{\|x + y\|}{2} \leq 1 - \delta\left(\frac{\delta}{\varepsilon}\right)^2.$$

We say that a transformation $T: K \rightarrow K, K \subset X$, is uniformly $k$-Lipschitzian if for each $x, y \in K$,

$$\|T^i x - T^i y\| \leq k\|x - y\|, \quad i = 1, 2, \ldots.$$

**Theorem 1.** Let $X$ be a uniformly convex Banach space. Then there exists a constant $\gamma > 1$ such that if $K$ is a nonempty, bounded, and convex subset of $X$, and if $T: K \rightarrow K$ is uniformly $k$-Lipschitzian for $k < \gamma$, then $T$ has a fixed point in $K$.

**Proof.** Take $\gamma$ to be the solution of the equation $\gamma^2 (1 - \delta(1/\gamma)) = 1$ and assume $1 < k < \gamma$, i.e., assume $k$ satisfies the inequality (*).

For $x \in K$ let $d(x) = \lim_{n \to \infty} \|x - T^n x\|$ and let $S(x; r)$ denote the closed spherical ball centered at $x$ with radius $r > 0$. Fix $x \in K$ and let $R$ consist of those real numbers $\varepsilon > 0$ for which there exists an integer $n$ such that

$$K \cap \bigcap_{i=0}^{n-1} S(T^i x; \varepsilon) \neq \emptyset.$$

Then $R \neq \emptyset$ (because $R$ contains the diameter of $K$) so we can define $\varepsilon_0 = \varepsilon_0(x) > 0$ be $\varepsilon_0$. For each $\varepsilon > 0$ define

$$C_\varepsilon = \bigcap_{i=0}^{n-1} S(T^i x; \varepsilon_\varepsilon + \varepsilon).$$

Then for each $\varepsilon > 0$ the sets $C_\varepsilon$ are nonempty and convex, so reflexivity of $X$ implies

$$C = \bigcap_{\varepsilon > 0} (C_\varepsilon \cap K) \neq \emptyset.$$

Let $z = z(x) \in C$. Notice that $z$ and $\varepsilon_0$ have the properties:

(i) for each $\varepsilon > 0$, $S(x_\varepsilon + \varepsilon)$ contains almost all terms of the sequence $(T^i x)$;
and a transformation \( T : K \to K \) nonexpansive with respect to \( s(x, y) \),
then for \( i = 1, 2, \ldots \),
\[
||T^i x - T^i y|| \leq \frac{1}{a} s(T^ix, T^iy) \leq \frac{1}{a} s(x, y) \leq \frac{\beta}{a} ||x - y||.
\]
Therefore \( T \) is uniformly \( \frac{\beta}{a} \)-lipschitzian and Theorem 1 implies:

**Theorem 3.** Let \( X \) be uniformly convex and \( K \subset X \) closed bounded and convex. If \( T : K \to K \) is nonexpansive with respect to a metric \( s(x, y) \) on \( K \) satisfying (**), where \( \frac{\beta}{a} < \gamma \) for \( \gamma \) as in Theorem 1, then \( T \) has a fixed point in \( K \).

4. Suppose \( X \) is uniformly convex and \( K \subset X \) is a lipschitzian retract of some closed bounded convex subset \( H \) of \( X \). Theorem 1 assures that if the lipschitz constant \( k \) of the retraction is sufficiently near \( 1 \) (but larger than \( 1 \)) then every nonexpansive mapping of \( K \) into itself has a fixed point. Thus, fixed point theorems hold for nonexpansive mappings defined on non-convex domains provided these domains are sufficiently "nice" retracts of convex sets.

5. Theorem 1 actually has a slightly more general formulation. If \( T : K \to K \) is continuous then one only need assume in Theorem 1 that \( T \) is "eventually" uniformly \( k \)-lipschitzian, i.e., that there exist an integer \( N \) such that if \( x, y \in K \) and \( i \geq N \), then
\[
||T^i x - T^i y|| \leq k ||x - y||.
\]
The proof carries over immediately.

6. We remark that Theorem 1 is an explicit generalization of the theorem of Browder–Göhde–Kirk described in the opening paragraph. In fact, if \( T \) is nonexpansive then \( T \) is uniformly 1-lipschitzian, and the inequality (*) in the proof of Theorem 1 obviously holds for \( k = 1 \) when \( X \) is uniformly convex. Moreover, one can show (see [5]) that for any \( x \in K \) the point \( s(x) \) as constructed in the proof of Theorem 1 is fixed under \( T \).

7. The fundamental open question raised by Theorem 1 is, of course, whether or not the solution of \( \gamma (1 - \delta(1/\gamma)) = 1 \) is the largest number \( \gamma \) for which the theorem holds. We note that this number is only slightly larger than 1; for example, in Hilbert space \( \gamma = \frac{\sqrt{2}}{2} \).
In general, Bernoulli convolutions have independent powers

by

Gavin Brown and William Moran (Liverpool)

Abstract. For a sequence \( b = (b_n) \) of non-negative real numbers such that \( \sum_{n=1}^{\infty} b_n < 1 \), let \( \nu(b) \) be the measure on the circle \( T \) represented by the infinite convolution
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta(-b_i) + \delta(b_i).
\]
It is shown that for a residual set of such \( b \)'s, the closure in the \( (L^\infty(\nu(b)), L^\infty(\nu(b))) \)-topology of the characters of \( T \) contains all constant functions with values in \([-1,1]\). It follows that for these measures \( \nu(b) \) is singular to \( \nu(b) \) unless \( r = s \).

1. Introduction. Let \( T \) be the circle group \( R/Z \) and let \( M(T) \) denote the convolution algebra of bounded regular Borel measures on \( T \). Further, let \( B \) denote the set of sequences \( (b_n) \) of real numbers satisfying \( b_n \geq 0 \) (\( n = 1, 2, 3, \ldots \)) and \( \sum_{n=1}^{\infty} b_n < 1 \). For each sequence \( b = (b_n) \) in \( B \), we write \( \nu(b) \) for the infinite convolution product

\[
\nu(b) = \lim_{n \to \infty} \frac{1}{n} \left( \delta(-b_n) + \delta(b_n) \right)
\]

where \( \delta(z) \) is the positive measure of mass 1 concentrated at \( z \in T \). The infinite convolution product converges in the weak* topology by [10], p. 121), so that we have defined a map \( \nu : B \to M(T) \). Let \( \mathcal{B} \) denote the image of \( \nu \) (thus \( \mathcal{B} \) is a set of symmetric Bernoulli convolutions). We regard \( B \) as a subspace of the compact space \([0, 1]^\mathbb{N}\). As such \( B \) is a compact Hausdorff space. We shall say that a subset \( \mathcal{A} \) of \( \mathcal{B} \) is virtually all of \( \mathcal{B} \) if \( \nu^{-1}(\mathcal{A}) \) is residual in \( B \) (i.e. \( B \setminus \nu^{-1}(\mathcal{A}) \) is of first category).

Let \( A(M(T)) \) denote the maximal ideal space of \( M(T) \) which we regard as a topological subspace of \( [L^\infty(\mu) : \mu \in M(T)] \) where \( L^\infty(\mu) \) has the \( (L^\infty(\mu), L^\infty(\mu)) \)-topology (see [9]). For \( \chi \in A(M(T)) \) and \( \mu \in M(T) \), \( \chi \mu \) indicates the \( \mu \)-coordinate of \( \chi \). The dual group \( T^* \) (or \( Z \)) of \( T \) is embedded in \( A(M(T)) \) in an obvious and natural way. Using this embedding, we define a subset \( \mathcal{A} \) of \( \mathcal{B} \) to consist of all measures \( \mu \in \mathcal{A} \) having the following property: