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The compact endomorphisms of the metric linear spaces $\mathscr{L}_\varphi$

by

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Abstract. For a topological linear space X, the relationship between the existence of non-trivial continuous linear functionals and the existence of non-trivial compact endomorphisms for X is studied. In this connection, several examples of topological linear spaces having only the trivial compact endomorphism are given.

0. Introduction. Starting-point of this paper is a result, found by J. H. Williamson in 1953, according to which for every topological linear space the existence of a compact endomorphism with an eigenvalue different from zero always implicates the existence of a non-trivial continuous linear functional. In this connection the question arises if there exist non-trivial compact endomorphisms for topological linear spaces with no continuous linear functionals but zero.

In this paper we treat this question for the metric linear spaces $\mathscr{L}_\varphi$ of the φ-integrable functions defined on some measure space $(X, \boldsymbol{A}, \mu)$ (see S. Cater [1], B. Gramsch [4] and W. Orlicz [6]). We begin with some general results concerning the relations between compact endomorphisms and continuous linear functionals. A chapter on $\mathscr{L}_\varphi$-spaces follows. The main part of this paper is concerned with the problem of the existence of compact endomorphisms for these spaces.

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1. General results. Let E and F denote real topological linear spaces. Then a linear mapping

$$T\colon E \to F$$

is called "*bounded*" (resp. "*compact*") if it maps a 0-neighborhood of E into a bounded (resp. compact) subset of F.

A real topological linear space E is said to be "*uniformly bounded*" if for any 0-neighborhood U of E there exists a positive integer n such that

$$U^n := \underbrace{U + \ldots + U}_{n} = E.$$

Then we have:

THEOREM 1.1. *Let F be a real topological linear space and E a uniformly bounded real topological linear space.*

If a linear mapping

$$T\colon E\to F$$

is bounded, then

$$T=0.$$

Proof. Let us suppose that there is a non-trivial bounded linear map

$$T\colon E\to F.$$

Then there exists a 0-neighborhood U in E such that $T(U)$ is a bounded subset of F and an element $x\in E\setminus\{0\}$ such that

$$y:=Tx\in F\setminus\{0\}.$$

Since E is uniformly bounded there is a positive integer n such that for any positive integer k there exist elements

$$x_1^k,\ldots,x_n^k\in U$$

with

$$kx=\sum_{i=1}^{n}x_i^k.$$

Consequently we have

$$ky=T(kx)=\sum_{i=1}^{n}T(x_i^k)\in\sum_{i=1}^{n}T(U),$$

and therefore

$$\{ky\mid k\in N\}\subseteq\sum_{i=1}^{n}T(U),$$

this being contrary to the boundedness of $\sum_{i=1}^{n}T(U)$. ■

COROLLARY. *The only compact linear mapping of a uniformly bounded real topological linear space into a real topological linear space is the zero-operator.*

In order to enter into the relation between the compact endomorphisms of a topological linear space and its continuous linear functionals, we recall the following notion introduced by A. Pełczyński.

A real topological linear space E is said to be "*transitive*" if for any two points

$$x,y\in E\setminus\{0\}$$

there exists a continuous endomorphism

$$A\colon E\to E$$

with

$$Ax=y.$$

The following result is due to A. Pełczyński:

THEOREM 1.2. *Let E be a transitive real topological linear space. Then the following assertions are equivalent:*

(i) *There exists a non-trivial continuous linear functional on E.*

(ii) *There exists a non-trivial compact endomorphism of E.*

Proof. i) ⇒ ii). Let f be a non-trivial continous linear functional on E and $x_0\in E\setminus\{0\}$. Then

$$T\colon E\to E\quad\text{with}\quad T(x):=f(x)x_0$$

is a non-trivial compact endomorphism of E.

(ii) ⇒ (i). Let

$$T\colon E\to E$$

be a non-trivial compact endomorphism of E. Then there is an element $x\in E\setminus\{0\}$, such that

$$y:=Tx$$

is also in $E\setminus\{0\}$.

Since E is transitive there is a continuous linear mapping

$$A\colon E\to E$$

with

$$Ay=x$$

and consequently

$$ATx=x.$$

By this we have found a compact endomorphism with 1 as an eigenvalue. According to the result of J. H. Williamson [10] the existence of a non-trivial continuous linear functional f is proved. ■

2. The Spaces $\mathcal{L}_\varphi$. In the following let (X,A,μ) be a σ-finite measure space on a set X. Furthermore let L denote the real linear space of all μ-measurable functions on X, which is partially ordered by the relation $\leqslant$ induced from $\boldsymbol{R}$, let Θ be the linear subspace of L consisting of all functions, which are equal to the zero almost everywhere on X with respect to μ, and finally let be $L_X:=L/_\Theta$.

Now we call two elements x and y of L_X "*orthogonal*"-indicated by the symbol $x \perp y$ — if there are representants x' and y' of the classes x and y respectively such that

$$\mu\{t \in X \mid x'(t)y'(t) \neq 0\} = 0;$$

$x, y \in L_X$ are said to be "*equi-measurable*"-denoted by $x \sim y$ — if there exists representants x' and y' of x and y respectively such that

$$\mu\{t \in X \mid x'(t) > r\} = \mu\{t \in X \mid y'(t) > r\}$$

holds for all $r \in \boldsymbol{R}$.

A sequence $(x_n)_{n \in \boldsymbol{N}}$ of elements of L_X is said to be "*monotonely increasing*" (resp. "*monotonely descending*") if $x_n \leqslant x_{n+1}$ (resp. $x_n \geqslant x_{n+1}$) holds for any $n \in N$.

DEFINITION 2.1. Let $(X, \boldsymbol{A}, \mu)$ be a σ-finite measure space on a set X. A linear subspace $L(X, \mu)$ of L_X endowed with a metric d, invariant under translations is called an "*$\mathscr{L}$-space on X*", if the following conditions hold:

(i) $(L(X, \mu), d)$ is a real complete metric linear space;

(ii) for any $A \in \boldsymbol{A}$ with $\mu(A) < \infty$ the characteristic function $\varkappa_A$ satisfies $\varkappa_A \in L(X, \mu)$, and if $y \in L(X, \mu)$, $x \in L_X$ and $|x| \leqslant |y|$ then $x \in L(X, \mu)$;

(iii) if $x, y \in L(X, \mu)$ and $|x| < |y|$, then $d(x, 0) < d(y, 0)$;

(iv) if $y \in L(X, \mu)$, $x \in L_X$ and $x \sim y$, then $x \in L(X, \mu)$ and $d(x, 0) = d(y, 0)$;

(v) if $x, y \in L(X, \mu)$ and $x \perp y$, then $d(x, y) = d(x, 0) + d(y, 0)$;

(vi) any monotonely descending sequence $(x_n)_{n \in \boldsymbol{N}}$, $x_n \in L(X, \mu)$ converging to 0 almost everywhere satisfies

$$\lim_n x_n = 0.$$

In [2] L. Drewnowski and W. Orlicz gave an axiomatic description of modular lattices. Definition 2.1 originates from that paper and characterizes the metric linear spaces $\mathscr{L}_\varphi$ studied by S. Cater [1], B. Gramsch [4] and W. Orlicz [6]. According to [2] the metric of an $\mathscr{L}$-space is monotonic, additive and absolutely continuous with respect to the measure μ and it follows from the theorem of Radon-Nikodym that d can be represented by an integral.

The following theorem is given in [2].

THEOREM 2.2. *Let $(X, \boldsymbol{A}, \mu)$ be a non-atomic σ-finite measure space on a set X and $(L(X, \mu), d)$ an $\mathscr{L}$-space on X.*

Then there is a unique continuous monotonely increasing subadditive function

$$\varphi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+ \quad \textit{with} \quad \varphi(0) = 0,$$

such that

$$d(x, 0) = \int_X \varphi(|x|)\, d\mu$$

holds for any $x \in L(X, \mu)$.

The function $\varphi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$, specified in the proposition of this theorem is called the "*modular function of the metric d*". Its computation is rather easy (see [2]), since we can take an element $A \in \boldsymbol{A}$ with $0 < \mu(A) < \infty$ and determine

$$\varphi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+ \quad \text{by} \quad \varphi(t) := \big(\mu(A)\big)^{-1} d(t\chi_A, 0).$$

The concept of the $\mathscr{L}_\varphi$-spaces is derived from the representation of the metric as an integral by means of the modular function. Therefore an $\mathscr{L}$-space $(L(X, \mu), d)$, to the metric d of which the modular function

$$\varphi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$$

belongs, is shortly called an "*$\mathscr{L}_\varphi$-space*".

3. Special compact endomorphisms. If $(X, \boldsymbol{A}, \mu)$ is a non-atomic measure algebra, i.e. a non-atomic measure space with $\mu(X) < \infty$, and if $(L(X, \mu), d)$ is an $\mathscr{L}$-space with a bounded metric, we easily realize that $L(X, \mu)$ is topologically isomorphic to the space $S(X, \mu)$ of all classes of, μ-measurable functions on X endowed with the topology of convergence in measure. Since this space is uniformly bounded we get in particular:

THEOREM 3.1. *Let $(X, \boldsymbol{A}, \mu)$ be a non-atomic measure algebra defined on a set X and let $S(X, \mu)$ denote the real topological linear space consisting of all classes of μ-measurable functions on X endowed with the topology of convergence in measure.*

Then there exists no non-trivial bounded linear mapping of $S(X, \mu)$ into a real topological linear space, especially the ideal of the compact endomorphisms of $S(X; \mu)$ is identical with the zero-operator.

Now we give more examples for the Theorems 1.1 and 1.2. Let for this purpose be $([0,1], \boldsymbol{A}, \lambda)$ the measure algebra generated by the Lebesgue-measure λ on the interval $[0,1]$. For the complete metric linear space $S([0,1], \lambda)$ of all classes of λ-measurable functions on $[0,1]$, N. T. Peck [7] constructed a decreasing sequence $(\tau_n)_{n \in \boldsymbol{N}}$ of metrizable linear topologies, all of them being weaker than the topology of convergence in measure.

Especially the spaces $(S([0,1], \lambda), \tau_n)$ as well as the completions of them are uniformly bounded and hence do not have any non-trivial compact endomorphisms.

If we consider $0 < p < 1$ and the mapping

$$\varphi\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+ \quad \text{given by} \quad \varphi(t) := t^p$$

we are able to show that the appropriate $\mathscr{L}_\varphi$-space $L^p([0,1], \lambda)$ is transitive. (See [8], Theorem IX. 6.4.).

THEOREM 3.2. *The complete metric linear space $L^p([0,1],\lambda)$ with $0<p<1$ has no non-trivial compact endomorphisms.*

More complicated is the case for general $\mathscr{L}_\varphi$-spaces. Under certain restrictions concerning the measure space S. Rolewicz found the following result:

THEOREM 3.3. *Let X be a separable metric space, such that it is borelian in its completion X. Let μ be a non-atomic borelian measure defined on X and suppose that $\big(L(X,\mu),d\big)$ is an $\mathscr{L}$-space on X.*

Then the following assertions are equivalent:

(i) *There exists a non-trivial continuous linear functional on $L(X,\mu)$.*

(ii) *There exists a non-trivial compact endomorphism of $L(X,\mu)$.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ i) is based on the following lemma:

LEMMA. *Let A and B be two separable metric spaces borelian in their completions and let μ_A,μ_B be non-atomic borelian measures on A and B respectively such that $\mu_A(A)=\mu_B(B)<\infty$.*

Then there is a one-to-one mapping

$$F\colon A\to B$$

such that F and F^{-1} are borelian and moreover $\mu_B\big(F(E)\big)=\mu_A(E)$ for each measurable borelian set E in A.

(See for example [9], Ch. VI. § 5 Ex. 27.)

Let

$$T\colon L(X,\mu)\to L(X,\mu)$$

be a non-trivial compact endomorphism of $L(X,\mu)$. Since the simple functions are dense in $L(X,\mu)$ and the measure is borelian, there is a borelian set A of finite positive measure such that

$$z:=T\chi_A\in L(X,\mu)\setminus\{0\}.$$

Moreover we may assume that there is a $c>0$ and a borelian set B of finite positive measure such that for any $t\in B$, $z(t)>c$. Let us put

$$j(t):=\begin{cases}0 & \text{for } t\in X\setminus B,\\ \dfrac{1}{z(t)} & \text{for } t\in B.\end{cases}$$

The operator

$$J\colon L(X,\mu)\to L(X,\mu)$$

given by the product

$$J(x):=jx$$

is continuous and we have

$$JT\chi_A=\chi_B.$$

Let us write $b:=\dfrac{\mu(B)}{\mu(A)}$. Let μ_A be the measure $b\mu$ restricted to the set A and let μ_B be the measure μ restricted to B. The sets A,B and the measures μ_A,μ_B satisfy the hypothesis of the lemma. Therefore there exists a measure preserving mapping

$$F\colon A\to B$$

of A onto B.

Let us define a linear map

$$P\colon L(X,\mu)\to L(X,\mu)$$

as follows

$$Px|_t:=\begin{cases}0 & \text{for } t\in X\setminus A\\ x\big(F(t)\big) & \text{for } t\in A.\end{cases}$$

It is easy to verify that P is continuous and that $P\chi_B=\chi_A$.

Thus $PJT\chi_A=\chi_A$ and the compact endomorphism PJT has 1 as an eigenvalue. According to the result of J. H. Williamson [10] the existence of an $f\in L(X,\mu)'\setminus\{0\}$ is proved. ∎

For a compact subset of an $\mathscr{L}_\varphi$-space on any measure space we have:

LEMMA 3.4. *Let $(X,\boldsymbol{A},\mu)$ be a σ-finite measure space on a set X. Furthermore let $\big(L(X,\mu),d\big)$ be an $\mathscr{L}$-space on X with an unbounded metric and $K\subseteq L(X,\mu)$ a compact subset.*

Then the following statements hold:

i) $\sup\limits_{x\in K} d(x,0)<\infty$,

ii) *for each $\varepsilon>0$ there exists a $\delta>0$ such that for each $A\in\boldsymbol{A}$ with $\mu(A)<\delta$*

$$\sup_{x\in K} \mathrm{d}(x\chi_A,0)\leqslant\varepsilon.$$

Using this lemma we get:

THEOREM 3.5. *Let $(X,\boldsymbol{A},\mu)$ be a non-atomic σ-finite measure space on a set X. Furthermore suppose that $\big(L(X,\mu),d\big)$ is an $\mathscr{L}$-space on X such that the modular function*

$$\varphi\colon \boldsymbol{R}^+\to\boldsymbol{R}^+$$

belonging to d satisfies the following conditon

$$\varliminf_n \frac{1}{n}\varphi\left(\frac{\varphi^{-1}(n)}{n}\right)=a>0.$$

Then the ideal of compact endomorphisms of $L(X,\mu)$ consists only of the zero operator.

Proof. Let

$$T\colon L(X,\mu)\to L(X,\mu)$$

denote a non-trivial compact endomorphism. Then, without restricting generality, we may suppose that the image of the open unit ball $B(0,1)$ under T is contained in a compact subset of $L(X,\mu)$. Since the simple functions are dense in $L(X,\mu)$, there exists an $A\in\boldsymbol{A}$ with $1>\mu(A)>0$ such that

$$z:=T\chi_A\in L(X,\mu)\setminus\{0\}.$$

Since $z\neq 0$ there is a real number $c>0$, such that

$$0<\mu\{t\in X \mid |z(t)|>c\}$$

and we define

$$B:=\{t\in X \mid |z(t)|>c\}.$$

The mapping

$$J:L(X,\mu)\to L(X,\mu),$$

given by the product

$$J(x):=jx,$$

where

$$j:X\to\boldsymbol{R}$$

denotes the bounded μ-measurable function defined by

$$j(t):=\begin{cases}\operatorname{sign} z(t), & t\in X\setminus B\\ c^{-1}\operatorname{sign} z(t), & t\in B,\end{cases}$$

is a continuous endomorphism. Therefore the composition

$$T_1:=J\circ T:\ L(X,\mu)\to L(X,\mu)$$

is again a compact endomorphism, which maps the open unit ball $B(0,1)$ into a compact subset K_1 of $L(X,\mu)$ and for which

$$\chi_B\leqslant T_1\chi_A$$

is valid.

Since $\mu(A)=a<1$ there is a partition $A_1^n,\ldots,A_n^n$ of A for each $n\in\boldsymbol{N}$, such that for each $i\in\{1,\ldots,n\}$

$$\mu(A_i^n)=\frac{a}{n};$$

consequently, for each $n\in\boldsymbol{N}$ and each $i\in\{1,\ldots,n\}$

$$y_i^n:=\varphi^{-1}(n)\chi_{A_i^n}\in B(0,1)$$

and the equation

$$\frac{\varphi^{-1}(n)}{n}\chi_A=\frac{1}{n}(y_1^n+\ldots+y_n^n)$$

holds for each $n\in\boldsymbol{N}$.

If we set

$$I_k^n:=\left\{t\in X \;\middle|\; |(T_1y_k^n)(t)|\geqslant\frac{\varphi^{-1}(n)}{n}\right\}$$

for each $n\in\boldsymbol{N}$ and each $k\in\{1,\ldots,n\}$, then we obtain from

$$\frac{\varphi^{-1}(n)}{n}\chi_B\leqslant T_1\left(\frac{\varphi^{-1}(n)}{n}\chi_A\right)=\frac{1}{n}\left(\sum_{i=1}^{n}T_1y_i^n\right)$$

that

$$\frac{\varphi^{-1}(n)}{n}\chi_B\leqslant\max\{|T_1y_1^n|\chi_{I_1^n},\ldots,|T_1y_n^n|\chi_{I_n^n}\}.$$

By Lemma 3.4 (i)

$$M:=\sup_{x\in K_1}d(x,0)<\infty$$

and this implies

$$\varphi\left(\frac{\varphi^{-1}(n)}{n}\right)\mu(I_k^n)\leqslant M$$

for each $n\in\boldsymbol{N}$ and each $k\in\{1,\ldots,n\}$ and therefore

$$\mu(I_k^n)=\frac{M}{\varphi\left(\frac{\varphi^{-1}(n)}{n}\right)}.$$

As consequence of the assumption

$$\lim_n\frac{1}{n}\varphi\left(\frac{\varphi^{-1}(n)}{n}\right)=a>0$$

there exists a subsequence $(n_r)_{r\in\boldsymbol{N}}$ of the positive integers such that the sequence

$$\left(\varphi\left(\frac{\varphi^{-1}(n_r)}{n_r}\right)\right)_{r\in\boldsymbol{N}}$$

is strictly monotonely increasing and unbounded.

Now, for any positive $\varepsilon<\frac{1}{2}(a\mu(B))$, let us select according to Lemma 3.4 (ii) the positive real number δ proper to K_1. For $\delta>0$ there exists again an $r_0\in\boldsymbol{N}$ — set $m:=n_{r_0}$ — such that for each $k\in\{1,\ldots,m\}$

$$\mu(I_k^m)\leqslant\delta$$

holds. Since

$$T_1y_k^m\in K_1$$

for each $k \in \{1, \ldots, m\}$, it follows from

$$\frac{\varphi^{-1}(m)}{m} \chi_B \leqslant \max\{|T_1 y_1^m| \chi_{I_1^m}, \ldots, |T_1 y_m^m| \chi_{I_m^m}\}$$

that

$$\varphi\left(\frac{\varphi^{-1}(m)}{m}\right)\mu(B) \leqslant \sum_{k=1}^{m} d(|T_1 y_k^m| \chi_{I_k^m}, 0) \leqslant m\varepsilon,$$

hence

$$\frac{1}{m}\varphi\left(\frac{\varphi^{-1}(m)}{m}\right) < \frac{a}{2}$$

and this contradicts the assumption.■

Providing an example for such an $\mathcal{L}$-space we take for X the unit intervall $[0, 1]$ and for μ the Lebesgue measure λ. The desired $\mathcal{L}$-space then is the $\mathcal{L}_\varphi$-space $(L([0, 1], \lambda), d)$ studied in detail by B. Gramsch [4], which belongs to the modular function

$$\varphi : \boldsymbol{R}^+ \to \boldsymbol{R}^+$$

with

$$\varphi(t) := \log(1+t).$$

Here we have for each $n \in \boldsymbol{N}$

$$\frac{1}{n}\varphi\left(\frac{\varphi^{-1}(n)}{n}\right) = \frac{1}{n}\log\left(\frac{n-1}{n} + \frac{e^n}{n}\right) \geqslant \frac{1}{n}\log\left(\frac{e^n}{n}\right) = 1 - \frac{\log n}{n},$$

hence

$$\varliminf_n \frac{1}{n}\varphi\left(\frac{\varphi^{-1}(n)}{n}\right) \geqslant 1.$$

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