The compact endomorphisms of the metric linear spaces $X$

by

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Abstract. For a topological linear space $X$, the relationship between the existence of non-trivial continuous linear functionals and the existence of non-trivial compact endomorphisms for $X$ is studied. In this connection, several examples of topological linear spaces having only the trivial compact endomorphism are given.

0. Introduction. Starting-point of this paper is a result, found by J. H. Williamson in 1963, according to which for every topological linear space the existence of a compact endomorphism with an eigenvalue different from zero always implicates the existence of a non-trivial continuous linear functional. In this connection the question arises if there exist non-trivial compact endomorphisms for topological linear spaces with no continuous linear functionals but zero.

In this paper we treat this question for the metric linear spaces $X_0$ of the $p$-integrable functions defined on some measure space $(X, A, \mu)$ (see S. Czer [1], B. Gramach [4] and W. Orlicz [6]). We begin with some general results concerning the relations between compact endomorphisms and continuous linear functionals. A chapter on $X_0$-spaces follows. The main part of this paper is concerned with the problem of the existence of compact endomorphisms for these spaces.

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1. General results. Let $E$ and $F$ denote real topological linear spaces. Then a linear mapping

$T : E \to F$

is called "bounded" (resp. "compact") if it maps a 0-neighborhood of $E$ into a bounded (resp. compact) subset of $F$.

A real topological linear space $E$ is said to be "uniformly bounded" if for any $0$-neighborhood $U$ of $E$ there exists a positive integer $n$ such that

$U^n := U + \ldots + U = E$.
Then we have:

**Theorem 1.1.** Let $E$ be a real topological linear space and $F$ a uniformly bounded real topological linear space.

If a linear mapping
\[ T: E \to F \]
is bounded, then
\[ T = 0. \]

*Proof.* Let us suppose that there is a non-trivial bounded linear map
\[ T: E \to F. \]
Then there exists a $0$-neighborhood $U$ in $E$ such that $T(U)$ is a bounded subset of $F$ and an element $x \in E \setminus \{0\}$ such that
\[ y := Tx \notin F \setminus \{0\}. \]
Since $E$ is uniformly bounded there is a positive integer $n$ such that for any positive integer $k$ there exist elements
\[ x_1^k, \ldots, x_n^k \in U \]
with
\[ kx = \sum_{i=1}^n x_i^k. \]
Consequently we have
\[ ky = T(kx) = \sum_{i=1}^n T(x_i^k) \in \sum_{i=1}^n T(U), \]
and therefore
\[ \{ky \mid k \in \mathbb{N}\} \subseteq \sum_{i=1}^n T(U), \]
this being contrary to the boundedness of $\sum_{i=1}^n T(U)$. ■

**Corollary.** The only compact linear mapping of a uniformly bounded real topological linear space into a real topological linear space is the zero-operator.

In order to enter into the relation between the compact endomorphisms of a topological linear space and its continuous linear functionals, we recall the following notion introduced by A. Pelczynski.

A real topological linear space $E$ is said to be "transitive" if for any two points
\[ x, y \in E \setminus \{0\} \]
there exists a continuous endomorphism
\[ A: E \to E \]
with
\[ Ax = y. \]

The following result is due to A. Pelczynski:

**Theorem 1.2.** Let $E$ be a transitive real topological linear space. Then the following assertions are equivalent:

(i) There exists a non-trivial continuous linear functional on $E$.

(ii) There exists a non-trivial compact endomorphism of $E$.

*Proof.* (i) $\Rightarrow$ (ii). Let $f$ be a non-trivial continuous linear functional on $E$ and $x \in E \setminus \{0\}$. Then
\[ T: E \to E \quad \text{with} \quad T(x) = f(x)x. \]
is a non-trivial compact endomorphism of $E$.

(ii) $\Rightarrow$ (i). Let
\[ T: E \to E \]
be a non-trivial compact endomorphism of $E$. Then there is an element $x \in E \setminus \{0\}$, such that
\[ y := Tx \]
is also in $E \setminus \{0\}$.
Since $E$ is transitive there is a continuous linear mapping
\[ A: E \to E \]
with
\[ Ay = x \]
and consequently
\[ ATx = x. \]

By this we have found a compact endomorphism with $1$ as an eigenvalue. According to the result of J. H. Williamson [10] the existence of a non-trivial continuous linear functional $f$ is proved. ■

2. The Spaces $\mathcal{S}_e$. In the following let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space on a set $X$. Furthermore let $L$ denote the real linear space of all $\mu$-measurable functions on $X$, which is partially ordered by the relation $\leq$ induced from $R$, let $\sigma$ be the linear subspace of $L$ consisting of all functions, which are equal to the zero almost everywhere on $X$ with respect to $\mu$, and finally let be $L_X := L/\sigma$.

Now we call two elements $x$ and $y$ of $L_X$ "orthogonal"-indicated
by the symbol $\equiv$ — if there are representatives $x'$ and $y'$ of the classes $x$ and $y$ respectively such that $\mu(\{t \in X \mid x'(t) \equiv y'(t)\}) \neq 0$;

$x, y \in L_X$ are said to be "equi-measurable" denoted by $x \sim y$ — if there exists representatives $x'$ and $y'$ of $x$ and $y$ respectively such that $\mu(\{t \in X \mid x'(t) > r\}) = \mu(\{t \in X \mid y'(t) > r\})$

holds for all $r \in R$.

A sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $L_X$ is said to be "monotonically increasing" (resp. "monotonically decreasing") if $x_n \leq x_{n+1}$ (resp. $x_n \geq x_{n+1}$) holds for any $n \in \mathbb{N}$.

**Definition 2.1.** Let $(X, A, \mu)$ be a $\sigma$-finite measure space on a set $X$. A linear subspace $L(X, \mu)$ of $L_X$ endowed with a metric $d$, invariant under translations is called an "$L$-space on $X$", if the following conditions hold:

(i) $(L(X, \mu), d)$ is a real complete metric linear space;

(ii) for any $f, g \in A$ with $\mu(A) < \infty$ the characteristic function $\chi_A$ satisfies $\chi_A \in L(X, \mu)$, and if $f, g \in L(X, \mu)$ and $|f| \leq |g|$ then $\chi_A \in L(X, \mu)$;

(iii) if $f, g \in L(X, \mu)$ and $|f| \leq |g|$, then $d(f, 0) < d(g, 0)$;

(iv) if $y \in L(X, \mu)$, $x \in L_X$ and $x \sim y$, then $x \in L(X, \mu)$ and $d(x, 0) = d(y, 0)$;

(v) if $x, y \in L(X, \mu)$ and $x \equiv y$, then $d(x, y) = d(x, 0) + d(y, 0)$;

(vi) any monotonically decreasing sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \in L(X, \mu)$ converging to $0$ almost everywhere satisfies $\lim_{n \to \infty} x_n = 0$.

In [2] L. Drewnowski and W. Orlicz gave an axiomatic description of modular lattices. Definition 2.1 originates from that paper and characterizes the metric linear spaces $L_\varphi$ studied by S. Cater [1], B. Grzegczak [4] and W. Orlicz [5]. According to [2] the metric of an $L$-space is monotonic, additive and absolutely continuous with respect to the measure $\mu$ and it follows from the theorem of Radon-Nikodym that $d$ can be represented by an integral. The following theorem is given in [2].

**Theorem 3.1.** Let $(X, A, \mu)$ be a $\sigma$-finite measure algebra defined on a set $X$ and let $S(X, \mu)$ denote the real topological linear space consisting of all classes of $\mu$-measurable functions on $X$ endowed with the topology of convergence in measure.

Then there exists no non-trivial bounded linear mapping of $S(X, \mu)$ into a real topological linear space, especially the ideal of the compact endomorphisms of $S(X, \mu)$ is identical with the zero-operator.

Now we give more examples for the Theorems 1.1 and 1.2. Let for this purpose be $(0, 1, A, \lambda)$ the measure algebra generated by the Lebesgue-measure $\lambda$ on the interval $[0, 1]$. For the complete metric linear space $S([0, 1], \lambda)$ all classes of $\lambda$-measurable functions on $[0, 1]$, N. T. Peck [7] constructed a decreasing sequence $(r_n)_{n \in \mathbb{N}}$ of metrizable linear topologies, all of them being weaker than the topology of convergence in measure.

Especially the spaces $S([0, 1], \lambda)$ as well as the completions of them are uniformly bounded and hence do not have any non-trivial compact endomorphisms.

If we consider $0 < p < 1$ and the mapping

$$\varphi: R^+ \to R^+ \quad \text{given by} \quad \varphi(t) := t^p$$

we are able to show that the appropriate $L_\varphi$-space $L_\varphi([0, 1], \lambda)$ is transitive.

(See [8], Theorem IX. 6.4.).
Theorem 3.2. The complete metric linear space $L^p([0, 1], \lambda)$ with $0 < p < 1$ has no non-trivial compact endomorphisms.

More complicated is the case for general $\mathcal{L}_p$-spaces. Under certain restrictions concerning the measure space $S$, Rolewicz found the following result:

Theorem 3.3. Let $X$ be a separable metric space, such that it is borelian in its completion $\bar{X}$. Let $\mu$ be a non-atomic borelian measure defined on $X$ and suppose that $(L(X, \mu), d)$ is an $\mathcal{L}_p$-space on $X$.

Then the following assertions are equivalent:

(i) There exists a non-trivial continuous linear functional on $L(X, \mu)$.

(ii) There exists a non-trivial compact endomorphism of $L(X, \mu)$.

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) is based on the following lemma:

Lemma. Let $A$ and $B$ be two separable metric spaces borelian in their completions and let $\mu_A, \mu_B$ be non-atomic borelian measures on $A$ and $B$ respectively such that $\mu_A(A) = \mu_B(B) < \infty$.

Then there is a one-to-one mapping

$$ F: A \to B $$

such that $F$ and $F^{-1}$ are borelian and moreover $\mu_B(F(B)) = \mu_A(A)$ for each measurable borelian set $E$ in $A$.

(See for example [9], Ch. VI, § 5 Ex. 27.)

Let

$$ T: L(X, \mu) \to L(X, \mu) $$

be a non-trivial compact endomorphism of $L(X, \mu)$. Since the simple functions are dense in $L(X, \mu)$ and the measure is borelian, there is a borelian set $E$ of finite positive measure such that

$$ T: E \to L(X, \mu) \setminus \{0\}. $$

Moreover we may assume that there is an $e > 0$ and a borelian set $B$ of finite positive measure such that for any $t \in B$, $T(t) > e$. Let us put

$$ j(t):= \begin{cases} 0 & \text{for } t \in X \setminus B, \\ \frac{1}{s(t)} & \text{for } t \in B. \end{cases} $$

The operator

$$ J: L(X, \mu) \to L(X, \mu) $$

given by the product

$$ J(x) = jx $$

is continuous and we have

$$ JT_A = \chi_B. $$
denote a non-trivial compact endomorphism. Then, without restricting generality, we may suppose that the image of the open unit ball \( B(0, 1) \) under \( T \) is contained in a compact subset of \( L(X, \mu) \). Since the simple functions are dense in \( L(X, \mu) \), there exists an \( A \in A \) with \( 1 > \mu(A) > 0 \) such that

\[ z := T_{t \chi_A} L(X, \mu) \setminus \{0\} . \]

Since \( z \neq 0 \) there is a real number \( c > 0 \), such that

\[ 0 < \mu(\{ t \chi_X \mid |z(t)| > c \}) \]

and we define

\[ B := \{ t \chi_X \mid |z(t)| > c \} . \]

The mapping

\[ J : L(X, \mu) \to L(X, \mu) , \]

given by the product

\[ J(x) := j_x , \]

where

\[ j : X \to B \]

denotes the bounded \( \mu \)-measurable function defined by

\[ j(t) := \begin{cases} \text{sign}(t), & t \in X \setminus B, \\ \frac{\text{sign}(t)}{c}, & t \in B , \end{cases} \]

is a continuous endomorphism. Therefore the composition

\[ T_1 := J \circ T : L(X, \mu) \to L(X, \mu) \]

is again a compact endomorphism, which maps the open unit ball \( B(0, 1) \) into a compact subset \( K_1 \) of \( L(X, \mu) \) and for which

\[ Z_1 \leq T_1 Z_1 \]

is valid.

Since \( \mu(A) = a < 1 \) there is a partition \( A^1, \ldots, A^n \) of \( A \) for each \( n \in \mathbb{N} \) such that for each \( i \in \{1, \ldots, n\} \)

\[ \mu(A^i) = \frac{a}{n} ; \]

consequently, for each \( n \in \mathbb{N} \) and each \( i \in \{1, \ldots, n\} \)

\[ y_i := \frac{1}{n} x_i B(0, 1) \]

and the equation

\[ \frac{\varphi^{-1}(n)}{n} Z_1 = \frac{1}{n} (y_1^* + \ldots + y_n^*) \]

holds for each \( n \in \mathbb{N} \).

If we set

\[ \mathcal{T}^n := \left\{ t \in X \mid \langle T_1 g, t \rangle \geq \frac{\varphi^{-1}(n)}{n} \right\} \]

for each \( n \in \mathbb{N} \) and each \( k \in \{1, \ldots, n\} \), then we obtain from

\[ \frac{\varphi^{-1}(n)}{n} Z_1 \leq T_1 \left( \frac{\varphi^{-1}(n)}{n} Z_1 \right) \leq \frac{1}{n} \sum_{i=1}^n T_1 g_i^* \]

that

\[ \frac{\varphi^{-1}(n)}{n} Z_1 \leq \max \left( \langle T_1 g_1^* g_1, \ldots, \langle T_1 g_n^* g_n \rangle \right) . \]

By Lemma 3.4 (i)

\[ M := \sup_{x \in K_1} \| x \| < \infty ; \]

and this implies

\[ \frac{\varphi^{-1}(n)}{n} \mu(T_1^n) \leq M \]

for each \( n \in \mathbb{N} \) and each \( k \in \{1, \ldots, n\} \) and therefore

\[ \mu(T_1^n) = \frac{M}{\varphi^{-1}(n)} . \]

As consequence of the assumption

\[ \lim_{n \to \infty} \varphi^{-1}(n) = a > 0 \]

there exists a subsequence \( (n_i)_{i \in \mathbb{N}} \) of the positive integers such that the sequence

\[ \left( \frac{1}{n_i} \varphi^{-1}(n_i) \right) \in \mathfrak{K}_1 \]

is strictly monotonically increasing and unbounded.

Now, for any positive \( \varepsilon \leq \frac{1}{2} (\mu(B)) \), let us select according to Lemma 3.4 (ii) the positive real number \( \delta \) proper to \( K_1 \). For \( \delta > 0 \) there exists again an \( n \in \mathbb{N} \) such that for each \( k \in \{1, \ldots, n\} \)

\[ \mu(T_1^n) \leq \delta \]

holds. Since

\[ T_1 y_n^* \in \mathfrak{K}_1 , \]

we have
for each \( k \in \{1, \ldots, m\} \), it follows from

\[
\frac{\varphi^{-1}(m)}{m} x_k \leq \max \{\|T_1 y_1^m x_k \|, \ldots, \|T_m y_m^m x_k \|\}
\]

that

\[
\varphi \left( \frac{\varphi^{-1}(m)}{m} \right) \mu(B) \leq \sum_{k=1}^m \delta(\|T_i y_i^m x_k \|, 0) \leq m \varepsilon,
\]

hence

\[
\frac{1}{m} \varphi \left( \frac{\varphi^{-1}(m)}{m} \right) < \frac{\alpha}{2}
\]

and this contradicts the assumption. \( \blacksquare \)

Providing an example for such an \( \mathcal{R} \)-space we take for \( X \) the unit interval \([0, 1]\) and for \( \mu \) the Lebesgue measure \( \lambda \). The desired \( \mathcal{R} \)-space then is the \( \mathcal{R}_\psi \)-space \((\mathbb{R}, [0, 1], \lambda, d)\) studied in detail by B. Gramsch [4], which belongs to the modular function

\[
\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+
\]

with

\[
\varphi(t) := \log(1 + t).
\]

Here we have for each \( n \in \mathbb{N} \)

\[
\frac{1}{n} \varphi \left( \frac{\varphi^{-1}(n)}{n} \right) = \frac{1}{n} \log \left( \frac{n - 1}{n} + \frac{\varepsilon}{\lambda} \right) = \frac{1}{n} \log \left( \frac{\varepsilon}{\lambda} \right) = 1 - \frac{\log n}{n},
\]

hence

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \varphi \left( \frac{\varphi^{-1}(n)}{n} \right) = 1.
\]

References