STUDIA MATHEMATICA, T. XLVI. (1973)

On generalized eigenfunctions of operators in a Hilbert space

by

KAZIMIERZ NAPIÓRKOWSKI (Warszawa)

Abstract. A triplet $\Phi \subset H \subset \Phi'$ is constructed for a given commutative C^*-algebra of bounded operators in the Hilbert space H. It is shown that the generalized eigenfunctions exist iff the corresponding eigenvalue belongs to the spectrum of the algebra. Some sufficient conditions of this kind are found in the case of an arbitrary triplet.

Let H be a separable Hilbert space and A a bounded selfadjoint operator in H. It is well-known that, in general, A does not possess a complete system of eigenvectors in H. Generalizations of the notion of eigenvector have been considered by many authors and can be found in books [1]–[4]. One of the ways is to take a locally convex space Φ and a continuous linear injection $\Phi \subset H$ [1] such that Φ is dense in H. The scalar product in H defines then an antilinear injection $H \subset \Phi$. If $A(\Phi) \subset \Phi$ and the restriction of A to Φ is continuous as a mapping $\Phi \to \Phi$ then one can consider the eigenvalues- and eigenfunction problem for the operator $A'\colon \Phi' \to \Phi'$, i. e. to find $\lambda \in C^1$ and $e_\lambda \in \Phi'$ such that $A' e_\lambda = \lambda e_\lambda$. The functionals e_λ are called *generalized* (*strong*) *eigenvectors*.

The authors mentioned above found various sufficient conditions for the space Φ to obtain a "large enough" collection of generalized eigenvectors i. e. for almost every $\lambda \in \mathrm{Sp}(A)$ there exists a generalized eigenvector corresponding to λ.

It can happen that this collection is "too large", i. e. it contains also eigenvectors e_λ for $\lambda \notin \mathrm{Sp}(A)$. We consider the problem of finding a space Φ for which the collection of generalized eigenvectors is "large enough" but not "too large". Because we do not use direct integral decompositions of H, all the conditions are fulfilled for every λ, not only for almost every λ.

Let $\mathscr{A}$ be a commutative C^*-algebra with unit, elements of which are bounded linear operators in H. Since each commutative C^*-algebra possesses a fundamental vector, we can, by using the Zorn's lemma, obtain an ordered family of vectors $(h_i)_{i \in (I, <)}$ satisfying the following conditions:

---

[1] We identify Φ and its image in H.

a) $h_i \in \big(\bigcup_{j<i} \mathscr{A} h_j\big)^\perp$; $\|h_i\| = 1$,

b) h_i is a fundamental vector for $\mathscr{A}\,|_{(\bigcup_{j<i} \mathscr{A} h_j)^\perp}$,

c) $\big(\bigcup_{i\in I} \mathscr{A} h_i\big)^\perp = \{0\}$, I has a minimal element 1.

The separability of H implies that I must be countable. It can be easily seen [4, 5] that I can be chosen either finite or the set of natural numbers.

We define

$$\Phi_i := \mathscr{A} h_i,$$
$$\|\varphi_i\|_i := \inf\{\|A\|:\ \varphi_i = Ah_i,\ A \in \mathscr{A}\}.$$

Let Φ be the algebraic direct sum $\bigoplus_{i\in I} \Phi_i$. The norm in Φ is defined by

$$\|\varphi\|_\Phi := \Big(\sum_{i\in I} \|\varphi_i\|_i^2\Big)^{1/2}.$$

Since $\Phi_i \subset (\Phi_j)^\perp$ for $i \neq j$, we may consider Φ as a subspace of H. From the definition of $\|\varphi_i\|_i$ we get $\|\varphi_i\|_i \geqslant \|\varphi_i\|_H$ and $\|\varphi\|_\Phi \geqslant \|\varphi\|_H$. This means that the injection $\Phi \subset H$ is continuous and c) implies that Φ is dense in H. From the construction of Φ it follows that $A(\Phi) \subset \Phi$ for every $A \in \mathscr{A}$ and that the restriction of A is continuous as a mapping Φ into Φ.

Let $\mathrm{Sp}(\mathscr{A})$ denote the set of all continuous linear multiplicative functionals on $\mathscr{A}$.

THEOREM 1. *Let $\mathscr{A}$, Φ be as above. Then*

1° *$\lambda \in \mathrm{Sp}(\mathscr{A})$ if and only if there exists $e_\lambda \in \Phi'$ such that for each $A \in \mathscr{A}$ $A' e_\lambda = \lambda(A) e_\lambda$,*

2° *Let $H_\lambda = \{e_\lambda \in \Phi' :\ A' e_\lambda = \lambda(A) e_\lambda$ for each $A \in \mathscr{A}\}$. H_λ is a linear closed subspace of Φ'. The norm on H_λ induced from Φ' is a Hilbert space norm. Let H_i be the closed subspace of H spanned by Φ_i and let $H_{i\lambda}$ be the set $\{e_\lambda \in H_\lambda :\ e_\lambda(\Phi_j) = 0$ for $j \neq i\}$. $H_{i\lambda}$ is a one-dimensional subspace of H_λ. Let $e_{i\lambda}$ be such an element of $H_{i\lambda}$ that $\langle h_i, e_{i\lambda}\rangle = 1$. Then the mapping $\varphi_i \to \varphi_i(\cdot)$ defined by $\hat{\varphi}_i(\lambda) = \langle \varphi_i, e_{i\lambda}\rangle$ can be uniquely extended to a unitary operator mapping H_i onto $L^2(\mathrm{Sp}(\mathscr{A}), \mu_i)$ where μ_i is the spectral measure generated by h_i. Besides $H = \bigoplus_{i\in I} H_i$.*

Proof. It follows from the construction of Φ that Φ' can be represented as the space of sequences $(f_i)_{i\in I}$, where $f_i \in \Phi_i'$ and $\sum_{i\in I} \|f_i\|^2 < \infty$. This way we can consider Φ_i' as subspaces of Φ'.

From b) it follows that h_1 is a separating vector for $\mathscr{A}$, hence the mapping $A \to Ah_1$ defines an isometry $J:\ \mathscr{A} \to \Phi_1$. Let now $\lambda \in \mathrm{Sp}(\mathscr{A})$.

The functional $e_\lambda = \lambda \circ J^{-1}$ is an element of Φ_1' and can be considered as an element of Φ'. Let $\varphi = Ah_1$ and $B \in \mathscr{A}$. Then

$$\langle B\varphi, e_\lambda\rangle = \langle BAh_1, e_\lambda\rangle = \lambda(BA) = \lambda(B)\lambda(A) = \lambda(B)\langle Ah_1, e_\lambda\rangle = \lambda(B)\langle \varphi, e_\lambda\rangle.$$

This means $B' e_\lambda = \lambda(B) e_\lambda$.

To prove the reverse we remark that a common generalized eigenvector e_λ defines a functional λ by the equation $A' e_\lambda = \lambda(A) e_\lambda$. The functional λ is automatically linear and multiplicative. $e_\lambda = \sum_{i\in I} e_{i\lambda}$, where $e_{i\lambda} \in \Phi_i'$. We take such i that $\langle h_i, e_{i\lambda}\rangle \neq 0$. Then

$$|\langle Ah_i, e_{i\lambda}\rangle| = |\lambda(A)|\,|\langle h_i, e_{i\lambda}\rangle| \leqslant \|Ah_i\|_\Phi \cdot \|e_{i\lambda}\|_{\Phi'} \leqq \|A\|_{\mathscr{A}} \cdot \|e_{i\lambda}\|_{\Phi'}.$$

We see that $|\lambda(A)| \leqslant \|A\|_{\mathscr{A}} \|e_{i\lambda}\|_{\Phi'} (|\langle h_i, e_{i\lambda}\rangle|)^{-1}$, hence λ is continuous. The proof of 1° is complete.

2° We show first that $\dim H_{i\lambda} = 1$. If it is not, we can pick two linearly independent generalized eigenvectors $e_{i\lambda}$ and $\hat{e}_{i\lambda}$. We can normalize them so that $\langle h_i, e_{i\lambda}\rangle = \langle h_i, \hat{e}_{i\lambda}\rangle = 1$. (If e. g. $\langle h_i, e_{i\lambda}\rangle = 0$ then $\langle Ah_i, e_{i\lambda}\rangle = \lambda(A)\langle h_i, e_{i\lambda}\rangle = 0$ and $e_{i\lambda} = 0$.) Then we get $\langle h_i, e_{i\lambda} - \hat{e}_{i\lambda}\rangle = 0$, whence $e_{i\lambda} = \hat{e}_{i\lambda}$. To see that H_λ is a Hilbert subspace of Φ' we compute the norm of e_λ

$$\|e_\lambda\| = \sup_{\|\varphi\|=1} |\langle \varphi, e_\lambda\rangle| = \sup_{\|\varphi\|=1} \Big|\sum_{i\in I} c_i \langle \varphi_i, e_{i\lambda}\rangle\Big|$$
$$= \sup_{\|\varphi\|=1} \sum_{i\in I} |c_i|\,\|\varphi_i\| = \Big(\sum_{i\in I} |c_i|^2\Big)^{1/2}.$$

Thus H_λ is isometric with l_J^2, where J is a subset of I.

For the proof of the rest of the theorem it is enough to remark that every $A \in \mathscr{A}\,|_{H_i}$ defines a function $\lambda \to \lambda(A) = \langle Ah_i, e_{i\lambda}\rangle$ which maps $\mathrm{Sp}(\mathscr{A}\,|_{H_i})$ continuously into the set of complex numbers. The mapping from $\mathscr{A}\,|_{H_i}$ into the space $\mathscr{C}(\mathrm{Sp}(\mathscr{A}\,|_{H_i}))$ is just the Gelfand isomorphism. The direct integral decomposition of H_i is obtained by repeating the method of [3].

THEOREM 2. *Let $\mathscr{A}$ be a commutative C^*-algebra of bounded operators in a Hilbert space H. Let $\Phi \subset H$ be a locally convex space dense and continuously embedded in H. Let Φ be invariant for $\mathscr{A}$ and let the restrictions of elements of $\mathscr{A}$ be continuous as mappings from Φ into Φ. If the mapping from $\mathscr{A}$ into $L(\Phi', \Phi')$ defined by $A \to A'$ is weakly continuous then the generalized eigenfunctions exist only for $\lambda \in \mathrm{Sp}(\mathscr{A})$.*

Proof. Let e_λ be a generalized eigenvector. Then for $\varphi \in \Phi$ and $A \in \mathscr{A}$

$$\langle \varphi, A' e_\lambda\rangle = \lambda(A)\langle \varphi, e_\lambda\rangle.$$

When $\|A\| \to 0$ then from our assumption the left-hand side tends to zero, and if φ is chosen such that $\langle \varphi, e_\lambda\rangle \neq 0$ we can conclude that $\lambda(A) \to 0$ i. e. $\lambda \in \mathrm{Sp}(\mathscr{A})$.

THEOREM 3. *Let H, $\mathscr{A}$ and Φ be as in Theorem 2. Let $\varphi \in \Phi$ possess the property that $\mathscr{A}\varphi$ is dense in Φ. If the mapping $A\varphi \mapsto A$ is continuous as a mapping of a subset of Φ into $\mathscr{A}$ then for each $\lambda \in \mathrm{Sp}(\mathscr{A})$ there exists a generalized eigenvector e_λ corresponding to λ.*

Proof. The mapping J satisfying $J(A\varphi) = A$ is well defined because φ is a separating vector for $\mathscr{A}$. For a given $\lambda \in \mathrm{Sp}(\mathscr{A})$ we define $\hat{e}_\lambda = \lambda \circ J$. $\hat{e}_\lambda$ is continuous as a composition of two continuous mappings, hence it can be (uniquely) extended onto the whole space Φ. We denote the extension by e_λ. It is to prove that $A'e_\lambda = \lambda(A)e_\lambda$ for $A \in \mathscr{A}$. We take an element of Φ of the form $B\varphi$ and compute:

$$\langle B\varphi, A'e_\lambda\rangle = \langle AB\varphi, e_\lambda\rangle = \langle AB\varphi, \hat{e}_\lambda\rangle = \lambda\big(J(AB\varphi)\big) = \lambda(AB)$$
$$= \lambda(A)\cdot\lambda(B) = \lambda(A)\,\langle B\varphi, \hat{e}_\lambda\rangle = \langle B\varphi, \lambda(A)e_\lambda\rangle.$$

Since we have assumed that the set of vectors $B\varphi$, $B \in \mathscr{A}$ is dense in Φ, the equality $\langle \psi, A'e_\lambda\rangle = \langle \psi, \lambda(A)e_\lambda\rangle$ holds for every $\psi \in \Phi$.

References

[1] Yu. M. Bieriezanskii, *Eigenfunction expansion for selfadjoint operators*. Kiev 1965.

[2] I. M. Gelfand, N. Ya. Vilenkin, *Some applications of harmonic analysis*, Moscow 1961.

[3] K. Maurin, *Methods of Hilbert spaces*, Warsaw 1967.

[4] — *General eigenfunction expansions and unitary representations of topological groups*, Warsaw 1968.

[5] M. H. Stone, *Linear transformations in Hilbert space*. Am. Math. Soc. Coll. Publ. vol. XV (1932).

DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS
WARSAW UNIVERSITY

Received February 22, 1972 (486)

STUDIA MATHEMATICA, T. XLIV. (1973)

Über die Existenz von Schauderbasen in Sobolev–Besov-Räumen. Isomorphiebeziehungen

von

HANS TRIEBEL (Jena)

Zusammenfassung. In der Arbeit wird die Struktur der Sobolev–Slobodeckij-Räume W_p^s, der Lebesgue-Räume H_p^s und der Besov-Räume $B_{p,q}^s$ untersucht: Isomorphie zu l_p, (l_q), $L_p((0,1))$, Existenz von Schauderbasen, Skaleneigenschaften.

1. Einleitung. Mit R_n wird der n-dimensionale reelle euklidische Raum bezeichnet. Ω ist ein beschränktes Gebiet, $\Omega \subset R_n$, der Klasse C^∞. Die Räume $L_p(R_n)$, $L_p\big((0,1)\big)$ und l_p haben die übliche Bedeutung, $1 < p < \infty$. In der Arbeit werden die Lebesgue-Räume $H_p^s(R_n)$, $H_p^s(\Omega)$, $\mathring{H}_p^s(\Omega)$, die Sobolev–Slobodeckij-Räume $W_p^s(R_n)$, $W_p^s(\Omega)$, $\mathring{W}_p^s(\Omega)$ und die Besov-Räume $B_{p,q}^t(R_n)$, $B_{p,q}^t(\Omega)$ untersucht; $0 \leqslant s < \infty$; $0 < t < \infty$; $1 < p < \infty$; $1 \leqslant q < \infty$. Sämtliche Räume sind komplex. Die genaue Definition erfolgt im zweiten Abschnitt. Das Ziel der Arbeit ist der Nachweis der Existenz von Schauderbasen in den genannten Räumen, sowie der Beweis von Isomorphiebeziehungen.

Um die Resultate besser formulieren zu können, führen wir den Begriff der *Skala* von Banachräumen ein (Lifteigenschaft). Es sei $-\infty < M_1 < M_2 < \infty$. $\{B_\lambda\}_{M_1 \leqslant \lambda \leqslant M_2}$ sei eine Menge von Banachräumen,

$$B_{\lambda_1} \subset B_{\lambda_2} \quad \text{für } M_1 \leqslant \lambda_2 \leqq \lambda_1 \leqslant M_2.$$

($\subset$ bezeichnet stets eine stetige Einbettung). $\{B_\lambda\}_{M_1 \leqslant \lambda \leqslant M_2}$ heißt Skala, falls es eine Menge linearer Operatoren $\{A_\varrho\}_{0 \leqslant \varrho \leqslant M_2 - M_1}$ gibt mit:

(a) A_ϱ ist auf $B_{M_1+\varrho}$ definiert und vermittelt eine isomorphe Abbildung von $B_{M_1+\varrho}$ auf ganz B_{M_1}.

(b) Ist $A_\varrho^{(\varkappa)}$ die Einschränkung von A_ϱ auf $B_{M_1+\varrho+\varkappa}$; $\varkappa \geqslant 0$; $M_1 + \varrho + \varkappa \leqslant M_2$; so vermittelt $A_\varrho^{(\varkappa)}$ eine isomorphe Abbildung von $B_{M_1+\varrho+\varkappa}$ auf ganz $B_{M_1+\varkappa}$.

(c) $A_{\varrho_1}^{(\varkappa)} A_{\varrho_2}^{(\varrho_1+\varkappa)} = A_{\varrho_1+\varrho_2}^{(\varkappa)}$; $(\varrho_1 + \varrho_2 + \varkappa \leqslant M_2 - M_1)$.

(Es ist ausreichend, die letzte Eigenschaft für $\varkappa = 0$ zu fordern).

Das Hauptziel der Arbeit ist der Beweis des folgenden Satzes.

SATZ. (a) *Es sei $N > 0$ und $1 < p < \infty$ vorgegeben. Dann sind $\{H_p^s(R_n)\}_{0 \leqslant s \leqslant N}$ und $\{H_p^s(\Omega)\}_{0 \leqslant s \leqslant N}$ Skalen. Sämtliche Räume $H_p^s(R_n)$,*