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On linear functionals in Hardy–Orlicz spaces, I

by

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Abstract. The paper can be regarded as a continuation of the paper "On Hardy–Orlicz spaces. I", Comm, Math. 15 (1971), pp. 3–56. The paper contains the study of spaces of linear functionals continuous in norm, continuous in modular and very weakly continuous on the Hardy–Orlicz space $H^{*\varphi}$ and on the space of finite elements $H^{\circ\varphi}$. Mutual relations between these spaces and the question of the extension of linear functionals from $H^{\circ\varphi}$ on $H^{*\varphi}$ are considered.

The main results of this paper were earlier announced in [4]. This paper can be regarded as a second part of the paper [6] which contains the study of Hardy–Orlicz spaces. Some results of the paper [6] and other papers will be needed here. We collect them in the first section.

I. ORLICZ AND HARDY-ORLICZ SPACES

1.1. *A φ-function* we call a real, nondecreasing and continuous for $u \geqslant 0$ function, equal 0 only at $u = 0$ and tending to ∞ when $u \to \infty$.

1.2. On φ-functions we impose sometimes the following conditions:

(Δ_2) $$\varphi(2u) \leqslant d\varphi(u),$$

(V_2) $$2\varphi(u) \leqslant \varphi(du),$$

(∇_2) $$2\varphi(u) \leqslant d^{-1}\varphi(du)$$

for $u \geqslant u_0$ with some constants $d > 1$ and $u_0 \geqslant 0$.

1.3. Among φ-functions we distinguish *log-convex φ-functions* which satisfy the inequality

$$\varphi(u^\alpha v^\beta) \leqslant \alpha\varphi(u) + \beta\varphi(v) \quad \text{for } u, v > 0 \text{ and } \alpha, \beta \geqslant 0,\ \alpha+\beta = 1,$$

and *convex φ-function* which satisfy the inequality

$$\varphi(\alpha u + \beta v) \leqslant \alpha\varphi(u) + \beta\varphi(v) \quad \text{for } u, v \geqslant 0 \text{ and } \alpha, \beta \geqslant 0,\ \alpha+\beta = 1.$$

Clearly, a φ-function φ is log-convex if and only if it can be represented in the form

(*) $$\varphi(u) = \Phi(\log u) \quad \text{for } u > 0,$$

where Φ is a convex function on the whole real axis. From this it follows that a log-convex φ-function φ is strictly increasing for $u \geqslant 0$ and so, it has an inverse function φ_{-1}. Convex φ-functions and more generally functions of the form $\varphi(u) = \psi(u^s)$ for $u \geqslant 0$, where ψ is a convex φ-function and $s > 0$, are a particular case of log-convex φ-functions.

1.4. On convex φ-function ψ we impose frequently the following conditions:

(0_1) $$\lim_{u\to 0+} u^{-1}\psi(u) = 0$$

and

(∞_1) $$\lim_{u\to\infty} u^{-1}\psi(u) = \infty.$$

Under these conditions for a convex φ-function ψ we define a function ψ' *complementary* to ψ by the formula

$$\psi'(v) = \sup\{uv - \psi(u):\ u \geqslant 0\}, \quad (v \geqslant 0).$$

The function ψ' is also convex φ-function, satisfies the conditions (0_1) (∞_1) and moreover, $(\psi')' = \psi$.

In the sequel only log-convex φ-functions φ for which a convex function Φ from their representation $(*)$ satisfies the condition (∞_1) will have applications and therefore the letter φ will be used only for these functions.

2.1. Let f be a complex-valued function, defined and measurable on the interval $[0, 2\pi)$. We define

$$\mathscr{I}_\varphi(f) = \int_0^{2\pi} \varphi(|f(t)|)\,dt.$$

In the space of all complex-valued functions, defined and measurable on $[0, 2\pi)$ the functional $\mathscr{I}_\varphi$ is a modular in the sense of Musielak and Orlicz.

2.2. By L^φ we denote the class of all complex-valued functions f, measurable on $[0, 2\pi)$ for which $\mathscr{I}_\varphi(f) < \infty$, by $L^{*\varphi}$ the class of all functions f such that $\alpha f \in L^\varphi$ for a certain $\alpha > 0$ (in general dependent on f) and by $L^{\circ\varphi}$ the class of all functions f such that $\alpha f \in L^\varphi$ for every $\alpha > 0$.

In the space of measurable on $[0, 2\pi)$ complex-valued functions the class L^φ is an absolutely convex set and the classes $L^{*\varphi}$ and $L^{\circ\varphi}$ are linear subspaces. The class L^φ is called *Orlicz class*, $L^{*\varphi}$ *Orlicz spaces* and $L^{\circ\varphi}$ *the space of finite elements* of $L^{*\varphi}$, ([7], [8]).

2.3. Generally, in the space $L^{*\varphi}$ the functional

$$\|f\|_\varphi^* = \inf\{\varepsilon > 0:\ \mathscr{I}_\varphi(f/\varepsilon) \leqslant \varepsilon\}, \quad (f \in L^{*\varphi}),$$

is a complete F-norm and $L^{0\varphi}$ is identical with the closure of the space of all continuous function on $[0, 2\pi]$ in the space $[L^{*\varphi}, \|\cdot\|_\varphi^*]$.

If $\varphi(u) = \psi(u^s)$ for $u \geqslant 0$, where ψ is a convex φ-function and $0 < s \leqslant 1$, then an s-homogeneous norm can be defined in $L^{*\varphi}$ by the formula

$$\|f\|_{s\varphi}^* = \inf\{\varepsilon > 0:\ \mathscr{I}_\varphi(f/\varepsilon^{1/s}) \leqslant 1\}, \quad (f \in L^{*\varphi}).$$

The norm $\|\cdot\|_\varphi^*$ and $\|\cdot\|_{s\varphi}^*$ are equivalent on $L^{*\varphi}$ ([7], [9]).

If ψ is a convex φ-function satisfying the conditions (0_1) and (∞_1), then besides the homogeneous norm $\|\cdot\|_{1\psi}^*$ in $L^*\psi$ another homogeneous norm can be introduced by the formula

$$\|f\|_{(\psi)}^* = \sup\Big\{\int_0^{2\pi} |f(t)g(t)|\,dt:\ \mathscr{I}_{\psi'}(g) \leqslant 1\Big\}, \quad (f \in L^{*\psi}).$$

The norms $\|\cdot\|_{1\psi}^*$ and $\|\cdot\|_{(\psi)}^*$ are equivalent; namely they satisfy the inequality

$$\|f\|_{1\psi}^* \leqslant \|f\|_{(\psi)}^* \leqslant 2\,\|f\|_{1\psi}^* \quad \text{for every } f \in L^{*\psi}.$$

The norm $\|\cdot\|_{(\psi)}^*$ may be written in the form

$$\|f\|_{(\psi)}^* = \inf\Big\{\frac{1}{\varepsilon}\big(1 + \mathscr{I}_\psi(\varepsilon f)\big) : \varepsilon > 0\Big\} \quad \text{for every } f \in L^{*\psi},$$

([2], Chap. II).

2.4. In general case in Orlicz space $L^{*\varphi}$ we have two concepts of convergence for sequences; one is a norm convergence and other is a modular convergence. We say that a sequence $\{f_n\} \subset L^{*\varphi}$ is *convergent in norm* to $f \in L^{*\varphi}$, if $\|f_n - f\|_\varphi^* \to 0$ as $n \to \infty$; this holds if and only if $\mathscr{I}_\varphi\big(\alpha(f_n - f)\big) \to 0$ as $n \to \infty$ for any $\alpha > 0$. Besides of this, we say that a sequence $\{f_n\} \subset L^{*\varphi}$ is *convergent in modular* to $f \in L^{*\varphi}$, if $\mathscr{I}_\varphi\big(\alpha(f_n - f)\big) \to 0$ as $n \to \infty$ for some $\alpha > 0$ (in general dependent on $\{f_n\}$ and f). In the case when φ satisfies the condition (Δ_2), $L^{*\varphi} = L^{\circ\varphi}$ and the norm and modular convergences are equivalent. Otherwise, we have only $L^{\circ\varphi} \subset L^{*\varphi}$ and only the norm convergence implies the modular convergence.

2.5. With respect to these two convergences in Orlicz space $L^{*\varphi}$ we have norm continuous and modular continuous linear functionals on $L^{*\varphi}$ and on $L^{\circ\varphi}$. In the case, when $\liminf_{u\to\infty} u^{-1}\varphi(u) = 0$, there does not exist a nontrivial modular continuous linear functional on $L^{*\varphi}$ (c. e. g. [12]). In the case, when ψ is a convex φ-function satisfying conditions (0_1) and (∞_1), the formula

$$\xi(f) = \int_0^{2\pi} f(t)g(t)\,dt, \quad (f \in L^{*\psi}),$$

where $g \in L^{*\psi'}$, yields the general form of a modular continuous linear functional on $L^{*\psi}$ ([12]). In the case this same formula yields also the general form of a norm continuous linear functional on $L^{\circ\psi}$ ([2]). It is known

([2]) that Orlicz space $[L^{*\psi}, \|\cdot\|_{1\psi}^{*}]$ where ψ is a convex φ-function, is reflexive if and only if ψ satisfies the conditions (Δ_2) and (∇_2).

2.6. Let f be a 2π-periodic function integrable on $[0, 2\pi)$. It is known ([14]) that the integral

$$\hat{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{f(x+t)-f(x-t)}{2\operatorname{tg}(t/2)}\,dt = \lim_{\varepsilon\to 0+}\left\{-\frac{1}{\pi}\int_{\varepsilon}^{\pi}\ldots\right\}$$

exists almost everywhere; the function $\hat{f}$, defined by this integral, is called the *conjugate function* to f.

R. Ryan has shown in the paper [13] that the mapping $f\to\hat{f}$ is a continuous linear operator from Orlicz space $[L^{*\psi}, \|\cdot\|_{1\psi}^{*}]$ into same Orlicz space, where ψ is a convex φ-function, (it holds iff this mapping sends $L^{*\psi}$ into $L^{*\psi}$, see [5]), if and only if ψ satisfies the conditions (Δ_2) and (∇_2).

3.1. Let F be an analytic function in the disc $D = \{z: |z| < 1\}$. We define

$$\mu_\varphi(r;\, F) = \int_0^{2\pi} \varphi(|F(re^{it})|)\,dt \quad \text{for } 0 \leqslant r < 1$$

and

$$\mu_\varphi(F) = \sup\{\mu_\varphi(r;\, F):\ 0 \leqslant r < 1\}.$$

$\mu_\varphi(r;\, F)$ is a nondecreasing function for $0 \leqslant r < 1$. For any function F analytic in D the following inequality

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{\mu_\varphi(F)}{\pi(1-|z|)}\right) \quad \text{for } z\in D$$

holds ([6]).

3.2. By H^φ we denote the class of all functions F analytic in D such that $\mu_\varphi(F) < \infty$ and by $H^{*\varphi}$ the class of all functions F such that $aF\in H^\varphi$ for a certain $a > 0$ (in general dependent on F). Moreover, by $H^{\circ\varphi}$ we denote the class of all functions F such that $aF\in H^\varphi$ for every $a > 0$.

The class H^φ is a convex set in the space of all analytic functions in D and the classes $H^{*\varphi}$ and $H^{\circ\varphi}$ are linear subspaces. The class H^φ we call *Hardy-Orlicz class*, $H^{*\varphi}$ *Hardy-Orlicz space and* $H^{\circ\varphi}$ *the space of finite elements in* $H^{*\varphi}$.

3.3. The class $H^{*\varphi}$ is a subset of the class N' of all functions F analytic in D for which the integrals

$$\int_0^{x} \log^+|F(re^{it})|\,dt \quad \text{for } 0 \leqslant r < 1,\ 0 \leqslant x \leqslant 2\pi$$

where $\log^+ u = \log\sup\{1, u\}$, are uniformly absolutely continuous functions of x. It is known ([14]) that functions $F\in N'$ have the nontangential limits

$$\lim_{z\to e^{it}} F(z) = F(e^{it})$$

almost everywhere on the circumference $\{z: |z| = 1\}$ and that for $F\in N'$ $F(e^{it}) = 0$ for almost all $t\in[0, 2\pi)$ implies $F(z) = 0$ for all $z\in D$.

For $F\in N'$, in particular for $F\in H^\varphi$, we have here

$$\mu_\varphi(F) = \int_0^{2\pi} \varphi(|F(e^{it})|)\,dt = \mathscr{I}_\varphi(F(e^{i\cdot})).$$

Identyfying functions $F\in N'$ with its boundary functions $F(e^{i\cdot})$ we can write

$$H^\varphi = L^\varphi \cap N', \quad H^{*\varphi} = L^{*\varphi}\cap N' \quad \text{and } H^{\circ\varphi} = L^{\circ\varphi}\cap N'.$$

3.4. Generally, in the space $H^{*\varphi}$ we introduce an F-norm by the formula

$$\|F\|_\varphi = \|F(e^{i\cdot})\|_\varphi^{*} = \inf\{\varepsilon > 0:\ \mu_\varphi(F/\varepsilon) \leqslant \varepsilon\},\ (F\in H^{*\varphi}).$$

The space $H^{*\varphi}$ is complete with respect to this norm and $H^{\circ\varphi}$ is identical with the closure of the space of all functions analytic in D and continuous in $\bar{D} = \{z: |z| \leqslant 1\}$ in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$. The space $[H^{\circ\varphi}, \|\cdot\|_\varphi]$ is separable; polynomials with rational coefficients form a dense set in this space.

For a fixed $F\in H^{*\varphi}$, $a^{-1}\|aF\|_\varphi$ is a nonincreasing function for $a > 0$. From this it follows that for $F\in H^{*\varphi}$ such that $0 < \|F\|_\varphi \leqslant R$ we have

$$\left\|\frac{RF}{\|F\|_\varphi}\right\|_\varphi \leqslant R.$$

By Fatou Lemma it follows that $\mu_\varphi(F/\|F\|_\varphi) \leqslant \|F\|_\varphi$ for $F \neq 0$.

If $\varphi(u) = \psi(u^s)$ for $u \geqslant 0$, where ψ is a convex φ-function and $0 < s \leqslant 1$, then an s-homogenous norm can be defined in $H^{*\varphi}$ by the formula

$$\|F\|_{s\varphi} = \|F(e^{i\cdot})\|_{s\varphi}^{*} = \inf\{\varepsilon > 0:\ \mu_\varphi(F/\varepsilon^{1/s}) \leqslant 1\},\ (F\in H^{*\varphi}).$$

The norms $\|\cdot\|_\varphi$ and $\|\cdot\|_{s\varphi}$ are equivalent on $H^{*\varphi}$.

If ψ is a convex φ-function satisfying the conditions (0_1) and (∞_1), then besides the homogeneous norm $\|\cdot\|_{1\psi}$ in $H^{*\psi}$ we can introduce another homogeneous norm by the formula

$$\|F\|_{(\psi)} = \|F(e^{i\cdot})\|_{(\psi)}^{*} = \sup\int_0^{2\pi} |F(e^{it})g(t)|\,dt,$$

where the supremum is taken over all functions $g\in L^{\psi'}$ such that $\mathscr{I}_{\psi'}(g) \leqslant 1$. The norms $\|\cdot\|_{1\psi}$ and $\|\cdot\|_{(\psi)}$ are equivalent on $H^{*\psi}$; namely

$$\|F\|_{1\psi} \leqslant \|F\|_{(\psi)} \leqslant 2\,\|F\|_{1\psi} \quad \text{for every } F\in H^{*\psi} \quad ([6]).$$

3.5. For $F \in H^{*\varphi}$ we define

$$[F]_\varphi = \inf\{\varepsilon > 0:\ \mu_\varphi(F/\varepsilon) < \infty\}.$$

Functional $[\cdot]_\varphi$ is a homogeneous pseudonorm on $H^{*\varphi}$ such that

1° $[F]_\varphi = 0$ if and only if $F \in H^{\circ\varphi}$,

2° $[F]_\varphi \leqslant \|F\|_\varphi$,

3° $[F]_\varphi = \lim\limits_{a\to\infty} a^{-1}\|aF\|_\varphi$ ([6]).

3.6. For functions F analytic in D following two linear operators

$$T_rF(z) = F(rz), \quad \text{where } 0 \leqslant r \leqslant 1,$$

and

$$S_hF(z) = F(e^{ih}z), \quad \text{where } h \text{ is a real number},$$

are interesting. Namely, for these operators we have:

1° $\mu_\varphi(T_rF) = \mu_\varphi(r; F)$ and $\mu_\varphi(S_hF) = \mu_\varphi(F)$,

2° $\mu_\varphi(F) < \infty$ implies $\mu_\varphi\big(\tfrac{1}{2}(T_rF - F)\big) \to 0$ as $r \to 1-$,

3° $\mu_\varphi(F) < \infty$ implies $\mu_\varphi\big(\tfrac{1}{2}(S_hF - F)\big) \to 0$ as $h \to 0$.

From this it follows that

1° $\|T_rF\|_\varphi \leqslant \|F\|_\varphi$ for $0 \leqslant r < 1$ and $\lim\limits_{r\to 1-} \|T_rF\|_\varphi = \|F\|_\varphi$,

2° $\|S_hF\|_\varphi = \|F\|_\varphi$,

3° $\|T_rF - F\|_\varphi \to 0$ as $r \to 1-$ for $F \in H^{\circ\varphi}$,

4° $\|S_hF - F\|_\varphi \to 0$ as $h \to 0$ for $F \in H^{\circ\varphi}$.

Analogical statements hold for norms $\|\cdot\|_{s\varphi}$ and $\|\cdot\|_{(\varphi)}$ when these norms can be introduced in $H^{*\varphi}$ by the before given formulas ([6]).

Moreover, for $F \in H^{*\varphi}$ we have here

$$[F]_\varphi \leqslant \inf\{\|F-G\|_\varphi:\ G \in H^{\circ\varphi}\} \leqslant \limsup_{r\to 1-} \|T_rF - F\|_\varphi \leqslant 2\,[F]_\varphi.$$

3.7. A set $X \subset H^{*\varphi}$ is called a *bounded set* in the space $[H^{*\varphi}, \|\cdot\|_\varphi]$ if $a_n \to 0$ and $\{F_n\} \subset X$ implies always $\|a_nF_n\|_\varphi \to 0$; this holds if and only if for every $\varepsilon > 0$ there exists $a > 0$ such that $\|aF\|_\varphi \leqslant \varepsilon$ for all $F \in X$.

A ball $\{F \in H^{*\varphi}:\ \|F\|_\varphi \leqslant R\}$, $R > 0$, is a bounded set in $[H^{*\varphi}, \|\cdot\|_\varphi]$ if and only if φ satisfies the condition (∇_2) ([6]).

3.8. In Hardy–Orlicz space $H^{*\varphi}$, similarly as in Orlicz space $L^{*\varphi}$, we have two convergences for sequences, one is a norm convergence and other a modular convergence. So we say that a sequence $\{F_n\} \subset H^{*\varphi}$ is convergent in norm to $F \in H^{*\varphi}$, if $\|F_n - F\|_\varphi \to 0$ as $n \to \infty$; this holds if and only if $\mu_\varphi\big(a(F_n - F)\big) \to 0$ as $n \to \infty$ for any $a > 0$. Moreover, we say that a sequence $\{F_n\} \subset H^{*\varphi}$ is convergent in modular to $F \in H^{*\varphi}$, if $\mu_\varphi\big(a(F_n - F)\big) \to 0$ as $n \to \infty$ for some $a > 0$ (in general dependent on $\{F_n - F\}$).

In the case when φ satisfies the condition (Δ_2), $H^{*\varphi} = H^{\circ\varphi}$ and the norm and modular convergences are equivalent. Otherwise, we have only $H^{\circ\varphi} \subset H^{*\varphi}$ and only the norm convergence implies the modular convergence.

4.1. Besides above mentioned two convergences in Hardy–Orlicz space $H^{*\varphi}$ we distinguish a third convergence. Namely, we say that a sequence $\{F_n\} \subset H^{*\varphi}$ is *convergent very weakly to* $F \in H^{*\varphi}$, if

$$\sup_n \|F_n - F\|_\varphi < \infty \quad \text{and} \quad \sup\{|F_n(z) - F(z)|:\ z \in E\} \to 0 \quad \text{as } n \to \infty$$

for any closed set $E \subset D$. This definition of very weak convergence does not change when the norm $\|\cdot\|_\varphi$ is replaced by the norm $\|\cdot\|_{s\varphi}$ or $\|\cdot\|_{(\varphi)}$.

4.2. *Modular convergence implies very weak convergence.*

Proof. Let be $\mu_\varphi\big(a(F_n - F)\big) \to 0$ as $n \to \infty$ for $a > 0$. Then there exists n_0 such that $\mu_\varphi\big(a(F_n - F)\big) \leqslant \dfrac{1}{a}$ for $n \geqslant n_0$. From this get

$$\sup_n \|F_n - F\|_\varphi \leqslant \sup\left\{\frac{1}{a},\ \|F_1 - F\|_\varphi, \ldots, \|F_{n_0} - F\|_\varphi\right\} < \infty.$$

From this and from the inequality given in 3.1 the theorem follows.

4.3. *If a sequence $\{F_n\} \subset H^{*\varphi}$ converges very weakly to $F \in H^{*\varphi}$, then*

$$\mu_\varphi(F) \leqslant \liminf_{n\to\infty} \mu_\varphi(F_n)$$

and

$$\|F\|_\varphi \leqslant \liminf_{n\to\infty} \|F_n\|_\varphi.$$

Proof. Because for $0 \leqslant r < 1$ a sequence $\{F_n(re^{it})\}$ tends to $F(re^{it})$ uniformly on t, we have

$$\mu_\varphi(r;\ aF) = \lim_{n\to\infty} \mu_\varphi(r;\ aF_n) \leqslant \liminf_{n\to\infty} \mu_\varphi(aF_n) \quad \text{for all } a > 0.$$

From this we get the inequalities of the theorem.

4.4. *A sequence $\{F_n\} \subset H^{*\varphi}$ is convergent very weakly if and only if $\sup\limits_n \|F_n\|_\varphi < \infty$ and a sequence $\{F_n(z)\}$ is convergent on a set $E \subset D$ which has a cluster point in D.*

Proof. If a sequence $\{F_n\} \subset H^{*\varphi}$ is convergent very weakly to $F \in H^{*\varphi}$, then

$$\sup_n \|F_n\|_\varphi \leqslant \sup_n \|F_n - F\|_\varphi + \|F\|_\varphi < \infty$$

and a sequence $\{F_n(z)\}$ is convergent (to $F(z)$) for all $z \in D$. Now, let $\{F_n\} \subset H^{*\varphi}$ be a sequence such that $\sup_n \|F_n\|_\varphi < R < \infty$ and a sequence $\{F_n(z)\}$ is convergent on a set which has a cluster point in D. Then by 3.1 we have

$$|F_n(z)| \leqslant \varphi_{-1}\left(\frac{R}{\pi(1-|z|)}\right) R \quad \text{for all } z \in D$$

and by the Vitali Theorem we obtain that a sequence $\{F_n(z)\}$ is convergent uniformly on every closed set $E \subset D$. Let F be a limit function for this sequence. Since for $0 \leqslant r < 1$ a sequence $\{F_n(re^{it})\}$ tends to $F(re^{it})$ uniformly on t, we have

$$\mu_\varphi(r;\ F/R) = \lim_{n\to\infty} \mu_\varphi(r;\ F_n/R) \leqslant R \quad \text{for } 0 \leqslant r < 1$$

and

$$\mu_\varphi(F/R) \leqslant R < \infty.$$

This proofs that $F \in H^{*\varphi}$. Now, we have

$$\sup_n \|F_n - F\|_\varphi \leqslant \sup_n \|F_n\|_\varphi + \|F\|_\varphi < \infty$$

and $F_n(z) \to F(z)$ as $n \to \infty$ uniformly on any closed set $E \subset D$.

4.5. *Every ball $\{F \in H^{*\varphi}\colon \|F\|_\varphi \leqslant R\}$, $R > 0$ is sequentially very weakly compact set.*

Proof. Let $\{F_n\} \subset H^{*\varphi}$ be a sequence such that $\sup_n \|F_n\|_\varphi \leqslant R$. Then by 3.1 we have

$$|F_n(z)| \leqslant \varphi_{-1}\left(\frac{R}{\pi(1-|z|)}\right) R \quad \text{for all } z \in D.$$

From this it follows by the Montel Theorem that there exists a subsequence $\{F_{n_k}\}$ of a sequence $\{F_n\}$, which is convergent uniformly on any closed set $E \subset D$. Now, from 4.4 we get the theorem.

With respect to these three convergences in Hardy-Orlicz space $H^{*\varphi}$ in the sequel we shall deal with norm continuous, modular continuous and very weakly continuous linear functionals on $H^{*\varphi}$.

II. LINEARS FUNCTIONALS IN HARDY-ORLICZ SPACES

1.1. Let z be a fixed point of a circle D. For any function F analytic in D we define

$$\gamma_{0,z}(F) = F(z) \quad \text{and} \quad \gamma_{n,z}(F) = \frac{1}{n!} F^{(n)}(z), \quad (n = 1, 2, \ldots).$$

For $z = 0$ we shall write γ_n insted of $\gamma_{n,0}$.

It is clear that $\gamma_{0,z}$ and $\gamma_{n,z}$ are linear functionals on the space of analytic functions in D, and therefore, they are linear functionals on any Hardy-Orlicz space $H^{*\varphi}$. We note here that these functionals are very weakly continuous on $H^{*\varphi}$.

We see from this that on any Hardy-Orlicz space there exist very weakly continuous linear functionals, and so, also modular continuous and norm continuous ones. Moreover, from the above it follows that there exist fundamental systems of very weakly continuous (modular continuous, norm continuous) linear functionals on $H^{*\varphi}$, i. e. there exist sequences $\{\xi_n\}$ of very weakly continuous (modular continuous, norm continuous) linear functionals on $H^{*\varphi}$ such that $\xi_n(F) = 0$ for $n = 1, 2, \ldots$ and $F \in H^{*\varphi}$ imply $F = 0$.

1.2. *In a general case, we have for the functionals mentioned at* 1.1 *the following estimation with respect to the norm $\|\cdot\|_\varphi$:*

$$|\gamma_{0,z}(F)| \leqslant \varphi_{-1}\left(\frac{\|F\|_\varphi}{\pi(1-|z|)}\right) \|F\|_\varphi$$

and

$$|\gamma_{n,z}(F)| \leqslant \inf\left\{\varphi_{-1}\left(\frac{\|F\|_\varphi}{\pi(1-r)}\right)\frac{r}{(r-|z|)^{n+1}}\colon\ |z| < r < 1\right\}\|F\|_\varphi, \qquad n = 1, 2, \ldots$$

for any $F \in H^{*\varphi}$ and $z \in D$.

When $\varphi(u) = \psi(u^s)$, where ψ is a convex φ-function and $0 < s \leqslant 1$, we have the following estimation for these functionals with respect to the norm $\|\cdot\|_{s\varphi}$:

$$|\gamma_{0,z}(F)| \leqslant \varphi_{-1}\left(\frac{1}{\pi(1-|z|)}\right) \|F\|_{s\varphi}^{1/s}$$

and

$$|\gamma_{n,z}(F)| \leqslant \inf\left\{\varphi_{-1}\left(\frac{1}{\pi(1-r)}\right)\frac{r}{(r-|z|)^{n+1}}\colon\ |z| < r < 1\right\}\|F\|_{s\varphi}^{1/s}, \quad n = 1, 2, \ldots$$

for any $F \in H^{*\varphi}$.

Proof. Estimations for the functional $\gamma_{0,z}$ follow immediately from the inequality

$$|F(z)| \leqslant \varphi_{-1}\left(\frac{\mu_\varphi(F)}{\pi(1-|z|)}\right) \quad \text{for } z \in D,$$

and from definitions of corresponding norms $\|\cdot\|_\varphi$ and $\|\cdot\|_{s\varphi}$. Estimations for the functionals $\gamma_{n,z}$ we get now by Cauchy's Integral Formula. Indeed, for any r, $|z| < r < 1$ we have

$$|\gamma_{n,z}(F)| = \left|\frac{1}{2\pi i}\int_{C_r} \frac{F(\zeta)}{(\zeta - z)^{n+1}} d\zeta\right| \leqslant \sup\{|F(\zeta)|\colon\ |\zeta| = r\} \frac{r}{(r-|z|)^{n+1}}$$
$$\leqslant \varphi_{-1}\left(\frac{\mu_\varphi(F)}{\pi(1-r)}\right) \frac{r}{(r-|z|)^{n+1}},$$

where C_r designates the circumference $\{\zeta:\ |\zeta| = r\}$. Thus

$$|\gamma_{n,z}(F)| \leqslant \inf\left\{\varphi_{-1}\left(\frac{\mu_\varphi(F)}{\pi(1-r)}\right)\frac{r}{(r-|z|)^{n+1}}:\ |z| < r < 1\right\}.$$

From this and definitions of norms $\|\cdot\|_\varphi$ and $\|\cdot\|_{s\varphi}$ follow now postulates estimations.

1.3. In the sequel, we shall denote by $(H^{*\varphi})^{\#}$ a class of all norm continuous linear functionals on $H^{*\varphi}$, by $(H^{*\varphi}_m)^{\#}$ a class of all modular continuous linear functionals on $H^{*\varphi}$, and by $(H^{*\varphi}_{vw})^{\#}$ a class of all very weakly continuous linear functionals on $H^{*\varphi}$.

1.4. *A sequence $\{F_n\} \subset H^{*\varphi}$ very weakly converges to $F \in H^{*\varphi}$ if and only if*

$$\sup_n \|F_n - F\|_\varphi < \infty \quad \text{and} \quad \xi(F_n) \to \xi(F) \quad \text{when } n \to \infty$$

for every $\xi \in (H^{*\varphi}_{vw})^{\#}$.

Proof. If a sequence $\{F_n\} \subset H^{*\varphi}$ is very weakly convergent to $F \in H^{*\varphi}$, then obviously $\sup_n \|F_n - F\|_\varphi < \infty$ and $\xi(F_n) \to \xi(F)$ when $n \to \infty$ for every $\xi \in (H^{*\varphi}_{vw})^{\#}$.

Conversely, for $\{F_n\} \subset H^{*\varphi}$ and $F \in H^{*\varphi}$ let $\sup_n \|F_n - F\|_\varphi < \infty$ and $\xi(F_n) \to \xi(F)$ when $n \to \infty$ for every $\xi \in (H^{*\varphi}_{vw})^{\#}$. Since the functionals $\gamma_{0,z}$, where $z \in D$, belong to $(H^{*\varphi}_{vw})^{\#}$ so we have $F_n(z) \to F(z)$ when $n \to \infty$ for every $z \in D$. Now, by 4.4 of Section I we obtain that $\{F_n\}$ converges very weakly to F.

1.5. *A sequence $\{F_n\} \subset H^{*\varphi}$ is very weakly convergent to $F \in H^{*\varphi}$ if and only if*

$$\sup_n \|F_n - F\|_\varphi < \infty \quad \text{and} \quad \gamma_m(F_n) \to \gamma_m(F) \quad \text{when } n \to \infty$$

for $m = 0, 1, 2, \ldots$

Proof. If $\{F_n\}$ very weakly converges to F then, $\sup_n \|F_n - F\|_\varphi < \infty$ and $\gamma_m(F_n) \to \gamma_m(F)$ when $n \to \infty$ for $m = 0, 1, 2, \ldots$ Conversely, let us suppose that $\{F_n\} \subset H^{*\varphi}$, $F \in H^{*\varphi}$, $\sup_n \|F_n - F\|_\varphi = R < \infty$ and $\gamma_m(F_n) \to \gamma_m(F)$ when $n \to \infty$ for $m = 0, 1, 2, \ldots$ Let r be a number such that $0 < r < 1$. Since sequence $\left\{\left(\frac{2r}{1+r}\right)^m\right\}$ converges to 0, then for every $\varepsilon > 0$ there is a m_0 such that

$$\varphi_{-1}\left(\frac{2R}{\pi(1-r)}\right)R \cdot \frac{1+r}{1-r}\left(\frac{2r}{1+r}\right)^{m_0} < \frac{\varepsilon}{2}$$

By 1.2 we have for every n

$$|\gamma_0(F_n - F)| \leqslant \varphi_{-1}\left(\frac{R}{\pi}\right)R \leqslant \varphi_{-1}\left(\frac{2R}{\pi(1-r)}\right)R$$

and

$$|\gamma_m(F_n - F)| \leqslant \varphi_{-1}\left(\frac{R}{\pi\left(1-(1+r)/2\right)}\right)\frac{(1+r)/2}{\left((1+r)/2\right)^{m+1}}R$$
$$= \varphi_{-1}\left(\frac{2R}{\pi(1-r)}\right)\left(\frac{2}{1+r}\right)^m R \quad \text{for } m = 1, 2, \ldots$$

From this we get for $|z| \leqslant r$

$$\left|\sum_{m=m_0}^{\infty}\gamma_m(F_n - F)z^m\right| \leqslant \sum_{m=m_0}^{\infty}|\gamma_m(F_n - F)|r^m$$
$$\leqslant \varphi_{-1}\left(\frac{2R}{\pi(1-r)}\right)R\sum_{m=m_0}^{\infty}\left(\frac{2r}{1+r}\right)^m = \varphi_{-1}\left(\frac{2R}{\pi(1-r)}\right)R \cdot \frac{1+r}{1-r}\left(\frac{2r}{1+r}\right)^{m_0} \leqslant \frac{\varepsilon}{2}.$$

Now, since $\gamma_m(F_n - F) \to 0$ when $n \to \infty$ for $m = 0, 1, 2, \ldots$, it follows that for already fixed $\varepsilon > 0$ there is a n_0 such that

$$|\gamma_m(F_n - F)| \leqslant (1-r)\frac{\varepsilon}{2} \quad \text{for } n \geqslant n_0 \text{ and } m = 1, 2, \ldots, m_0 - 1.$$

Thus, for $|z| \leqslant r$ and $n \geqslant n_0$ we get

$$|F_n(z) - F(z)| = \left|\sum_{m=0}^{\infty}\gamma_m(F_n - F)z^m\right|$$
$$\leqslant \sum_{m=0}^{m_0-1}|\gamma_m(F_n - F)|r^m + \left|\sum_{m=m_0}^{\infty}\gamma_m(F_n - F)z^m\right| \leqslant (1-r)\frac{\varepsilon}{2}\cdot\sum_{m=0}^{m_0-1}r^m + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

From this we conclude that $\{F_n(z)\}$ converges to $F(z)$ uniformly in the circle $|z| \leqslant r$. This yields that $\{F_n(z)\}$ converges to $F(z)$ on every closed subset E of D. Thus $\{F_n\}$ is very weakly convergent to F.

1.6. *A sequence $\{F_n\} \subset H$ is very weakly convergent if and only if $\sup_n \|F_n\|_\varphi < \infty$ and for $m = 0, 1, 2, \ldots$ the sequences $\{\gamma_m(F_n)\}$ are convergent.*

Proof. If $\{F_n\} \subset H^{*\varphi}$ very weakly converges to $F \in H^{*\varphi}$ then

$$\sup_n \|F_n\|_\varphi \leqslant \sup_n \|F_n - F\|_\varphi + \|F\|_\varphi < \infty$$

and by 1.5 we see that the sequences $\{\gamma_m(F_n)\}$ converge for $m = 0, 1, \ldots$ Conversely, let $\{F_n\} \subset H^{*\varphi}$ be such a sequence that $\sup_n \|F_n\|_\varphi = \frac{R}{2} < \infty$

and for $m = 0, 1, 2, \ldots$ the sequences $\{\gamma_m(F_n)\}$ are convergent. Then we have $\sup_{n,k} \|F_n - F_k\|_\varphi \leqslant R$ and

$$\gamma_m(F_n - F_k) \to 0 \quad \text{when} \quad n, k \to \infty \text{ for } m = 0, 1, \ldots$$

Replacing now in the proof of 1.5 a function F by functions F_k we immediatelly obtain that $F_n(z) - F_k(z) \to 0$ when $n, k \to \infty$ uniformly on every circle $|z| \leqslant r < 1$. This combined with 4.4 of Section I implies that $\{F_n\}$ is very weakly convergent.

2.1. *Let ξ be a linear functional on $H^{*\varphi}$. We define*

$$\nu_\varphi(\xi;\, R) = \sup\{|\xi(F)|:\ \|F\|_\varphi \leqslant R;\ F \in H^{*\varphi}\} \quad \text{for } R > 0.$$

For every linear functional ξ on $H^{*\varphi}$ and for every $R > 0$ the following inequality

$$|\xi(F)| \leqslant R^{-1}\nu_\varphi(\xi, R)\, \|F\|_\varphi$$

holds for every $F \in H^{*\varphi}$ such that $0 < \|F\|_\varphi \leqslant R$.

Proof. If $\nu_\varphi(\xi; R) = \infty$ the inequality in question is obvious. Let us exclude this and suppose that $\nu_\varphi(\xi;\, R) < \infty$. Then

$$|\xi(F)| \leqslant \nu_\varphi(\xi;\, R) \quad \text{for every } F \in H^{*\varphi} \text{ such that } \|F\|_\varphi \leqslant R.$$

Since, in view of 3.4 of Section I, for $F \in H^{*\varphi}$ such that $0 < \|F\|_\varphi \leqslant R$ also $\|RF/\|F\|_\varphi\|_\varphi \leqslant R$, we get

$$|\xi(RF/\|F\|_\varphi)| \leqslant \nu_\varphi(\xi; R).$$

This implies the desired inequality.

2.2. *A linear functional ξ on $H^{*\varphi}$ belongs to $(H^{*\varphi})^\#$ if and only if $\nu_\varphi(\xi; R) < \infty$ for some $R > 0$.*

Proof. If $\xi \in (H^{*\varphi})^\#$ then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\xi(F)| \leqslant \varepsilon$ for all $F \in H^{*\varphi}$ such that $\|F\|_\varphi \leqslant \delta$ i. e. $\nu_\varphi(\xi;\, \delta) \leqslant \varepsilon$. Fixing ε and taking $R = \delta$ we prove the required implication.

Conversely, if $\nu_\varphi(\xi;\, R) < \infty$ for some $R > 0$ then, by the inequality proved in 2.1, ξ is norm continuous on $H^{*\varphi}$ and so $\xi \in (H^{*\varphi})^\#$.

2.3. *For a fixed $\xi \in (H^{*\varphi})^\#$, $R^{-1}\nu_\varphi(\xi;\, R)$ is nondecreasing for $R > 0$. More precisely*

$$R^{-1}\nu_\varphi(\xi; R) = \sup\{|\xi(F)|:\ \mu_\varphi(F) \leqslant R,\ F \in H^{*\varphi}\}.$$

Proof. Observe that

$$\sup\{|\xi(F)|:\ \mu_\varphi(F) \leqslant R,\ F \in H^{*\varphi}\} = \sup\{|\xi(R^{-1}G)|:\ \|G\|_\varphi \leqslant R,\ G \in H^{*\varphi}\} = R^{-1}\nu_\varphi(\xi; R).$$

2.4. *A functional $\xi \in (H^{*\varphi})^\#$ belongs to $(H^{*\varphi}_m)^\#$ if and only if*

$$\lim_{R\to 0+} R^{-1}\nu_\varphi(\xi;\, R) = 0.$$

Proof. If $\xi \in (H^{*\varphi}_m)^\#$ then obviously

$$\lim_{R\to 0+} R^{-1}\nu_\varphi(\xi;\, R) = \lim_{R\to 0+} \sup\{|\xi(F)|:\ \mu_\varphi(F) \leqslant R,\ F \in H^{*\varphi}\} = 0.$$

Conversely, let $\lim_{R\to 0+} R^{-1}\nu_\varphi(\xi;\, R) = 0$ for $\xi \in (H^{*\varphi})^\#$, Further, let $\{F_n\} \subset H^{*\varphi}$ be a sequence such that $\mu_\varphi(aF_n) \to 0$ when $n \to \infty$ for $a > 0$. Then $\xi(aF_n) \to 0$ when $n \to \infty$ and this implies that $\xi(F_n) \to 0$ when $n \to \infty$. This means that $\xi \in (H^{*\varphi}_m)^\#$.

2.5. *If $\xi \in (H^{*\varphi}_{vw})^\#$ then $\nu_\varphi(\xi;\, R) < \infty$ for every $R > 0$.*

Proof. By 4.5 of Section I every ball $\{F \in H^{*\varphi}:\ \|F\|_\varphi \leqslant R\}$ is sequentially very weakly compact. Thus for every $R > 0$ there is $F_R \in H^{*\varphi}$ such that $\|F_R\|_\varphi \leqslant R$ and $\nu_\varphi(\xi;\, R) = |\xi(F_R)|$. This yields the theorem.

2.6. For $R > 0$ we denote by $(H^{*\varphi})^\#_R$ a class of all functionals $\xi \in (H^{*\varphi})^\#$ for which $\nu_\varphi(\xi;\, R) < \infty$ and by $(H^{*\varphi}_m)^\#_R$ a class of all functionals $\xi \in (H^{*\varphi}_m)^\#$ for which $\nu_\varphi(\xi;\, R) < \infty$. Further $(H^{*\varphi})^\#_0$ will denote a class of all functionals $\xi \in (H^{*\varphi})^\#$ for which $\nu_\varphi(\xi;\, R) < \infty$ for every $R > 0$. The class $(H^{*\varphi}_m)^\#_0$ is defined similarly.

According to 2.2 we have

$$(H^{*\varphi})^\# = \bigcup_{n=1}^{\infty} (H^{*\varphi})^\#_{1/n} \quad \text{and} \quad (H^{*\varphi})^\#_0 = \bigcap_{n=1}^{\infty} (H^{*\varphi})^\#_n.$$

Analogous relations hold for $(H^{*\varphi}_m)^\#$ and $(H^{*\varphi}_m)^\#_0$.

Clearly $(H^{*\varphi})^\#_R$, $R > 0$ and $(H^{*\varphi})^\#_0$ are linear subspaces of $(H^{*\varphi})^\#$ and similarly $(H^{*\varphi}_m)^\#_R$, $R > 0$ and $(H^{*\varphi}_m)^\#_0$ are linear subspaces of $(H^{*\varphi}_m)^\#$. It is also evident that the functional $\nu_\varphi(\cdot;\, R)$ is a homogeneous norm in $(H^{*\varphi})^\#_R$, $R > 0$. We shall show that:

2.7. *$(H^{*\varphi})^\#_R$, $R > 0$ is complete relative to the norm $\nu_\varphi(\cdot;\, R)$.*

Proof. Le $\{\xi_n\} \subset (H^{*\varphi})^\#_R$ be such a sequence that $\nu_\varphi(\xi_k - \xi_l;\, R) \to 0$ when $k, l \to \infty$. Since for every $F \in H^{*\varphi}$ there is a $a > 0$ such that $\|aF\|_\varphi \leqslant R$ thus

$$|\xi_k(F) - \xi_l(F)| \leqslant a^{-1}\nu_\varphi(\xi_k - \xi_l;\, R)$$

for every k and l. From this we may deduce that $\xi_k(F) - \xi_l(F) \to 0$ when $k, l \to \infty$ for every $F \in H^{*\varphi}$. This means that for every $F \in H^{*\varphi}$ a sequence $\{\xi_n(F)\}$ is convergent; its limit we designate by $\xi(F)$. Obviously, ξ is a linear functional on $H^{*\varphi}$. By our assumption it follows that for every $\varepsilon > 0$ there is a n_0 such that

$$|\xi_k(F) - \xi_l(F)| \leqslant \nu_\varphi(\xi_k - \xi_l;\, R) \leqslant \varepsilon \quad \text{for } k, l \geqslant n_0$$

and for every $F \in H^{*\varphi}$ such that $\|F\|_\varphi \leqslant R$. Passing to the limit with $l \to \infty$ we obtain

$$|\xi_k(F) - \xi(F)| \leqslant \varepsilon \quad \text{for } k \geqslant n_0 \text{ and for } F \in H^{*\varphi} \text{ such that } \|F\|_\varphi \leqslant R.$$

This implies further that

$$\nu_\varphi(\xi;\ R) \leqslant \varepsilon + \nu_\varphi(\xi_{n_0};\ R) < \infty,$$

what proves, by 2.2, that $\xi \in (H^{*\varphi})^\#_R$ and $\nu_\varphi(\xi_k - \xi; R) \leqslant \varepsilon$ for $k \geqslant n_0$. This means that $\{\xi_n\}$ converges to ξ with respect to the norm $\nu_\varphi(\cdot;\ R)$.

2.8. $(H^{*\varphi}_m)^\#_R$, $R > 0$ *is a closed linear subspace of* $[(H^{*\varphi}_m)^\#_R,\ \nu_\varphi(\cdot;\ R)]$.

Proof. It is clear that $(H^{*\varphi}_m)^\#_R$ is a linear subspace of $(H^{*\varphi})^\#_R$. Let now $\{\xi_n\} \subset (H^{*\varphi}_m)^\#$ converge to $\xi \in (H^{*\varphi})^\#_R$ in the norm $\nu_\varphi(\cdot;\ R)$. Then for every $\varepsilon > 0$ there is a n_0 such that $\nu_\varphi(\xi - \xi_{n_0};\ R) \leqslant \frac{\varepsilon}{2} R$. $\xi_{n_0} \in (H^{*\varphi}_m)^\#_R$ and so by 2.4 for this $\varepsilon > 0$ there is a $R_0 > 0$ such that $R_1^{-1}\nu_\varphi(\xi_{n_0};\ R_1) \leqslant \frac{\varepsilon}{2}$ for $0 < R_1 \leqslant R_0$. By 2.3 we deduce now that for $0 < R_1 \leqslant \inf\{R, R_0\}$

$$\begin{aligned} R_1^{-1}\nu_\varphi(\xi;\ R_1) &\leqslant R_1^{-1}\nu_\varphi(\xi - \xi_{n_0};\ R_1) + R_1^{-1}\nu_\varphi(\xi_{n_0};\ R_1) \\ &\leqslant R^{-1}\nu_\varphi(\xi - \xi_{n_0};\ R) + R_1^{-1}\nu_\varphi(\xi_{n_0};\ R_1) \leqslant \varepsilon. \end{aligned}$$

In view of 2.4 this implies that $\xi \in (H^{*\varphi}_m)^\#_R$.

3.1. *For* $\xi \in (H^{*\varphi})^\#$ *we define*

$$\varkappa_\varphi(\xi) = \inf\{\varepsilon > 0:\ \nu_\varphi(\xi;\ 1/\varepsilon) \leqslant 1\}.$$

In view of 2.1 and 2.2 we note that for every $\xi \in (H^{*\varphi})^\#$ *holds* $\varkappa_\varphi(\xi) < \infty$.

We shall prove that the functional $\varkappa_\varphi$ *has the following properties on* $(H^{*\varphi})^\#$:

1° $\varkappa_\varphi(\xi) = 0$ *if and only if* $\xi = 0$,

2° $\varkappa_\varphi(a\xi) = \varkappa_\varphi(\xi)$ *for* $|a| = 1$,

3° $\varkappa_\varphi(\xi_1 + \xi_2) \leqslant \varkappa_\varphi(\xi_1) + \varkappa_\varphi(\xi_2)$.

By these properties $(H^{*\varphi})^\#$ *is a metric space if we define distance of* $\xi_1, \xi_2 \in (H^{*\varphi})^\#$ *by*

$$d(\xi_1, \xi_2) = \varkappa_\varphi(\xi_1 - \xi_2).$$

Proof. If $\xi = 0$ then for every $\varepsilon > 0$ we have $\nu_\varphi(\xi;\ 1/\varepsilon) = 0 \leqslant 1$ and so $\varkappa_\varphi(\xi) = 0$. On the other hand, if $\varkappa_\varphi(\xi) = 0$ then $\nu_\varphi(\xi;\ 1/\varepsilon) \leqslant 1$ for every $\varepsilon > 0$. Thus $|\xi(F)| \leqslant 1$ for every $F \in H^{*\varphi}$. If follows that $|\xi(aF)| \leqslant 1$ for every $a > 0$ and $F \in H^{*\varphi}$. Passing to the limit with $a \to \infty$ we get $\xi(F) = 0$ for every $F \in H^{*\varphi}$. Therefore $\xi = 0$.

The proof of 2° is obvious.

To prove 3° let us take $\varepsilon_1 > \varkappa_\varphi(\xi_1)$ and $\varepsilon_2 > \varkappa_\varphi(\xi_2)$. Then $\nu_\varphi(\xi_1;\ 1/\varepsilon_1) \leqslant 1$ and $\nu_\varphi(\xi_2;\ 1/\varepsilon_2) \leqslant 1$. This, by 2.3, implies that

$$\begin{aligned} \nu_\varphi(\xi_1 + \xi_2;\ 1/(\varepsilon_1+\varepsilon_2)) &\leqslant \nu_\varphi(\xi_1;\ 1/(\varepsilon_1+\varepsilon_2)) + \nu_\varphi(\xi_2;\ 1/(\varepsilon_1+\varepsilon_2)) \\ &= \big((\varepsilon_1+\varepsilon_2)\nu_\varphi(\xi_1;\ 1/(\varepsilon_1+\varepsilon_2)) + (\varepsilon_1+\varepsilon_2)\nu_\varphi(\xi_2;\ 1/(\varepsilon_1+\varepsilon_2))\big)/(\varepsilon_1+\varepsilon_2) \\ &\leqslant \big(\varepsilon_1\nu_\varphi(\xi_1;\ 1/\varepsilon_1) + \varepsilon_2\,\nu_\varphi(\xi_2;\ 1/\varepsilon_2)\big)/(\varepsilon_1+\varepsilon_2) \leqslant 1. \end{aligned}$$

Thus $\varkappa_\varphi(\xi_1+\xi_2) \leqslant \varepsilon_1 + \varepsilon_2$. This immediately gives us 3°.

3.2. *Any* $\xi \in (H^{*\varphi})^\#$ *and* $F \in H^{*\varphi}$ *such that* $\varkappa_\varphi(\xi)\,\|F\|_\varphi \leqslant 1$ *satisfy the following inequality*

$$|\xi(F)| \leqslant \varkappa_\varphi(\xi)\,\|F\|_\varphi.$$

Proof. Let us take $\varepsilon > \varkappa_\varphi(\xi)$. Thus $\nu_\varphi(\xi|;\ 1/\varepsilon) \leqslant 1$ and, further, by 2.1

$$|\xi(F)| \leqslant \varepsilon\nu_\varphi(\xi;\ 1/\varepsilon)\,\|F\|_\varphi \leqslant \varepsilon\,\|F\|_\varphi \quad \text{for } F \in H^{*\varphi} \text{ such that } \varepsilon\,\|F\|_\varphi \leqslant 1.$$

From this we get

$$|\xi(F)| \leqslant \varkappa_\varphi(\xi)\,\|F\|_\varphi \quad \text{for } F \in H^{*\varphi} \text{ such that } \varkappa_\varphi(\xi)\,\|F\|_\varphi < 1.$$

Now, by norm continuity of ξ we the desired inequality.

3.3. *For a sequence* $\{\xi_n\} \subset (H^{*\varphi})^\#$, $\varkappa_\varphi(\xi_n) \to 0$ *when* $n \to \infty$ *if and only if* $\nu_\varphi(\xi_n;\ R) \to 0$ *when* $n \to \infty$ *for every* $R > 0$.

It follows straight forwardly from 3.2.

3.4. *The space* $(H^{*\varphi})^\#$ *is complete relative to the metric* $d(\xi_1, \xi_2) = \varkappa_\varphi(\xi_1 - \xi_2)$.

Proof. Let $\{\xi_n\} \subset (H^{*\varphi})^\#$ be a sequence such that $\varkappa_\varphi(\xi_k - \xi_l) \to 0$ when $k, l \to \infty$. Then, by 3.3, $\nu_\varphi(\xi_k - \xi_l;\ R) \to 0$, when $k, l \to \infty$ for every $R > 0$. It means that the sequence $\{\xi_n(F)\}$ is uniformly convergent on every ball $\{F \in H^{*\varphi}:\ \|F\|_\varphi \leqslant R\}$, $R > 0$. The limit functional ξ is clearly linear and norm continuous on $H^{*\varphi}$. Further, $\nu_\varphi(\xi_k - \xi;\ R) \to 0$ when $k \to \infty$ for every $R > 0$. This, by 3.3 implies that $\varkappa_\varphi(\xi_k - \xi) \to 0$ for $k \to \infty$.

3.5. *The space* $(H^{*\varphi}_m)^\#$ *is a closed linear subspace of* $[(H^{*\varphi})^\#,\ \varkappa_\varphi]$.

Proof. Let $\{\xi_n\} \subset (H^{*\varphi}_m)^\#$ converge to $\xi \in (H^{*\varphi})^\#$ in the metric $\varkappa_\varphi$. Then, in view of 3.3 $\nu_\varphi(\xi_n - \xi;\ 1) \to 0$ when $n \to \infty$. Thus for every $\varepsilon > 0$ there exists a n_0 such that $\nu_\varphi(\xi_{n_0} - \xi;\ 1) \leqslant \frac{\varepsilon}{2}$. Since $\xi_{n_0} \in (H^{*\varphi}_m)^\#$, it follows now from 2.4 that for the already fixed $\varepsilon > 0$ there is a $R_0 > 0$ such that $R^{-1}\nu_\varphi(\xi_{n_0};\ R) \leqslant \frac{\varepsilon}{2}$ for $0 < R \leqslant R_0$. By virtue of 2.3 we now get for $0 < R \leqslant \inf\{1, R_0\}$

$$\begin{aligned} R^{-1}\nu_\varphi(\xi;\ R) &\leqslant R^{-1}\nu_\varphi(\xi - \xi_{n_0};\ R) + R^{-1}\nu_\varphi(\xi_{n_0};\ R) \\ &\leqslant \nu_\varphi(\xi - \xi_{n_0};\ 1) + R^{-1}\nu_\varphi(\xi_{n_0};\ R) \leqslant \varepsilon. \end{aligned}$$

Taking into account 2.4 we see that $\xi \in (H_m^{*\varphi})^{\#}$.

3.6. *The space* $(H^{*\varphi})_R^{\#}$ *(resp.* $(H_m^{*\varphi})_R^{\#}$*) is, for every* $R > 0$*, a closed linear subspace of* $[(H^{*\varphi})^{\#}, \varkappa_\varphi]$ *(resp.* $[(H_m^{*\varphi})^{\#}, \varkappa_\varphi]$*).*

It follows directly from 3.3, 2.7 and 2.8.

3.7. *The space* $(H^{*\varphi})_0^{\#}$ *(resp.* $(H_m^{*\varphi})_0^{\#}$*) is a closed linear subspace of* $[(H^{*\varphi})^{\#}, \varkappa_\varphi]$ *(resp.* $[(H_m^{*\varphi})^{\#}, \varkappa_\varphi]$*).*

Proof. Straight forward application of 3.6 and 2.6.

3.8. *A functional* $\xi \in (H^{*\varphi})^{\#}$ *is a member of* $(H^{*\varphi})_0^{\#}$ *if and only if* $\varkappa_\varphi(\alpha\xi) \to 0$ *when* $\alpha \to 0$.

Proof. Let us notice that if $\xi \in (H^{*\varphi})_0^{\#}$ then $\nu_\varphi(\xi; R) < \infty$ for all $R > 0$. This in turn implies for every $R > 0$ that $\nu_\varphi(\alpha\xi; R) = |\alpha|\nu_\varphi(\xi; R) \to 0$ when $\alpha \to 0$. By 3.3 we now get that $\varkappa_\varphi(\alpha\xi) \to 0$ when $\alpha \to 0$.

Conversely, let for $\xi \in (H^{*\varphi})^{\#}$ be $\varkappa_\varphi(\alpha\xi) \to 0$ when $\alpha \to 0$. Then, by 3.3, $\nu_\varphi(\alpha\xi; R) \to 0$ when $\alpha \to 0$ for every $R > 0$. It means that for every $R > 0$ there exists $a_R > 0$ such that $\nu_\varphi(a_R\xi; R) \leqslant 1$. Thus $\nu_\varphi(\xi; R) \leqslant a_R^{-1}$ and $\xi \in (H^{*\varphi})_0^{\#}$.

3.9. *The space* $[(H^{*\varphi})_0^{\#}, \varkappa_\varphi]$ *is a Fréchet space and* $(H_m^{*\varphi})_0^{\#}$ *is its closed linear subspace.*

It follows clearly from 3.1, 3.8, 3.7 and 3.5.

3.10. *The space* $(H_{vw}^{*\varphi})^{\#}$ *is a closed linear subspace of* $[(H_m^{*\varphi})_0^{\#}, \varkappa_\varphi]$.

Proof. That $(H_{vw}^{*\varphi})^{\#}$ is a subspace of $(H_m^{*\varphi})_0^{\#}$ is an immediate consequence of 4.2 of Section I and 2.5. Now, let $\{\xi_n\} \subset (H_{vw}^{*\varphi})^{\#}$ be a sequence convergent is norm $\varkappa_\varphi$ to $\xi \in (H_m^{*\varphi})_0^{\#}$. Let, further, $\{F_m\} \subset H^{*\varphi}$ converge very weakly to $F \in H^{*\varphi}$ and, besides, let $\sup\limits_m \|F_m - F\|_\varphi \leqslant R$. Then, in view of 3.3, for every $\varepsilon > 0$ there is a n_0 such that $\nu_\varphi(\xi - \xi_{n_0}; R) \leqslant \varepsilon/2$. For, this given $\varepsilon > 0$, by virtue of the fact that ξ_{n_0} is a member of $(H_{vw}^{*\varphi})^{\#}$, there is a m_0 such that $|\xi_{n_0}(F - F_m)| \leqslant \varepsilon/2$ for $m \geqslant m_0$. Thus, for $m \geqslant m_0$ we obtain

$$\begin{aligned}|\xi(F - F_m)| &\leqslant |\xi(F - F_m) - \xi_{n_0}(F - F_m)| + |\xi_{n_0}(F - F_m)| \\ &\leqslant \nu_\varphi(\xi - \xi_{n_0}; R) + |\xi_{n_0}(F - F_m)| \leqslant \varepsilon.\end{aligned}$$

It follows then $\xi(F_m) \to \xi(F)$ when $m \to \infty$, and $\xi \in (H_{vw}^{*\varphi})^{\#}$.

4.1. Let ξ be a linear functional on $H^{*\varphi}$. We define

$$(T_r^{\#}\xi)(F) = \xi(T_r F) \quad \text{for every } F \in H^{*\varphi} \text{ and } 0 \leqslant r \leqslant 1$$

and

$$(S_h^{\#}\xi)(F) = \xi(S_h F) \quad \text{for every } F \in H^{*\varphi} \text{ and a real } h.$$

Simple consequences of the above definitions and 3.6 of Section I are the following:

$$\nu_\varphi(T_r^{\#}\xi; R) \leqslant \nu_\varphi(\xi; R) \quad \text{for } 0 \leqslant r < 1 \text{ and } R > 0,$$

$$\nu_\varphi(S_h^{\#}\xi; R) = \nu_\varphi(\xi; R) \quad \text{for } h \text{ real and } R > 0.$$

Further, for $\xi \in (H^{*\varphi})^{\#}$,

$$\varkappa_\varphi(T_r^{\#}\xi) \leqslant \varkappa_\varphi(\xi) \quad \text{for } 0 \leqslant r < 1$$

and

$$\varkappa_\varphi(S_h^{\#}\xi) = \varkappa_\varphi(\xi) \quad \text{for } h \text{ real}.$$

4.2. *If* $\xi \in (H^{*\varphi})^{\#}$ *then* $T_r^{\#}\xi \in (H_{vw}^{*\varphi})^{\#}$ *for every* r, $0 \leqslant r < 1$.

Proof. Let the sequence $\{F_n\} \subset H^{*\varphi}$ very weakly converge to $F \in H^{*\varphi}$. Then the sequence $\{F_n(z)\}$ converges uniformly on the circumference $\{z\colon |z| = r\}$, $0 \leqslant r < 1$. Now, the sequence $\{T_r F_n\}$ norm converges to $T_r F$. Thus

$$\lim_{n\to\infty} T_r^{\#}\xi(F_n) = \lim_{n\to\infty} \xi(T_r F_n) = \xi(T_r F) = T_r^{\#}\xi(F).$$

This proves that $T_r^{\#}\xi \in (H_{vw}^{*\varphi})^{\#}$ for every $0 \leqslant r < 1$.

4.3. *A functional* $\xi \in (H^{*\varphi})^{\#}$ *belongs to* $(H_{vw}^{*\varphi})^{\#}$ *if and only if*

$$\lim_{r\to 1-} \varkappa_\varphi(T_r^{\#}\xi - \xi) = 0.$$

Proof. If $\xi \in (H^{*\varphi})^{\#}$ and $\varkappa_\varphi(T_r^{\#}\xi - \xi) \to 0$ for $r \to 1-$ then, by 4.2 and 3.10, $\xi \in (H_{vw}^{*\varphi})^{\#}$.

On the other hand, let $\xi \in (H_{vw}^{*\varphi})^{\#}$. Let us take on arbitrary $R > 0$, and let $\{r_n\}$ be such a sequence that $0 \leqslant r_n < 1$, $r_n \to 1-$ and

$$\limsup_{r\to 1-} \nu_\varphi(T_r^{\#}\xi - \xi; R) = \lim_{n\to\infty} \nu_\varphi(T_{r_n}^{\#}\xi - \xi; R).$$

Since $T_{r_n}^{\#}\xi - \xi \in (H_{vw}^{*\varphi})^{\#}$ and the ball $\{F \in H^{*\varphi}\colon \|F\|_\varphi \leqslant R\}$ is, by 4.5, of Section I, sequentialy very weakly complete, then for every n there is a $F_n \in H^{*\varphi}$ such that

$$\|F_n\|_\varphi \leqslant R \quad \text{and} \quad \nu_\varphi(T_{r_n}^{\#}\xi - \xi; R) = |(T_{r_n}^{\#}\xi - \xi)(F_n)| = |\xi(T_{r_n}F_n - F_n)|.$$

Now, because $\|F_n\|_\varphi \leqslant R$ for $n = 1, 2, \ldots$, we can find a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ very weakly converging to some $F_0 \in H^{*\varphi}$. Then also the sequence $\{T_{r_{n_k}}F_{n_k}\}$ is very weakly converges to F_0. In fact, $\sup\limits_k \|T_{r_{n_k}}F_{n_k}\|_\varphi \leqslant \sup\limits_k \|F_{n_k}\|_\varphi \leqslant R$. We further get, by Maximum Principle,

$$\begin{aligned}&\sup\{|T_{r_{n_k}}F_{n_k}(z) - F_0(z)|\colon |z| \leqslant \varrho\} \\ &\quad = \sup\{|F_{n_k}(r_{n_k}z) - F_0(z)|\colon |z| \leqslant \varrho\} \\ &\quad \leqslant \sup\{|F_{n_k}(r_{n_k}z) - F_0(r_{n_k}z)|\colon |z| \leqslant \varrho\} + \sup\{|F_0(r_{n_k}z) - F_0(z)|\colon |z| \leqslant \varrho\} \\ &\quad \leqslant \sup\{|F_{n_k}(z) - F_0(z)|\colon |z| \leqslant \varrho\} + \sup\{|F_0(r_{n_k}z) - F_0(z)|\colon |z| \leqslant \varrho\}.\end{aligned}$$

It follows now

$$\sup\{|T_{r_n}F_{n_k}(z)-F_0(z)|:\ |z|\leqslant\varrho\}\to 0 \quad \text{when } k\to\infty$$

for every ϱ, $0\leqslant\varrho<1$.

Thus, by 4.4 of Section I $\{T_{r_{n_k}}F_{n_k}\}$ very weakly converges to F_0. Now,

$$\lim_{r\to 1-}\sup\nu_\varphi(T_r^{\#}\xi-\xi;\ R)=\lim_{k\to\infty}|\xi(T_{r_n}F_{n_k}-F_{n_k})|=0.$$

This leads to a conclusion

$$\lim_{r\to 1-}\nu_\varphi(T_r^{\#}\xi-\xi;\ R)=0$$

for every $R>0$, and in virtue of 3.3 $\varkappa_\varphi(T_r^{\#}\xi-\xi)\to 0$ when $r\to 1-$.

4.4. *If $\xi\in(H_m^{*\varphi})^{\#}$, then for every $F\in H^{*\varphi}$*

$$\lim_{r\to 1-}T_r^{\#}\xi(F)=\lim_{r\to 1-}\xi(T_rF)=\xi(F)$$

and

$$\lim_{h\to 0}S_h^{\#}\xi(F)=\lim_{h\to 0}\xi(S_hF)=\xi(F).$$

This is an immediate consequence of 3.6 of Section I.

5.1. A sequence $\{\xi_n\}$ of functionals defined on $H^{*\varphi}$ is said to be *pointwise convergent* to a functional ξ, if for every $F\in H^{*\varphi}$

$$\lim_{n\to\infty}\xi_n(F)=\xi(F).$$

The space $H^{*\varphi}$ equipped with a norm $\|\cdot\|_\varphi$ is a Fréchet space. Thus, by Banach Theorem ([1]) every sequence of functionals $\{\xi_n\}\subset(H^{*\varphi})^{\#}$, such that the sequence $\{\xi_n(F)\}$ converges for every $F\in H^{*\varphi}$, is pointwise convergent to a functional $\xi\in(H^{*\varphi})^{\#}$.

5.2. *A sequence $\{\xi_n\}\subset(H^{*\varphi})^{\#}$ is pointwise convergent if and only if $\sup_n\varkappa_\varphi(\xi_n)<\infty$ and the sequence $\{\xi_n(F)\}$ converges for every F belonging to a certain linearly dense subset of $[H^{*\varphi},\ \|\cdot\|_\varphi]$.*

Proof. Let $\{\xi_n\}\subset(H^{*\varphi})^{\#}$ be pointwise convergent. Then, in virtue of Mazur–Orlicz Theorem ([10]), for every $\varepsilon>0$ there is a $\delta>0$ such that $|\xi_n(F)|\leqslant\varepsilon$ for every n and for every $F\in H^{*\varphi}$ such that $\|F\|_\varphi\leqslant\delta$. Thus there is an $R>0$ such that $\nu_\varphi(\xi_n;R)\leqslant 1$ for every n. Now, $\varkappa_\varphi(\xi_n)\leqslant\dfrac{1}{R}$ for every n, and so $\sup_n\varkappa_\varphi(\xi_n)\leqslant\dfrac{1}{R}<\infty$. Conversely, assume that $\{\xi_n\}\subset(H^{*\varphi})^{\#}$ be a sequence such that $\sup_n\varkappa_\varphi(\xi_n)<\infty$ and $\{\xi_n(F)\}$ converges for every F from a set X dense in $[H^{*\varphi},\|\cdot\|_\varphi]$. Since $\sup_n\varkappa_\varphi(\xi_n)<\infty$, by 3.2 it follows that for every $\varepsilon>0$ there is a $\delta>0$ such that $|\xi_n(F)|\leqslant\dfrac{\varepsilon}{3}$

for every n and every $F\in H^{*\varphi}$, $\|F\|_\varphi\leqslant\delta$. X is dense in $[H^{*\varphi},\ \|\cdot\|_\varphi]$ and so for every $F\in H^{*\varphi}$ there is a $G\in X$ such that $\|F-G\|_\varphi\leqslant\delta$. The sequence $\{\xi_n(G)\}$ is convergent, thus there is an n_0 such that for $k,l\geqslant n_0$, $|\xi_k(G)-\xi_l(G)|\leqslant\dfrac{\varepsilon}{3}$. For $k,l\geqslant n_0$ we have then $|\xi_k(F)-\xi_l(F)|\leqslant|\xi_k(F-G)|+|\xi_k(G)-\xi_l(G)|+|\xi_l(G-F)|\leqslant\varepsilon$. This means that $\{\xi_n(F)\}$ converges for every $F\in H^{*\varphi}$.

5.3. *If $\{\xi_n\}\subset(H^{*\varphi})^{\#}$ is pointwise convergent to $\xi\in(H^{*\varphi})^{\#}$ then*

$$\nu_\varphi(\xi;R)\leqslant\liminf_{n\to\infty}\nu_\varphi(\xi_n;\ R)\quad \textit{for every } R>0,$$

and also

$$\varkappa_\varphi(\xi)\leqslant\liminf_{n\to\infty}\varkappa_\varphi(\xi_n).$$

Proof. Let for $R>0$ $\{\xi_{n_k}\}$ be such a subsequence of $\{\xi_n\}$ that

$$\liminf_{n\to\infty}\nu_\varphi(\xi_n;\ R)=\lim_{k\to\infty}\nu_\varphi(\xi_{n_k};\ R).$$

We can assume here that this above limit is finite. Then for every $\varepsilon>0$ there exists a k_0 such that

$$|\xi_{n_k}(F)|\leqslant\nu_\varphi(\xi_{n_k};\ R)\leqslant\lim_{k\to\infty}\nu_\varphi(\xi_{n_k};\ R)+\varepsilon$$

for every $k\geqslant k_0$ and every $F\in H^{*\varphi}$, $\|F\|_\varphi\leqslant R$. Passing with $k\to\infty$ we obtain

$$|\xi(F)|\leqslant\lim_{k\to\infty}\nu_\varphi(\xi_{n_k};\ R)+\varepsilon$$

for every $F\in H^{*\varphi}$, $\|F\|_\varphi\leqslant R$, and

$$\nu_\varphi(\xi;R)\leqslant\lim_{k\to\infty}\nu_\varphi(\xi_{n_k};R)+\varepsilon.$$

But $\varepsilon>0$ was an arbitrary number and so we proved the first of inequalities in question. Now, assume $\varepsilon>\liminf_{n\to\infty}\varkappa_\varphi(\xi_n)$. Then there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\varkappa_\varphi(\xi_{n_k})<\varepsilon$ for $k=1,2,\ldots$ This implies that $\nu_\varphi(\xi_{n_k};\ 1/\varepsilon)\leqslant 1$ for $k=1,2,\ldots$ Application of the first inequality now yields $\nu_\varphi(\xi;1/\varepsilon)\leqslant 1$, and $\varkappa_\varphi(\xi)\leqslant\varepsilon$. This accomplishes the proof.

6.1. Let us now consider linear functionals on $H^{\circ\varphi}$. Similarly as for $H^{*\varphi}$ we can define classes: $(H^{\circ\varphi})^{\#}$ of norm continuous functionals on $H^{\circ\varphi}$, $(H_m^{\circ\varphi})^{\#}$ of modular continuous functionals and $(H_{vw}^{\circ\varphi})^{\#}$ of very weakly continuous functionals on $H^{\circ\varphi}$ and others. In a similar fashion we define functionals $\nu_\varphi(\xi;\ R)$, $R>0$, and $\varkappa_\varphi(\xi)$ with respect to $H^{\circ\varphi}$. Let us observe here that all so far proved for linear functionals on $H^{*\varphi}$ results with exception of 1.6, 2.5 and 4.5 hold also when $H^{*\varphi}$ is replaced by $H^{\circ\varphi}$. From now on, to distinguish functionals relative to the space $H^{\circ\varphi}$ from those relative to the space $H^{*\varphi}$, the former will be marked $\circ$, e. g. $\nu_\varphi^\circ(\xi^\circ; R)$ will denote $\sup\{|\xi^\circ(F)|:\ F\in H^{\circ\varphi},\ \|F\|_\varphi\leqslant R\}$ for ξ° defined on $H^{\circ\varphi}$.

6.2. Let ξ be a linear functional on $H^{*\varphi}$. Since $H^{\circ\varphi}\subset H^{*\varphi}$, then the range of ξ can be restricted to $H^{\circ\varphi}$. Thus we can construct a linear functional ξ° on $H^{\circ\varphi}$ out of a linear one ξ on $H^{*\varphi}$ in such a way that:

$$\xi^\circ(F)=\xi(F)\quad \text{for every } F\in H^{\circ\varphi}.$$

Clearly, for every $\xi\in(H^{*\varphi}_{\square})^{\#}_{\square}$, $\xi^\circ\in(H^{\circ\varphi}_{\square})^{\#}_{\square}$ and

$$\nu_\varphi^\circ(\xi^\circ; R)\leqslant\nu_\varphi(\xi; R)\quad \text{for every } R>0$$

and then

$$\varkappa_\varphi^\circ(\xi^\circ)\leqslant\varkappa_\varphi(\xi).$$

6.3. *For every functional $\xi^\circ\in(H^{\circ\varphi})^{\#}$ there exists a functional $\xi\in(H^{*\varphi})^{\#}$ such that*

$$\xi^\circ(F)=\xi(F)\quad \textit{for every } F\in H^{\circ\varphi}$$

and

$$\nu_\varphi^\circ(\xi^\circ; R)=\nu_\varphi(\xi; R)\quad \textit{for every } R>0.$$

Proof. If $F\in H^{*\varphi}$ then $T_rF\in H^{\circ\varphi}$ for $0\leqslant r<1$. We define a nonnegative functional on $H^{*\varphi}$ by

$$p(F)=\sup\{|\xi^\circ(T_rF)|:\ 0\leqslant r<1\},\qquad (F\in H^{*\varphi}).$$

Let $\nu_\varphi^\circ(\xi^\circ; R)<\infty$ for $R>0$. Then, by 2.1 and 3.6 of Section I for every $F\in H^{*\varphi}$, $\|F\|_\varphi\leqslant R$ we have for every $0\leqslant r<1$

$$|\xi^\circ(T_rF)|\leqslant R^{-1}\nu_\varphi^\circ(\xi^\circ; R)\,\|T_rF\|_\varphi\leqslant R^{-1}\nu_\varphi^\circ(\xi^\circ; R)\,\|F\|_\varphi.$$

This implies that

$$(*)\qquad p(F)\leqslant R^{-1}\nu_\varphi^\circ(\xi^\circ; R)\,\|F\|_\varphi$$

for every $F\in H^{*\varphi}$, $\|F\|_\varphi\leqslant R$. From this we conclude that $p(F)<\infty$ for every $F\in H^{*\varphi}$. This functional $p(\cdot)$ is, obviously, a homogeneous pseudonorm on $H^{*\varphi}$. By 3.6 of Section I for every $F\in H^{\circ\varphi}$, $\xi^\circ(T_rF)\to\xi^\circ(F)$ for $r\to 1-$. From this as well as from definition of $p(\cdot)$ it follows that

$$|\xi^\circ(F)|\leqslant p(F)\quad \text{for every } F\in H^{\circ\varphi}.$$

By Hahn–Banach Theorem there exists a linear functional ξ on $H^{*\varphi}$ such that

$$\xi(F)=\xi^\circ(F)\quad \text{for } F\in H^{\circ\varphi}$$

and

$$|\xi(F)|\leqslant p(F)\quad \text{for } F\in H^{*\varphi}.$$

The functional ξ has the desired properties since for every $R>0$ such that $\nu_\varphi^\circ(\xi^\circ; R)<\infty$ in view of $(*)$ we have

$$\nu_\varphi^\circ(\xi^\circ; R)\leqslant\nu_\varphi(\xi; R)\leqslant\sup\{p(F):\ F\in H^{*\varphi},\ \|F\|_\varphi\leqslant R\}\leqslant\nu_\varphi^\circ(\xi^\circ; R)$$

and for every $R>0$ such that $\nu_\varphi^\circ(\xi^\circ; R)=\infty$, by the inequality $\nu_\varphi^\circ(\xi^\circ; R)\leqslant\nu_\varphi(\xi; R)$ also $\nu_\varphi(\xi; R)=\infty$.

6.4. *For every $\xi^\circ\in(H_m^{\circ\varphi})^{\#}$ there exists a unique functional $\xi\in(H_m^{*\varphi})^{\#}$ such that*

$$\xi(F)=\xi^\circ(F)\quad \textit{for every } F\in H^{\circ\varphi}.$$

Moreover, for every $R>0$,

$$\nu_\varphi(\xi; R)=\nu_\varphi^\circ(\xi^\circ; R).$$

Proof. In view of 6.3, for $\xi^\circ\in(H_m^{\circ\varphi})^{\#}$, there is a $\xi\in(H^{*\varphi})^{\#}$ such that $\xi(F)=\xi^\circ(F)$ for $F\in H^{\circ\varphi}$ and $\nu_\varphi(\xi; R)=\nu_\varphi^\circ(\xi^\circ; R)$ for $R>0$. By 2.4 $R^{-1}\nu_\varphi^\circ(\xi^\circ; R)\to 0$ as $R\to 0$ since $\xi^\circ\in(H_m^{\circ\varphi})^{\#}$. Thus also $R^{-1}\nu_\varphi(\xi; R)\to 0$ as $R\to 0$ and by 2.4 $\xi\in(H_m^{*\varphi})^{\#}$. Now, let $\xi_1\in(H_m^{*\varphi})^{\#}$ be another functional such that $\xi_1(F)=\xi^\circ(F)$ for every $F\in H^{\circ\varphi}$. Take an arbitrary $F\in H^{*\varphi}$. Since $T_rF\in H^{\circ\varphi}$ for $0\leqslant r<1$, then $(\xi_1-\xi)(T_rF)=0$. By 4.4 it follows that $(\xi_1-\xi)(F)=0$ and so $\xi_1=\xi$.

6.5. *For every $\xi^\circ\in(H_{vw}^{\circ\varphi})^{\#}$ there exists a unique $\xi\in(H_{vw}^{*\varphi})^{\#}$ such that*

$$\xi(F)=\xi^\circ(F)\quad \textit{for every } F\in H^{\circ\varphi}.$$

Moreover, for every $R>0$,

$$\nu_\varphi(\xi; R)=\nu_\varphi^\circ(\xi^\circ; R).$$

Proof. By 6.4, in view of the inclusion $(H_{vw}^{\circ\varphi})^{\#}\subset(H_m^{\circ\varphi})^{\#}$, for a $\xi^\circ\in(H_{vw}^{\circ\varphi})^{\#}$ there is a unique $\xi\in(H_m^{*\varphi})^{\#}$ such that $\xi(F)=\xi^\circ(F)$ for every $F\in H^{\circ\varphi}$. For this functional $\nu_\varphi(\xi; R)=\nu_\varphi^\circ(\xi^\circ; R)$ for every $R>0$. We shall show that $\xi\in(H_{vw}^{*\varphi})^{\#}$. Assume, on the contrary, that $\xi\notin(H_{vw}^{*\varphi})^{\#}$. Then there is a $\varepsilon_0>0$ and a sequence $\{F_n\}\subset H^{*\varphi}$ very weakly converging to 0 such that $|\xi(F_n)|\geqslant 2\varepsilon_0$. Since $\xi\in(H_m^{*\varphi})^{\#}$ then by 4.4 $\lim\limits_{r\to 1-}\xi(T_rF_n)=\xi(F_n)$ for $n=1,2,\ldots$ It follows that for every n there exists an r_n, $0\leqslant r_n<1$ such that

$$|\xi(T_{r_n}F_n)-\xi(F_n)|\leqslant\varepsilon_0.$$

Thus for $n=1,2,\ldots$

$$\varepsilon_0\leqslant|\xi(F_n)|-|\xi(T_{r_n}F_n)-\xi(F_n)|\leqslant|\xi(T_{r_n}F_n)|.$$

Elements of $\{T_{r_n}F_n\}$ belong to $H^{\circ\varphi}$. Since $\{F_n\}$ very weakly converges to 0, by 3.6 of Section I

$$\sup_n\|T_{r_n}F_n\|_\varphi\leqslant\sup_n\|F_n\|_\varphi<\infty.$$

Applying now the Maximum Principle we get for any $0 \leqslant \varrho < 1$

$$\sup\{|T_{r_n}F_n(z)|:\ |z| \leqslant \varrho\} = \sup\{|F_n(r_n z)|:\ |z| \leqslant \varrho\} \leqslant \sup\{|F_n(z)|:\ |z| \leqslant \varrho\}.$$

Thus $\{T_{r_n}F_n\}$ very weakly converges to 0. Now, since

$$|\xi^\circ(T_{r_n}F_n)| = |\xi(T_{r_n}F_n)| \geqslant \varepsilon_0$$

we get $\xi^\circ \notin (H^{\circ\varphi}_{vw})^\#$. This contradiction accomplishes the proof.

6.6. *If $\xi^\circ \in (H^{\circ\varphi})^\#$, then for every $F \in H^{\circ\varphi}$*

$$\lim_{r\to 1-} T_r^\# \xi^\circ(F) = \lim_{r\to 1-} \xi^\circ(T_r F) = \xi^\circ(F)$$

and

$$\lim_{h\to 0} S_h^\# \xi^\circ(F) = \lim_{h\to 0} \xi^\circ(S_h F) = \xi^\circ(F).$$

This is an immediate consequence of 3.6 of Section I

6.7. *A sequence of functionals $\{\xi_n^\circ\} \subset (H^{\circ\varphi})^\#$ converges pointwise (on $H^{\circ\varphi}$) if and only if $\sup_n \varkappa_\varphi^\circ(\xi_n^\circ) < \infty$ and for $m = 0, 1, 2, \ldots$ converges the sequence $\{\xi_n(U_m)\}$ where $U_m(z) = z^m$.*

This theorem follows from 5.2 reformulated for $H^{\circ\varphi}$ and from the fact that polynomials form a dense subset of $[H^{\circ\varphi}, \|\cdot\|_\varphi]$.

7.1. If φ satisfies condition (Δ_2), then, as is known, $H^{*\varphi} = H^{\circ\varphi}$ and modular convergence and norm convergence are equivalent. Then we have

$$(H^{*\varphi}_m)^\# = (H^{*\varphi})^\# = (H^{\circ\varphi})^\# = (H^{\circ\varphi}_m)^\#.$$

Also

$$(H^{*\varphi}_m)^\#_0 = (H^{*\varphi})^\#_0 = (H^{\circ\varphi})^\#_0 = (H^{\circ\varphi}_m)^\#_0$$

and

$$(H^{*\varphi}_m)^\#_R = (H^{*\varphi})^\#_R = (H^{\circ\varphi})^\#_R = (H^{\circ\varphi}_m)^\#_R$$

for every $R > 0$.

7.2. *If φ does not satisfy (Δ_2) there exist non-trivial functionals $\xi \in (H^{*\varphi})^\#_0$ such that*

$$\xi(F) = 0 \quad \text{for every } F \in H^{\circ\varphi}.$$

Proof. The functional $[\cdot]_\varphi$ is, by 3.5, of Section I a homogeneous pseudonorm satysfying conditions: $[F]_\varphi = 0$ if and only if $F \in H^{\circ\varphi}$ and $[F]_\varphi \leqslant \|F\|_\varphi$ for $F \in H^{*\varphi}$. If φ does not satisfy (Δ_2) then $H^{*\varphi} \neq H^{\circ\varphi}$. Take $F_0 \in H^{*\varphi} \setminus H^{\circ\varphi}$ and put

$$\xi(aF_0) = a[F_0]_\varphi \quad \text{for any number } a.$$

Further, by Hahn–Banach Theorem we get a linear functional ξ on $H^{*\varphi}$ such that $\xi(aF_0) = a[F_0]_\varphi$ for any number a and such that $|\xi(F)| \leqslant [F]_\varphi$ for every $F \in H^{*\varphi}$. Observe that ξ posesses the required properties.

7.3. *If φ does not satisfy (Δ_2) then $(H^{*\varphi})^\#_0 \neq (H^{*\varphi}_m)^\#_0$.*

Proof. By 7.2 there is then a non-trivial linear functional $\xi \in (H^{*\varphi})^\#_0$ such that $\xi(F) = 0$ for every $F \in H^{\circ\varphi}$. Suppose that $(H^{*\varphi})^\#_0 = (H^{*\varphi}_m)^\#_0$. Then $\xi \in (H^{*\varphi}_m)^\#_0$. By 4.5 we get now for every $F \in H^{*\varphi}$

$$\xi(F) = \lim_{r\to 1-} \xi(T_r F) = \lim_{i\to 1-} \xi^\circ(T_r F) = 0.$$

This contradicts the assumption that ξ is non-trivial on $H^{*\varphi}$.

7.4. *If $\xi \in (H^{*\varphi})^\#$ is such a functional that*

$$\xi(F) = 0 \quad \text{for every } F \in H^{\circ\varphi}$$

then $\xi \in (H^{*\varphi})^\#_0$. More precisely there is a constant $M \geqslant 0$ such that

$$|\xi(F)| \leqslant M[F]_\varphi \leqslant M\|F\|_\varphi \quad \text{for every } F \in H^{*\varphi}.$$

Proof. Since $\xi \in (H^{*\varphi})^\#$ then there exists a $\delta > 0$ such that for every $F \in H^{*\varphi}$, $\|F\|_\varphi \leqslant \delta$ the condition $|\xi(F)| \leqslant 1$ holds. Now, let for $F \in H^{*\varphi}$ be $[F]_\varphi \leqslant \frac{1}{3}\delta$. Then, in view of 3.6 of Section I there is a $G \in H^{\circ\varphi}$ such that $\|F-G\|_\varphi \leqslant \delta$. Thus $|\xi(F)| = |\xi(F-G)| \leqslant 1$. So for every $F \in H^{*\varphi}$ such that $[F]_\varphi \leqslant \frac{1}{3}\delta$ is true that $|\xi(F)| \leqslant 1$. This yields that

$$\xi(F)| \leqslant \frac{3}{\delta}[F]_\varphi \leqslant \frac{3}{\delta}\|F\|_\varphi$$

for every $F \in H^{*\varphi}$.

7.5. $(\tilde{H}^{*\varphi})^\#$ will designate a class of all functionals $\xi \in (H^{*\varphi})^\#$ such that $\xi(F) = 0$ for every $F \in H^{\circ\varphi}$.

Clearly, if a sequence $\{\xi_n\} \subset (\tilde{H}^{*\varphi})^\#$ pointwise converges on $H^{*\varphi}$ to a $\xi \in (H^{*\varphi})^\#$ then $\xi \in (\tilde{H}^{*\varphi})^\#$. This, together with 7.4, implies that $(\tilde{H}^{*\varphi})^\#$ is a closed linear subspace of a Fréchet space $[(H^{*\varphi})^\#_0, \varkappa_\varphi]$.

7.6. *The space $(\tilde{H}^{*\varphi})^\#$ can be endowed with a homogeneous norm defined by*

$$\|\xi\|^\#_\varphi = \nu_\varphi(\xi;\ 1) = \sup\{|\xi(F)|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant 1\}, \quad (\xi \in (\tilde{H}^{*\varphi})^\#).$$

This norm is equivalent on $(\tilde{H}^{*\varphi})^\#$ with the norm $\varkappa_\varphi$.

Proof. By 7.4 $(\tilde{H}^{*\varphi})^\# \subset (H^{*\varphi})^\#_1$. From this we conclude that $\|\xi\|^\#_\varphi = \nu_\varphi(\xi;\ 1)$ is a homogeneous norm on $(\tilde{H}^{*\varphi})^\#$. Notice that $(\tilde{H}^{*\varphi})^\#$ is a closed linear subspace of a Banach space $[(H^{*\varphi})^\#_1;\ \nu_\varphi(\cdot;\ 1)]$. Thus, $(\tilde{H}^{*\varphi})^\#$ is a Fréchet space when equipped with any of the norms $\varkappa_\varphi$ and $\|\cdot\|^\#_\varphi$. By 3.3, for a sequence $\{\xi_n\} \subset (\tilde{H}^{*\varphi})^\#$, a convergence $\varkappa_\varphi(\xi_n) \to 0$ implies convergence $\|\xi_n\|_\varphi^\# = \nu_\varphi(\xi_n;1) \to 0$. Now, the Closed Graph Theorem applied to the identity operator says that for a sequence $\{\xi_n\} \subset (\tilde{H}^{*\varphi})^\#$ a convergence $\|\xi_n\|_\varphi^\# = \nu_\varphi(\xi_n;\ 1) \to 0$ implies $\varkappa_\varphi(\xi_n) \to 0$. It means that the norms $\|\cdot\|_\varphi^\#$ and $\varkappa_\varphi$ are equivalent on $(\tilde{H}^{*\varphi})^\#$.

8.1. *If* $(H^{*\varphi})^\# = (H^{*\varphi})_0^\#$ *then the space* $(H^{*\varphi})^\#$ *can be endowed with the homogeneous norm defined by*

$$\|\xi\|_\varphi^\# = \nu_\varphi(\xi;\ 1) = \sup\{|\xi(F)|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant 1\},\ (\xi \in (H^{*\varphi})^\#).$$

This norm is then equivalent with $\varkappa_\varphi$.
Besides, for every set $Y \subset (H^{*\varphi})^\#$ $\sup\{\|\xi_\varphi^\#\|:\ \xi \in Y\} < \infty$ *if and only if* $\sup\{\varkappa_\varphi(\xi):\ \xi \in Y\} < \infty$.

Proof. By 3.7 the space $(H^{*\varphi})^\#$ with the norm $\varkappa_\varphi$ is then a Fréchet space. Since then also $(H^{*\varphi})^\# = (H^{*\varphi})_1^\#$, thus by 2.7 the space $(H^{*\varphi})^\#$ with the norm $\nu_\varphi(\cdot;\ 1)$ is a Banach space. Finally, 3.3 says that for a sequence $\{\xi_n\} \subset (H^{*\varphi})^\#$ a convergence $\varkappa_\varphi(\xi_n) \to 0$ implies $\nu_\varphi(\xi_n; 1) \to 0$. Again, the Closed Graph Theorem applied to the identity mapping gives the reversed implication, and so the norms $\|\cdot\|_\varphi^\#$ and $\varkappa_\varphi$ are equivalent on $(H^{*\varphi})^\#$.

Let now $\sup\{\|\xi\|_\varphi^\#:\ \xi \in Y\} < \infty$. Then the set Y is bounded in the space $[(H^{*\varphi})^\#,\ \|\cdot\|_\varphi^\#]$. Hence Y is also bounded in the space $[(H^{*\varphi})^\#,\ \varkappa_\varphi]$. It follows then that $\sup\{\varkappa_\varphi(\xi):\ \xi \in Y\} < \infty$. Conversely let $\sup\{\varkappa_\varphi(\xi):\ \xi \in Y\} < \infty$. Then there exists an $R > 0$ such that $\sup\{\nu_\varphi(\xi;\ R):\ \xi \in Y\} \leqslant 1$. This means that Y is bounded in the space $[(H^{*\varphi})_R^\#,\ \nu_\varphi(\cdot;\ R)]$. Since here $(H^{*\varphi})_R^\# = (H^{*\varphi})_1^\# = (H^{*\varphi})^\#$ it follows that the space $(H^{*\varphi})^\#$ is complete with respect to both norms $\nu_\varphi(\cdot;\ R)$ and $\|\cdot\|_\varphi^\# = \nu_\varphi(\cdot;\ 1)$. Since then for every $\xi \in (H^{*\varphi})^\#$ we have $\nu_\varphi(\xi;\ 1) \leqslant \nu_\varphi(\xi;\ R)$ for $R \geqslant 1$ and $\nu_\varphi(\xi;\ R) \leqslant \nu_\varphi(\xi;\ 1)$ for $0 < R < 1$, thus, by the Closed Graph Theorem applied to the identity operator, the norms $\nu_\varphi(\cdot;\ R)$ and $\|\cdot\|_\varphi^\# = \nu_\varphi(\cdot;\ 1)$ are equivalent on $(H^{*\varphi})^\#$. Thus Y is bounded in the space $[(H^{*\varphi})^\#,\ \|\cdot\|_\varphi^\#]$. Hence $\sup\{\|\xi\|_\varphi^\#:\ \xi \in Y\} < \infty$.

If $(H^{*\varphi})^\# = (H^{*\varphi})_0^\#$ then, obviously, $(H_m^{*\varphi})^\# = (H_m^{*\varphi})_0^\#$.

Theorem 8.1 remains valid if $H^{*\varphi}$ is replaced by $H^{\circ\varphi}$.

8.2. $(H^{*\varphi})^\# = (H^{*\varphi})_0^\#$ *if and only if* $(H^{\circ\varphi})^\# = (H^{\circ\varphi})_0^\#$.

Proof. If $(H^{*\varphi})^\# = (H^{*\varphi})_0^\#$, then by 6.2 and 3.8 also $(H^{\circ\varphi})^\# = (H^{\circ\varphi})_0^\#$. Conversely, let $(H^{\circ\varphi})^\# = (H^{\circ\varphi})_0^\#$. Let us take an arbitrary $\xi \in (H^{*\varphi})^\#$. Then the functional ξ° linear on $H^{\circ\varphi}$ and such that $\xi^\circ(F) = \xi(F)$ for every $F \in H^{\circ\varphi}$ by our hypothesis belongs to $(H^{\circ\varphi})_0^\#$. In view of 6.3 for ξ° there exists a ξ_1 in $(H^{*\varphi})^\#$ such that $\xi_1(F) = \xi^\circ(F)$ for every $F \in H^{\circ\varphi}$ and $\nu_\varphi^\circ(\xi^\circ;\ R) = \nu_\varphi(\xi_1;\ R)$ for every $R > 0$. This yields $\xi_1 \in (H^{*\varphi})_0^\#$. Now, $\xi - \xi_1 \in (H^{*\varphi})^\#$ and $(\xi - \xi_1)(F) = 0$ for every $F \in H^{\circ\varphi}$. Applying 7.4 we get $\xi - \xi_1 \in (H^{*\varphi})_0^\#$. Thus $\xi = (\xi - \xi_1) + \xi_1 \in (H^{*\varphi})_0^\#$, and it means that $(H^{*\varphi})^\# = (H^{*\varphi})_0^\#$.

8.3. *If* φ *satisfies condition* (V_2) *then* $(H^{*\varphi}) = (H^{*\varphi})_0^\#$.

Proof. Let $\xi \in (H^{*\varphi})^\#$. By 2.2 there is then an $R_0 > 0$ such that $\nu_\varphi(\xi;\ R_0) < \infty$. Let now R be any positive number. Since φ satisfies (V_2), then by 3.7 of Section I there is a $\alpha > 0$ such that $\|\alpha F\|_\varphi \leqslant R_0$ for every $F \in H^{*\varphi}$, $\|F\|_\varphi \leqslant R$. From this follows that

$$\begin{aligned}\nu_\varphi(\xi;\ R) &= \sup\{|\xi(F)|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant R\}\\ &= \alpha^{-1}\sup\{|\xi(\alpha F)|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant R\}\\ &\leqslant \alpha^{-1}\sup\{|\xi(F):\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant R_0\}\\ &= \alpha^{-1}\nu_\varphi(\xi, R_0) < \infty.\end{aligned}$$

This implies that $\nu_\varphi(\xi;\ R) < \infty$ for every $R > 0$ and, further, $\xi \in (H^{*\varphi})_0^\#$. Thus $(H^{*\varphi})^\# = (H^{*\varphi})_0^\#$.

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