On linear functionals in Hardy-Orlicz spaces, I

by

R. LEŚNIEWSKIĆ (Poznań)

Abstract. The paper can be regarded as a continuation of the paper "On Hardy-Orlicz spaces. I", Colloq. Math. 15 (1967), pp. 3-66. The paper contains the study of spaces of linear functionals continuous in norm, continuous in modular and very weakly continuous on the Hardy-Orlicz space $H^p$ and on the space of finite elements $H^p$. Mutual relations between these spaces and the question of the extension of linear functionals from $H^p$ on $H^p$ are considered.

The main results of this paper were earlier announced in [4]. This paper can be regarded as a second part of the paper [6] which contains the study of Hardy-Orlicz spaces. Some results of the paper [5] and other papers will be needed here. We collect them in the first section.

I. ORLICZ AND HARDY-ORLICZ SPACES

1.1. A $\varphi$-function we call a real, nondecreasing and continuous for $u \geq 0$ function, equal 0 only at $u = 0$ and tending to $\infty$ when $u \to \infty$.

1.2. On $\varphi$-functions we impose sometimes the following conditions:

$$(A_1) \quad \varphi(2u) \leq d\varphi(u),$$

$$(V_1) \quad 2\varphi(u) \leq \varphi(2u),$$

$$(V_2) \quad 2d\varphi(u) \leq d^{-1}\varphi(du)$$

for $u \geq u_0$ with some constants $d > 1$ and $u_0 > 0$.

1.3. Among $\varphi$-functions we distinguish log-convex $\varphi$-functions which satisfy the inequality

$$\varphi(u^n) \leq n\varphi(u) + \beta \varphi(v)$$

for $u, v > 0$ and $a, \beta \geq 0$, $a + \beta = 1$.

and convex $\varphi$-function which satisfy the inequality

$$\varphi(u^n + \beta v) \leq n\varphi(u) + \beta \varphi(v)$$

for $u, v > 0$ and $a, \beta \geq 0$, $a + \beta = 1$.

Clearly, a $\varphi$-function $\varphi$ is log-convex if and only if it can be represented in the form

$$\varphi(u) = \Phi(\log u)$$

for $u > 0$. 
where \( \Phi \) is a convex function on the whole real axis. From this it follows that a log-convex \( \varphi \)-function \( \varphi \) is strictly increasing for \( u \geq 0 \) and so, it has an inverse function \( \varphi^{-1} \). Convex \( \varphi \)-functions and more generally functions of the form \( \varphi(u) - \varphi(u') \) for \( u \geq 0 \), where \( \varphi \) is a convex \( \varphi \)-function and \( u' \geq 0 \), are a particular case of log-convex \( \varphi \)-functions.

1.4. On convex \( \varphi \)-function \( \varphi \) we impose frequently the following conditions:

\[
\begin{align*}
(0_1) & \quad \lim_{u \to 0^+} u^{-1} \varphi(u) = 0 \\
(\infty_1) & \quad \lim_{u \to \infty} u^{-1} \varphi(u) = \infty.
\end{align*}
\]

Under these conditions for a convex \( \varphi \)-function \( \varphi \) we define a function

\[ \varphi'(t) = \sup \{ w \in \mathbb{R} : \varphi(u) \geq w u, \quad u \geq 0 \}, \quad (\varphi \geq 0). \]

The function \( \varphi' \) is also convex \( \varphi \)-function, satisfies the condition \((0_1)\), \((\infty_1)\) and moreover, \( (\varphi') = \varphi \).

In the sequel only log-convex \( \varphi \)-functions \( \varphi \) for which a convex function \( \Phi \) from their representation \((\ast)\) satisfies the condition \((\infty_1)\) will have applications and therefore the letter \( \varphi \) will be used only for these functions.

2.1. Let \( f \) be a complex-valued function, defined and measurable on the interval \([0, 2\pi]\). We define

\[ J_\varepsilon(f) = \int_0^{2\pi} \varphi(|f(t)|) dt. \]

In the space of all complex-valued functions, defined and measurable on \([0, 2\pi]\) the functional \( J_\varepsilon \) is a modular in the sense of Mushaal and Orlicz.

2.2. By \( L^\varepsilon \) we denote the class of all complex-valued functions \( f \), measurable on \([0, 2\pi]\) for which \( J_\varepsilon(f) < \infty \), by \( L^{\infty} \) the class of all functions \( f \) such that \( a f \) is \( L^\varepsilon \) for a certain \( a > 0 \) (in general dependent on \( f \)) and by \( L^{\infty} \) the class of all functions \( f \) such that \( a f \) is \( L^\varepsilon \) for every \( a > 0 \).

In the space of measurable on \([0, 2\pi]\) complex-valued functions the class \( L^\varepsilon \) is an absolutely convex set and the classes \( L^\varepsilon \) and \( L^{\infty} \) are linear subspaces. The class \( L^\varepsilon \) is called \( \text{Orlicz class} \), \( L^{\infty} \) \( \text{Orlicz spaces} \) and \( L^\varepsilon \) the space of finite elements of \( L^{\infty} \), ([7], [3]).

2.3. Generally, in the space \( L^\varepsilon \) the functional

\[ J_\varepsilon(f) = \inf \{ \varepsilon > 0 : \| f \|_\varepsilon \leq \varepsilon \}, \quad (f \in L^\varepsilon) \]

is a complete \( F \)-norm and \( L^\varepsilon \) is identical with the closure of the space of all continuous function on \([0, 2\pi]\) in the space \([L^\varepsilon, \| \cdot \|_\varepsilon]\).
where \( \log^+ u = \log \sup \{1, u\} \), are uniformly absolutely continuous functions of \( u \). It is known ([14]) that functions \( F \in \mathcal{N} \) have the non-tangential limits

\[
\lim_{u \to 0^+} F(u) = F(0^+)
\]

almost everywhere on the circumference \( \{z: |z| = 1\} \) and that for \( F \in \mathcal{N} \)

\[
F(0^+) = 0
\]

for almost all \( \tau \in (0, 2\pi) \) implies \( F(0) = 0 \) for all \( \tau \in D \).

For \( F \in \mathcal{N} \), in particular for all \( F \in \mathcal{H}^* \), we have here

\[
\mu_a(F) = \int_0^{2\pi} \varphi(|F(e^{it})|)dt = \varphi \int_0^{2\pi} \bigg\{ \frac{1}{\pi(1-| \tau |)} \bigg\}
\]

Identifying functions \( F \in \mathcal{N} \) with its boundary functions \( F|\tau| \) we can write

\[
\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{N}^* \quad \mathcal{H}^* = \mathcal{L}^* \cap \mathcal{N}^* \quad \text{and} \quad \mathcal{H}^* = \mathcal{L}^* \cap \mathcal{N}^*.
\]

3.4. Generally, in the space \( \mathcal{H}^* \) we introduce an \( F \)-norm by the formula

\[
||F||_F = \|F(\tau)|^\alpha dt = \inf \{\epsilon > 0: \mu_a(F|\tau|) \leq \epsilon\}, (F \in \mathcal{H}^*)
\]

The space \( \mathcal{H}^* \) is complete with respect to this norm and \( \mathcal{H}^* \) is identical with the closure of the space of all functions analytic in \( D \) and continuous in \( D \) \( \{z: |z| < 1\} \) in the space \( ||F||_F \). The space \( \mathcal{H}^* \) is separable; polynomials with rational coefficients form a dense set in this space.

For a fixed \( \mathcal{F} \in \mathcal{H}^* \), \( a^{-1}||F||_F \) is a non-increasing function for \( a > 0 \). From this it follows that for \( \mathcal{F} \in \mathcal{H}^* \) such that \( 0 < ||\mathcal{F}||_F \leq R \) we have

\[
\frac{R}{||\mathcal{F}||_F} \leq R
\]

By Fatou Lemma it follows that \( \mu_a(F||F||_F) \leq ||F||_F \) for \( F \neq 0 \).

If \( \varphi(u) = \varphi(e^{it}) \) for \( u > 0 \), where \( \varphi \) is a convex \( g \)-function and \( 0 < \alpha < 1 \), then an \( \alpha \)-homogeneous norm can be defined in \( \mathcal{H}^* \) by the formula

\[
||F||_\alpha = \|F(\tau)|^\alpha dt = \inf \{\epsilon > 0: \mu_a(F|\tau|)^\alpha \leq \epsilon\}, (F \in \mathcal{H}^*)
\]

The norms \( |||\cdot|||_F \) and \( |||\cdot|||_\alpha \) are equivalent on \( \mathcal{H}^* \).

If \( \varphi \) is a convex \( g \)-function satisfying the conditions (0) and (\( \infty \)), then besides the homogeneous norm \( |||\cdot|||_\alpha \) in \( \mathcal{H}^* \) we can introduce another homogeneous norm by the formula

\[
|||F|||_{\alpha} = ||F(\tau)|^\alpha dt = \sup \{\int_0^{2\pi} |F(\tau)|^\alpha dt\}
\]

where the supremum is taken over all functions \( g \in \mathcal{L}^* \) such that \( \mathcal{J} \geq 1 \). The norms \( |||\cdot|||_F \) and \( |||\cdot|||_\alpha \) are equivalent on \( \mathcal{H}^* \); namely

\[
||F||_F \leq ||F||_\alpha \leq 2||F||_F \text{ for every } F \in \mathcal{H}^* \quad \text{([1])}
\]
3.5. For $F \in H^{**}$ we define

$$[F]_p = \inf \{ \varepsilon > 0 : \mu_\varepsilon(F) < \infty \}. $$

Functional $[\cdot]_p$ is a homogeneous pseudonorm on $H^{**}$ such that

1° $[F]_p = 0$ if and only if $F \in H^p$,

2° $[F]_p \leq ||F||_p$,

3° $[F]_p = \lim_{r \to \infty} ||aF||_p$ [9].

3.6. For functions $F$ analytic in $D$ following two linear operators

$$T_c F(z) = F(zr), \quad 0 \leq r \leq 1,$$

and

$$S_\delta F(z) = F(\delta z), \quad \delta \in \mathbb{R},$$

are interesting. Namely, for these operators we have:

1° $\mu_r(T_c F) = \mu_r(F)$ and $\mu_r(S_\delta F) = \mu_r(F)$,

2° $\mu_r(F) < \infty$ implies $\mu_r(\{T_c F(z) - F(z)\} \to 0$ as $r \to 1$,

3° $\mu_r(F) < \infty$ implies $\mu_r(\{S_\delta F(z) - F(z)\} \to 0$ as $\delta \to 0$.

From this it follows that

1° $||T_c F||_p \leq ||F||_p$ for $0 \leq r < 1$ and lim $||T_c F||_p = ||F||_p$,

2° $||S_\delta F||_p = ||F||_p$,

3° $||T_c F(z) - F(z)||_p \to 0$ as $r \to 1$ for $F \in H^p$,

4° $||S_\delta F(z) - F(z)||_p \to 0$ as $\delta \to 0$ for $F \in H^p$.

Analagous statements hold for norms $||\cdot||_p$ and $||\cdot||_q$ when these norms can be introduced in $H^{**}$ by the before given formulas (26).

Moreover, for $F \in H^{**}$ we have here

$$[F]_p \leq \inf \{ ||F - G||_p : G \in H^{**} \} \leq \sup_{r < 1} ||T_c F||_p \leq 2 [F]_p.$$

3.7. A set $X \subset H^{**}$ is called a bounded set in the space $[H^{**}, ||\cdot||_p]$ if $a_n \to 0$ and $F_n \subset X$ implies always $||a_n F_n||_p \to 0$; this holds if and only if for every $\varepsilon > 0$ there exists $a > 0$ such that $||aF||_p \leq \varepsilon$ for all $F \in X$.

A ball $\{F \in H^{**} : ||F||_p \leq R\}$, $R > 0$, is a bounded set in $[H^{**}, ||\cdot||_p]$ if and only $||F||_p \leq \sup_{r < 1} ||T_c F||_p$.

3.8. In Hardy–Orlicz space $H^{**}$, similarly as in Orlicz space $L^{**}$, we have two convergences for sequences, one is a norm convergence and other a modular convergence. So we say that a sequence $\{F_n\} \subset H^{**}$ is convergent in norm to $F \in H^{**}$, if $||F_n - F||_p \to 0$ as $n \to \infty$; this holds if and only if $\mu_n(a(F_n - F)) \to 0$ as $n \to \infty$ for any $a > 0$. Moreover, we say that a sequence $\{F_n\} \subset H^{**}$ is convergent in modular to $F \in H^{**}$, if $\mu_n(a(F_n - F)) \to 0$ as $n \to \infty$ for some $a > 0$ (in general dependent on $[F_n - F]$).

In the case when $\varphi$ satisfies the condition $(\Delta_2)$, $H^{**} = H^p$ and the norm and modular convergences are equivalent. Otherwise, we have only $H^{**} \subset H^p$ and only the norm convergence implies the modular convergence.

4.1. Besides above mentioned two convergences in Hardy–Orlicz space $H^{**}$ we distinguish a third convergence. Namely, we say that a sequence $\{F_n\} \subset H^{**}$ is convergent very weakly to $F \in H^p$, if

$$\sup ||F_n - F||_p < \infty \quad \text{and} \quad \sup_n \{ ||F_n(z) - F(z)||_p : z \in \overline{E} \} \to 0 \quad \text{as} \quad n \to \infty$$

for any closed set $E \subset D$. This definition of very weak convergence does not change when the norm $||\cdot||_p$ is replaced by the norm $||\cdot||_q$ or $||\cdot||_p$.

4.2. Modular convergence implies very weak convergence.

Proof. Let $\mu_n(a(F_n - F)) \to 0$ as $n \to \infty$ for $a > 0$. Then there exists $n_0$ such that $\mu_n(a(F_n - F)) \leq \frac{1}{a}$ for $n \geq n_0$. From this get

$$\sup_n ||F_n - F||_p \leq \sup \left\{ \frac{1}{a}, ||F_1 - F||_p, \ldots, ||F_{n_0} - F||_p \right\} < \infty.$$ 

From this and from the inequality given in 3.1 the theorem follows.

4.3. If a sequence $\{F_n\} \subset H^{**}$ converges very weakly to $F \in H^{**}$, then

$$\mu_n(F) \leq \inf_{n \to \infty} \mu_n(F_n)$$

and

$$||F||_p \leq \inf_{n \to \infty} ||F_n||_p.$$ 

Proof. Because for $0 \leq r < 1$ a sequence $\{F_n(re^i)\}$ tends to $F(re^i)$ uniformly on $t$, we have

$$\mu_n(r; aF) = \lim_{n \to \infty} \mu_n(r; aF_n) \leq \inf_{n \to \infty} \mu_n(aF_n)$$

for all $a > 0$.

From this we get the inequalities of the theorem.

4.4. A sequence $\{F_n\} \subset H^{**}$ is convergent very weakly if and only if $\sup ||F_n||_p < \infty$ and a sequence $\{F_n(z)\}$ is convergent on a set $E \subset D$ which has a cluster point in $D$.

Proof. If a sequence $\{F_n\} \subset H^{**}$ is convergent very weakly to $F \in H^{**}$, then

$$\sup_n ||F_n||_p \leq \sup_n ||F_n - F||_p + ||F||_p < \infty.$$
and a sequence \( \{ F_n(s) \} \) is convergent (to \( F(s) \)) for all \( s \in D \). Now, let \( \{ F_n \} \) \( \in H^* \) be a sequence such that \( \sup_n \| F_n \| < R < \infty \) and a sequence \( \{ F_n(s) \} \) is convergent on a set which has a cluster point in \( D \). Then by 3.1 we have

\[
|F_n(s)| \leq \varphi_1 \left( \frac{R}{\pi(1-|s|)} \right) R \quad \text{for all } s \in D
\]

and by the Vitali Theorem we obtain that a sequence \( \{ F_n(s) \} \) is convergent uniformly on every closed set \( E \subset D \). Let \( F \) be a limit function for this sequence. Since for \( 0 \leq r < 1 \) a sequence \( \{ F_n(r) \} \) tends to \( F(r) \) uniformly on \( \Gamma \), we have

\[
\mu_n(r; F/E) = \lim_{\delta \to 0} \mu_n(r; F_n) \leq R \quad \text{for } 0 \leq r < 1
\]

and

\[
\mu_n(F/E) \leq R < \infty.
\]

This proves that \( F \in H^* \). Now, we have

\[
\sup_n \| F_n - F \| \leq \sup_n \| F_n \| + \| F \| < \infty
\]

and \( F_n \to F \) as \( n \to \infty \) uniformly on any closed set \( E \subset D \).

4.5. Every ball \( F \in H^* ; \| F \| \leq R, R > 0 \) is sequentially very weakly compact set.

Proof. Let \( \{ F_n \} \subset H^* \) be a sequence such that \( \sup_n \| F_n \| \leq R \). Then by 3.1 we have

\[
|F_n(s)| \leq \varphi_1 \left( \frac{E}{\pi(1-|s|)} \right) R \quad \text{for all } s \in D.
\]

From this it follows by the Montel Theorem that there exists a subsequence \( \{ F_{n_k} \} \) of a sequence \( \{ F_n \} \), which is convergent uniformly on any closed set \( E \subset D \). Now, from 4.4 we get the theorem.

With respect to these three convergences in Hardy–Orlicz space \( H^* \) in the sequel we shall deal with norm continuous, modular continuous and very weakly continuous linear functionals on \( H^* \).

II. LINEAR FUNCTIONALS IN HARDY-ORLICZ SPACES

1.1. Let \( z \) be a fixed point of a circle \( D \). For any function \( F \) analytic in \( D \) we define

\[
\gamma_{n,s}(F) = F(s) \quad \text{and} \quad \gamma_{n,s}(F) = \frac{1}{n} F^{(n)}(s), \quad (n = 1, 2, \ldots)
\]

For \( z = 0 \) we shall write \( \gamma_n \) instead of \( \gamma_{n,s} \).

It is clear that \( \gamma_{n,s} \) and \( \gamma_{n,s} \) are linear functionals on the space of analytic functions in \( D \), and therefore, they are linear functionals on any Hardy–Orlicz space \( H^* \). We note here that these functionals are very weakly continuous on \( H^* \).

We see from this that on any Hardy–Orlicz space there exist very weakly continuous linear functionals, and so, also modular continuous and norm continuous ones. Moreover, from the above it follows that there exist fundamental systems of very weakly continuous (modular continuous, norm continuous) linear functionals on \( H^* \), i.e., there exist sequences \( \{ \xi_n \} \) of very weakly continuous (modular continuous, norm continuous) linear functionals on \( H^* \) such that \( \xi_n(F) = 0 \) for \( n = 1, 2, \ldots \) and \( F \in H^* \) imply \( F = 0 \).

1.2. In a general case, we have for the functionals mentioned at 1.1 the following estimate with respect to the norm \( \| . \|_p \) and

\[
\| \gamma_{n,s}(F) \| \leq \varphi_1 \left( \frac{\| F \|_p}{\pi(1-|s|)} \right) \| F \|_p
\]

and

\[
\| \gamma_{n,s}(F) \| \leq \inf \left\{ \frac{\| F \|_p}{\pi(1-r)} \left( \frac{r}{|s|} \right)^{\alpha} : |s| < r < 1 \right\} \| F \|_p, \quad n = 1, 2, \ldots
\]

for any \( F \in H^* \) and \( s \in D \).

When \( \varphi(s) = \varphi(u^s) \), where \( \varphi \) is a convex \( \varphi \)-function and \( 0 < s \leq 1 \), we have the following estimation for these functionals with respect to the norm \( \| . \|_q \): and

\[
\| \gamma_{n,s}(F) \| \leq \varphi_1 \left( \frac{\| F \|_q}{\pi(1-|s|)} \right) \| F \|_q
\]

and

\[
\| \gamma_{n,s}(F) \| \leq \inf \left\{ \frac{1}{\pi(1-r)} \left( \frac{r}{|s|} \right)^{\alpha} : |s| < r < 1 \right\} \| F \|_q, \quad n = 1, 2, \ldots
\]

for any \( F \in H^* \).

Proof. Estimations for the functional \( \gamma_{n,s} \) follow immediately from the inequality

\[
\| F(s) \| \leq \varphi_1 \left( \frac{\mu_n(F)}{\pi(1-|s|)} \right) \quad \text{for } s \in D
\]

and from definitions of corresponding norms \( \| . \|_p \) and \( \| . \|_q \). Estimations for the functionals \( \gamma_{n,s} \) we get now by Cauchy's Integral Formula. Indeed, for any \( r, |s| < r < 1 \) we have

\[
\| \gamma_{n,s}(F) \| = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{(z-s)^{n+1}} \, dz \leq \sup \{ |F(z)| : |z| = r \} \frac{r}{(r-|s|)^{n+1}}
\]

\[
\leq \varphi_1 \left( \frac{\mu_n(F)}{\pi(1-r)} \right) \frac{r}{(r-|s|)^{n+1}},
\]
where \( C \) designates the circumference \( \{|z| = r\} \). Thus

\[
|\gamma_{n}(F)| \leq \inf \left\{ \frac{\mu(F)}{\pi(1-r)} \right\} \frac{r}{(r-s)|s-r|^2} : |s| < r < 1 \}
\]

From this and definitions of norms \( \| \cdot \|_{\rho} \) and \( \| \cdot \|_{\infty} \) follow now postulates estimations.

1.3. In the sequel, we shall denote by \((H^{\infty})^d\) a class of all norm continuous linear functionals on \(H^s\), by \((H^{\infty})_{\text{mod}}^d\) a class of all modular continuous linear functionals on \(H^s\), and by \((H^{\infty})_{\text{mod}}^d\) a class of all very weakly continuous linear functionals on \(H^s\).

1.4. A sequence \( \{F_n\} \in H^s \) very weakly converges to \( F \in H^s\) if and only if

\[
\sup_n \left\| F_n - F \right\|_{\rho} < \infty \quad \text{and} \quad \tilde{\xi}(F_n) \rightarrow \tilde{\xi}(F) \quad \text{when} \quad n \rightarrow \infty
\]

for every \( \xi \in (H^{\infty})_{\text{mod}}^d \).

Proof. If a sequence \( \{F_n\} \in H^s \) is very weakly convergent to \( F \in H^s\), then obviously \( \sup_n \| F_n - F \|_{\rho} < \infty \) and \( \tilde{\xi}(F_n) \rightarrow \tilde{\xi}(F) \) when \( n \rightarrow \infty \) for every \( \xi \in (H^{\infty})_{\text{mod}}^d \).

Conversely, for \( \{F_n\} \in H^s \) and \( F \in H^s\) let sup \( \| F_n - F \|_{\rho} < \infty \) and \( \tilde{\xi}(F_n) \rightarrow \tilde{\xi}(F) \) when \( n \rightarrow \infty \) for every \( \xi \in (H^{\infty})_{\text{mod}}^d \).

Since the functionals \( \gamma_{n} \), where \( n \in D \), belong to \((H^{\infty})_{\text{mod}}^d\), so we have \( \gamma_{n}(s) \rightarrow \gamma(s) \) when \( n \rightarrow \infty \) for every \( s \in D \). Now, by 4.4 of Section I we obtain that \( \{F_n\} \) converges very weakly to \( F \).

1.5. A sequence \( \{F_n\} \subset H^s \) is very weakly convergent to \( F \in H^s\) if and only if

\[
\sup_n \left\| F_n - F \right\|_{\rho} < \infty \quad \text{and} \quad \gamma_{m}(F_n) \rightarrow \gamma_m(F) \quad \text{when} \quad n \rightarrow \infty
\]

for \( m = 0, 1, 2, \ldots \)

Proof. If \( \{F_n\} \) very weakly converges to \( F \) then \( \sup_n \| F_n - F \|_{\rho} < \infty \) and \( \gamma_m(F_n) \rightarrow \gamma_m(F) \) when \( n \rightarrow \infty \) for \( m = 0, 1, 2, \ldots \). Conversely, let us suppose that \( \{F_n\} \subset H^s \), \( F \in H^s\), \( \sup_n \| F_n - F \|_{\rho} = R < \infty \) and \( \gamma_m(F_n) \rightarrow \gamma_m(F) \) when \( n \rightarrow \infty \) for \( m = 0, 1, 2, \ldots \). Let \( \varepsilon > 0 \) be a number such that \( 0 < r < 1 \). Since sequence \( \left\{ \frac{2r}{1+r} \right\} \) converges to 0, then for every \( \varepsilon > 0 \) there is a \( n_0 \) such that

\[
\varphi_{m-1}\left( \frac{2R}{\pi(1-r)} \right)^{\frac{1+r}{1+r}} \left( \frac{2r}{1+r} \right)^{m_0} < \frac{\varepsilon}{2}
\]

By 1.2 we have for every \( n \)

\[
|\gamma_m(F_n - F)| \leq \varphi_{m-1}\left( \frac{R}{\pi(1-r)} \right)^{\frac{1+r}{1+r}} \left( \frac{2R}{\pi(1-r)} \right)^{m_0} \varepsilon
\]

and

\[
|\gamma_m(F_n - F)| \leq \varphi_{m-1}\left( \frac{R}{\pi(1-r)} \right)^{\frac{1+r}{1+r}} \left( \frac{2R}{\pi(1-r)} \right)^{m_0} \varepsilon
\]

for \( m = 1, 2, \ldots \)

From this we get for \( |s| < r \)

\[
\sum_{m=0}^{\infty} \left| \gamma_m(F_n - F) \right| \leq \sum_{m=0}^{\infty} \left| \gamma_m(F_n - F) \right|^{m_0} \varepsilon
\]

\[
\leq \varepsilon \varphi_{m-1}\left( \frac{2R}{\pi(1-r)} \right)^{\frac{1+r}{1+r}} \left( \frac{2R}{\pi(1-r)} \right)^{m_0} \left( \frac{1+r}{1+r} \right) \left( \frac{2r}{1+r} \right)^{m_0} \leq \frac{\varepsilon}{2}
\]

Now, since \( \gamma_m(F_n - F) \rightarrow 0 \) when \( n \rightarrow \infty \) for \( m = 0, 1, 2, \ldots \), it follows that for already fixed \( \varepsilon > 0 \) there is a \( n_0 \) such that

\[
|\gamma_m(F_n - F)| \leq \frac{(1-r)}{2} \varepsilon \quad \text{for} \quad m > n_0 \quad \text{and} \quad m = 1, 2, \ldots, n_0 - 1.
\]

Thus, for \( |s| < r \) and \( n > n_0 \) we get

\[
|F_n(s) - F(s)| = \sum_{m=0}^{n_0-1} \left| \gamma_m(F_n - F) \right|^{m+1} \leq \sum_{m=0}^{n_0-1} \left| \gamma_m(F_n - F) \right|^{m+1} \leq \frac{\varepsilon}{2}
\]

From this we conclude that \( \{F_n(s)\} \) converges to \( F(s) \) uniformly in the circle \( |s| < r \). This yields that \( \{F_n(s)\} \) converges to \( F(s) \) on every closed subset \( D \) of \( D \). Thus \( \{F_n\} \) is very weakly convergent to \( F \).

1.6. A sequence \( \{F_n\} \subset H \) is very weakly convergent if and only if \( \sup_n \| F_n \|_{\rho} < \infty \) and for \( m = 0, 1, 2, \ldots \) the sequences \( \gamma_{m}(F_n) \) are convergent.

Proof. If \( \{F_n\} \subset H^s \) very weakly converges to \( F \in H^s\) then

\[
\sup_n \left\| F_n \right\|_{\rho} \leq \sup_n \left\| F_n - F \right\|_{\rho} + \left\| F \right\|_{\rho} < \infty
\]

and by 1.5 we see that the sequences \( \gamma_{m}(F_n) \) converge for \( m = 0, 1, \ldots \)

Conversely, let \( \{F_n\} \subset H^s \) be such a sequence that \( \sup_n \left\| F_n \right\|_{\rho} = \frac{R}{2} < \infty \)
and for \( m = 0, 1, 2, \ldots \) the sequences \( \{ y_m(F_n) \} \) are convergent. Then we have
\[
\lim_{n \to \infty} \| F_m - F_k \|_p \leq R
\]
Replacing now in the proof of 1.5 a function \( F \) by functions \( F_m \) we immediately obtain that \( F_m(x) - F_k(x) \to 0 \) when \( m, k \to \infty \) uniformly on every circle \( |x| \leq r < 1 \). This combined with 4.4 of Section 1 implies that \( \{ F_m \} \) is very weakly convergent.

2.1. Let \( \xi \) be a linear functional on \( H^\infty \). We define
\[
\nu_\xi(\xi; \bar{E}) = \sup \{ \| \xi(F) \|_p : \| F \|_p \leq R, F \in H^\infty \} \quad \text{for } R > 0.
\]

For every linear functional \( \xi \) on \( H^\infty \) and for every \( R > 0 \) the following inequality
\[
\| \xi(F) \|_p \leq R, \xi(\bar{E}) \in H^\infty
\]
holds for every \( F \in H^\infty \) such that \( 0 < \| F \|_p \leq R \).

Proof. If \( \nu_\xi(\xi; \bar{E}) = \infty \) the inequality in question is obvious. Let us exclude this and suppose that \( \nu_\xi(\xi; \bar{E}) < \infty \). Then
\[
\| \xi(F) \|_p \leq \nu_\xi(\xi; \bar{E}) \quad \text{for every } F \in H^\infty \text{ such that } \| F \|_p \leq R.
\]
Since, in view of 3.4 of Section I, for \( F \in H^\infty \) such that \( 0 < \| F \|_p \leq R \) also \( \| RF \|_p \leq R \), we get
\[
\| \xi(RF) \|_p \leq \nu_\xi(\xi; \bar{E}) R.
\]
This implies the desired inequality.

2.2. A linear functional \( \xi \) on \( H^\infty \) belongs to \( (H^\infty)^0 \) if and only if
\[
\lim_{R \to \infty} \| \xi(RF) \|_p \leq \nu_\xi(\xi; \bar{E}) < \infty \quad \text{for some } R > 0.
\]

Proof. If \( \xi \in (H^\infty)^0 \) then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that
\[
\| \xi(F) \|_p < \varepsilon \quad \text{for all } F \in H^\infty \text{ such that } \| F \|_p < \delta.
\]
Fixing \( \varepsilon = \varepsilon(\delta) \) and taking \( R = \delta \) we prove the required implication.

Conversely, if \( \xi(\xi; \bar{E}) < \infty \) for some \( R > 0 \) then, by the inequality proved in 2.1, \( \xi \) is norm continuous on \( H^\infty \) and so \( \xi \in (H^\infty)^0 \).

2.3. For a fixed \( \xi \in (H^\infty)^0 \), \( \xi^{-1}(\xi; \bar{E}) \) is nondecreasing for \( R > 0 \).

More precisely
\[
\lim_{R \to \infty} \nu_\xi(\xi; \bar{E}) = \sup \{ \| \xi(F) \|_p : \| F \|_p \leq R, F \in H^\infty \}.
\]

Proof. Observe that
\[
\sup \{ \| \xi(F) \|_p : \| F \|_p \leq R, F \in H^\infty \} = \sup \{ \| \xi(G) \|_p : \| G \|_p \leq R, G \in H^\infty \} = \nu_\xi(\xi; \bar{E}) R.
\]

2.4. A functional \( \xi \in (H^\infty)^0 \) belongs to \( (H^\infty)^0 \) if and only if
\[
\lim_{R \to \infty} \nu_\xi(\xi; \bar{E}) = 0.
\]

Proof. \( \xi \in (H^\infty)^0 \) then obviously
\[
\lim_{R \to \infty} \nu_\xi(\xi; \bar{E}) = 0.
\]

Conversely, let \( \xi(\xi; \bar{E}) \to 0 \) for \( \xi \in (H^\infty)^0 \). Further, let \( \xi(\xi) = \xi \) be a sequence such that \( \nu_\xi(aF_\xi) \to 0 \) when \( n \to \infty \) for \( a > 0 \). Then \( \xi(aF_\xi) \to 0 \) when \( n \to \infty \) and this implies that \( \xi(F_\xi) \to 0 \) when \( n \to \infty \). This means that \( \xi \in (H^\infty)^0 \).

2.5. If \( \xi \in (H^\infty)^0 \) then \( \nu_\xi(\xi; \bar{E}) < \infty \) for every \( \xi > 0 \).

Proof. By 4.5 of Section I every ball \( \xi = H^\infty \) is sequentially very weakly compact. Thus for every \( \xi > 0 \) there is \( F_\xi \in H^\infty \) such that \( \| F_\xi \|_p \leq R \) and \( \nu_\xi(\xi; \bar{E}) = \| F_\xi \|_p \). This yields the theorem.

2.6. For \( R > 0 \) we denote by \( (H^\infty)^0_R \) a class of all functionals \( \xi \in (H^\infty)^0 \) for which \( \nu_\xi(\xi; \bar{E}) < \infty \) and by \( (H^\infty)^0 \) a class of all functionals \( \xi \in (H^\infty)^0 \) for which \( \nu_\xi(\xi; \bar{E}) < \infty \). Further, \( (H^\infty)^0 \) will denote a class of all functionals \( \xi \in (H^\infty)^0 \) for which \( \nu_\xi(\xi; \bar{E}) < \infty \) for every \( E > 0 \). The class \( (H^\infty)^0 \) is defined similarly.

According to 2.2 we have
\[
(H^\infty)^0 = \bigcup_{R=1}^{\infty} (H^\infty)^0_R \quad \text{and} \quad (H^\infty)^0 = \bigcup_{R=1}^{\infty} (H^\infty)^0_R.
\]

Analogous relations hold for \( (H^\infty)^0_R \) and \( (H^\infty)^0 \).

Clearly \( (H^\infty)^0_R, R > 0 \) and \( (H^\infty)^0 \) are linear subspaces of \( (H^\infty)^0 \) and similarly \( (H^\infty)^0_R, R > 0 \) and \( (H^\infty)^0 \) are linear subspaces of \( (H^\infty)^0 \).

It is also evident that the functional \( \nu(\cdot; E) \) is a homogeneous norm in \( (H^\infty)^0_R, R > 0 \). We shall show that:

2.7. \( (H^\infty)^0_R, R > 0 \) is complete relative to the norm \( \nu(\cdot; E) \).

Proof. Let \( \xi(\xi, \bar{E}) \) be such a sequence that \( \nu_\xi(\xi, \bar{E}) < \infty \) when \( E \to \infty \). Since for every \( E \in H^\infty \) there is a \( \varepsilon > 0 \) such that \( \| aF \|_p \leq R \) this implies that
\[
\nu_\xi(\xi, \bar{E}) \leq \varepsilon \nu_\xi(\xi, \bar{E}).
\]

for every \( k \) and \( F \). From this we may deduce that \( \xi(kF) \to \xi(F) \) when \( k \to \infty \) for every \( E \in H^\infty \). This means that for every \( E \in H^\infty \) a sequence \( \nu_\xi(\xi, \bar{E}) \) is convergent; its limit we designate by \( \xi(F) \). Obviously, \( \xi \) is a linear functional on \( H^\infty \). By our assumption it follows that for every \( \varepsilon > 0 \) there is a \( N \) such that
\[
\| \xi \|_p \leq \varepsilon \quad \text{for } k, l \geq N.
\]
and for every $F \in H^s$ such that $\|F\|_p \leq R$. Passing to the limit with $l \to \infty$ we obtain

$$|\xi_1(F) - \xi(F)| < \varepsilon$$

for $k > m$, and for $F \in H^s$ such that $\|F\|_p \leq R$.

This implies further that

$$\nu_\varphi(\xi; R) = \varepsilon + \nu_\varphi(\xi; R; E) < \infty,$$

what proves, by 2.2, that $\xi \in (H^s)^\#_p$ and $\nu_\varphi(\xi; \xi; E) = \varepsilon$ for $k > m$.

This means that $\{\xi_k\} \subset (H^s)^\#_p$ converges to $\xi$ with respect to the norm $\nu_\varphi(\cdot; E)$. Let

$$\nu_\varphi(\xi; \xi) = \nu_\varphi(\xi; \xi; E) < \varepsilon < \infty,$$

what proves, by 2.2, that $\xi = (H^s)^\#_p$. Let $\nu_\varphi(\cdot; E)$ converge to $\nu_\varphi(\cdot; E)$ in the norm $\nu_\varphi(\cdot; E)$. Then for every $\varepsilon > 0$ there is a $n_0$ such that $\nu_\varphi(\xi - \xi_k; E) = \varepsilon$ and $\nu_\varphi(\xi_k; E) < \varepsilon$ for $k > m$.

By 2.4 for this $\varepsilon > 0$ there is a $R_1 > 0$ such that $\nu_\varphi(\xi - \xi_k; E) < \varepsilon$ for $k > m$.

By 2.3 in view of the $\nu_\varphi(\xi; E)$ we get

$$\nu_\varphi(\xi; E) = \inf \{\varepsilon > 0 : \nu_\varphi(\xi; \xi; E) < \varepsilon\}.$$
Taking into account 2.4 we see that $\xi \in (H^s_t)_0$.

3.6. The space $(H^s_t)_0$ (resp. $(H^s_w)_0^2$) is, for every $R > 0$, a closed linear subspace of $[(H^s_t)_0]$, $\nu_0$ (resp. $[(H^s_w)_0^2]$, $\nu_0$).

It follows directly from 3.3, 2.7 and 2.8.

3.7. The space $(H^s_t)_0$ (resp. $(H^s_w)_0^2$) is a closed linear subspace of

$[(H^s_t)_0]$, $\nu_0$ (resp. $[(H^s_w)_0^2]$, $\nu_0$).

Proof. Straightforward application of 3.6 and 2.6.

3.8. A functional $\xi \in (H^s_t)_0$ is a member of $(H^s_t)_0$ if and only if $\nu_0(a\xi) \to 0$ when $a \to 0$.

Proof. Let us notice that if $\xi \in (H^s_t)_0$, then $\nu_0(\xi; R) < \infty$ for all $R > 0$. This in turn implies for every $R > 0$ that $\nu_0(a\xi; R) = |a|\nu_0(\xi; R) \to 0$ when $a \to 0$. By 3.3 we now get that $\nu_0(a\xi) \to 0$ when $a \to 0$.

Conversely, let for $\xi \in (H^s_t)_0$ be $\nu_0(a\xi) \to 0$ when $a \to 0$. Then, by 3.3, $\nu_0(a\xi; R) \to 0$ when $a \to 0$. It means that for every $R > 0$ there exists $a_0 > 0$ such that $\nu_0(a\xi; R) \leq 1$. Thus $\nu_0(\xi; R) \leq a_0^2$ and $\xi \in (H^s_t)_0$.

3.9. The space $[(H^s_t)_0]$, $\nu_0$ is a Fréchet space and $(H^s_t)_0$ is its closed linear subspace.

It follows clearly from 3.1, 3.8, 3.7 and 3.5.

3.10. The space $(H^s_t)_0$ is a closed linear subspace of $[(H^s_t)_0]$, $\nu_0$.

Proof. That $(H^s_t)_0$ is a subspace of $(H^s_t)_0$ is an immediate consequence of 4.3 of Section I and 2.5. Now, let $(\xi_0) \subseteq (H^s_t)_0$ be a sequence convergent in norm $\xi_0 \in (H^s_t)_0$. Let, further, $(F_0) \subseteq (H^s_t)$ converge very weakly to $F \in (H^s_t)$ and, besides, $\lim_{n \to \infty} \|F_n - F\|_E \leq R$. Then, in view of 3.3, for every $\varepsilon > 0$ there is a $a_0$ such that $\nu_0(a_0\xi - \xi_0; R) \leq \varepsilon/2$. For, given $\varepsilon > 0$, by virtue of the fact that $\xi_0$ is a member of $(H^s_t)_0$, there is a $m_0$ such that $|\xi_0(F - F_n)| \leq \varepsilon/2$ for $m \geq m_0$. Thus, for $m \geq m_0$ we obtain

$|\xi(F - F_n)| \leq |\xi(F - F_n) - \xi_0(F - F_n)| + |\xi_0(F - F_n)| \leq \nu_0(\xi - \xi_0; R) + |\xi_0(F - F_n)| \leq \varepsilon$.

It follows then $\xi(F_n) \to \xi(F)$ when $m \to \infty$ and $\xi \in (H^s_t)_0$.

4.1. Let $\xi$ be a linear functional on $H^s_t$. We define

$(T_{\varepsilon}^t)\xi(F) = \xi(T_{\varepsilon}F)$

for every $F \in H^s_t$ and $0 \leq \varepsilon \leq 1$.

and

$(S^t_\varepsilon)\xi(F) = \xi(S^t_\varepsilon F)$

for every $F \in H^s_t$ and a real $h$.

Simple consequences of the above definitions and 3.6 of Section I are the following:

Further, for $\xi \in (H^s_t)_0$, $\nu_0(T^r_{\varepsilon}\xi) \leq \nu_0(\xi)$ for $0 \leq \varepsilon \leq 1$ and $R > 0$.

and

$\nu_0(S^t_\varepsilon\xi) \leq \nu_0(\xi)$ for $h$ real and $R > 0$.

4.2. If $\xi \in (H^s_t)_0$ then $T^r_{\varepsilon} \xi \in (H^s_t)_0$ for every $r \geq \varepsilon > 1$.

Proof. Let the sequence $(F_n) \subseteq H^s_t$ very weakly converge to $F \in H^s_t$.

Then the sequence $(F_n) \subseteq H^s_t$ converges uniformly on the circumference $x = |x| = r$, $0 \leq r < 1$. Now, the sequence $(T_{\varepsilon}F_n)$ norm converges to $T_{\varepsilon}F$.

This proves that $T^r_{\varepsilon} \xi \in (H^s_t)_0$ for every $0 \leq r < 1$. 

4.3. A functional $\xi \in (H^s_t)_0$ belongs to $(H^s_t)_0$ if and only if

$\lim_{r \to 1} T^r_{\varepsilon} \xi - \xi = 0$.

Proof. If $\xi \in (H^s_t)_0$ and $\nu_0(T^r_{\varepsilon} \xi - \xi) \to 0$ for $r \to 1$ then, by 4.2 and 3.10, $\xi \in (H^s_t)_0$.

On the other hand, let $\xi \in (H^s_t)_0$. Let us take on arbitrary $R > 0$, and let $(\xi_n)$ be such a sequence that $0 \leq \nu_0(\xi_n) < 1$, $\tau_n \to 1$ and

$\lim_{n \to \infty} \nu_0(T^r_{\varepsilon} \xi - \xi; R) = \lim_{n \to \infty} \nu_0(T^r_{\varepsilon} \xi - \xi; R)$.

Since $T^r_{\varepsilon} \xi - \xi \in (H^s_t)_0$ and the ball $(F \in H^s_t: \|F\| \leq R)$ is, by 4.5, of Section I, sequentially very weakly complete, then for every $n$ there is a $F_n \in H^s_t$ such that

$\|F_n\| \leq R$ and $\nu_0(T^r_{\varepsilon} \xi - \xi; R) = \nu_0(T^r_{\varepsilon} \xi - \xi; F_n) = |\xi(T_{\varepsilon}F_n - F_n)|$.

Now, because $\|F_n\| \leq R$ for $n = 1, 2, \ldots$, we can find a subsequence $(F_n)$ of $(F_n)$ very weakly converging to some $F \in H^s_t$. Then also the sequence $(T_{\varepsilon}F_n)$ is very weakly converges to $F_0$. In fact, $\sup_{n \to \infty} \|F_n - F\|_E \leq \|F\|_E$. We further get by Maximum Principle,

$\sup_{n \to \infty} \|F_n - F\|_E (\|F_n\|_E ^{\text{max}})$

$\leq \|F\|_E (\|F\|_E ^{\text{max}}) + \|F\|_E (\|F\|_E ^{\text{max}})$. 

$\leq \|F\|_E (\|F\|_E ^{\text{max}}) + \|F\|_E (\|F\|_E ^{\text{max}})$. 

$\leq \|F\|_E (\|F\|_E ^{\text{max}}) + \|F\|_E (\|F\|_E ^{\text{max}})$. 

$\leq \|F\|_E (\|F\|_E ^{\text{max}}) + \|F\|_E (\|F\|_E ^{\text{max}})$.
It follows now
\[ \sup \{ |T_n F_n(x) - F_n(x)| : |x| \leq \varepsilon \} \to 0 \quad \text{when } n \to \infty \]
for every \( \varepsilon > 0 \) and \( 0 < \varepsilon < 1 \).
Thus, by 4.4 of Section I, \( \{ T_n F_n \} \) very weakly converges to \( F_0 \). Now,
\[ \lim \sup_{n \to \infty} \nu_n \left( \frac{T_n}{n} \xi - \xi ; R \right) = \lim_{n \to \infty} \nu_n \left( \frac{T_n}{n} (F_n - F) \right) = 0. \]
This leads to a conclusion
\[ \lim_{n \to \infty} \nu_n \left( \frac{T_n}{n} \xi - \xi ; R \right) = 0 \]
for every \( R > 0 \), and in virtue of 3.3 \( \nu_n \left( \frac{T_n}{n} \xi - \xi \right) \to 0 \) when \( n \to \infty \).

4.4. If \( \xi \in (H^{\ast})^\# \), then for every \( F \in H^{\ast} \)
\[ \lim_{n \to \infty} \nu_n \left( \frac{T_n}{n} \xi ; R \right) = \lim_{n \to \infty} \nu_n \left( \xi ; R \right) = \nu \left( \xi ; R \right) \]
and
\[ \lim_{n \to \infty} \nu_n \left( \frac{T_n}{n} \xi ; R \right) = \lim_{n \to \infty} \nu_n \left( \xi ; R \right) = \nu \left( \xi ; R \right). \]
This is an immediate consequence of 3.6 of Section I.

5.1. A sequence \( \{ \xi_n \} \) of functionals defined on \( H^{\ast} \) is said to be pointwise convergent to a functional \( \xi \) if for every \( F \in H^{\ast} \)
\[ \lim_{n \to \infty} \nu_n \left( \xi_n ; F \right) = \nu \left( \xi ; F \right). \]
The space \( H^{\ast} \) equipped with a norm \( \| \cdot \| \) is a Fréchet space. Thus, by Banach Theorem (11) every sequence of functionals \( \{ \xi_n \} \in (H^{\ast})^\# \), such that the sequence \( \{ \xi_n (F) \} \) converges for every \( F \in H^{\ast} \), is pointwise convergent to a functional \( \xi \in (H^{\ast})^\# \).

5.2. A sequence \( \{ \xi_n \} \in (H^{\ast})^\# \) is pointwise convergent if and only if \( \sup \nu_n (\xi_n) < \infty \) and the sequence \( \{ \xi_n (F) \} \) converges for every \( F \) belonging to a certain linearly dense subset of \( (H^{\ast})^\# \).

Proof. Let \( \{ \xi_n \} \subset (H^{\ast})^\# \) be pointwise convergent. Then, in virtue of Mazur--Orlicz Theorem (10), for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |\xi_n (F)| < \varepsilon \) for every \( n \) and for every \( F \in H^{\ast} \) such that \( \|F\| < \delta \). Thus there is an \( R > 0 \) such that \( \nu_n (\xi_n ; R) < \frac{1}{R} \) for every \( n \). Now, \( \nu_n (\xi_n) \leq \frac{1}{R} \) for every \( n \), and so \( \sup \nu_n (\xi_n) < \frac{1}{R} \leq \infty \). Conversely, assume that \( \{ \xi_n \} \subset (H^{\ast})^\# \) be a sequence such that \( \sup \nu_n (\xi_n) < \infty \) and \( \{ \xi_n (F) \} \) converges for every \( F \) from a set \( X \) dense in \( (H^{\ast})^\# \). Since \( \sup \nu_n (\xi_n) < \infty \), by 3.2 it follows that for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( |\xi_n (F)| < \frac{\varepsilon}{3} \for every \( n \) and for every \( F \in H^{\ast} \), \( |F| \leq \delta \). X is dense in \( [H^{\ast}]^\# \) and so for every \( F \in H^{\ast} \) there is a \( \xi \in X \) such that \( |F - G| \leq \delta \). The sequence \( \{ \xi_n (G) \} \) is convergent, thus there is an \( n_0 \) such that for \( n \geq n_0, |\xi_n (G) - \xi (G)| < \frac{\varepsilon}{3} \). For \( k \geq n_0 \), we have then \( |\xi_n (F) - \xi (F)| < |\xi_n (F - G)| + |\xi_n (G) - \xi (G)| + |\xi (G) - \xi (F)| \leq \frac{\varepsilon}{3} \). This means that \( \{ \xi_n (F) \} \) converges for every \( F \in H^{\ast} \).

5.3. If \( \{ \xi_n \} \subset (H^{\ast})^\# \) is pointwise convergent to \( \xi \in (H^{\ast})^\# \) then
\[ \nu_n (\xi ; R) \leq \liminf \nu_n (\xi_n ; R) \quad \text{for every } R > 0, \]
and also
\[ \nu_n (\xi ; R) \leq \liminf \nu_n (\xi_n ; R). \]
Proof. Let for \( R > 0 \) \( \{ \xi_n \} \) be such a subsequence of \( \{ \xi_n \} \) that
\[ \lim \sup \nu_n (\xi_n ; R) = \lim \nu_n (\xi_n ; R). \]
We can assume here that this above limit is finite. Then for every \( \varepsilon > 0 \) there exists a \( k_0 \) such that
\[ \nu_k (\xi_k (F)) \leq \nu_k (\xi_k ; R) \leq \lim \nu_k (\xi_k ; R) + \varepsilon \]
for every \( k \geq k_0 \) and every \( F \in H^{\ast} \), \( |F| \leq R \). Passing with \( k \to \infty \) we obtain
\[ |\xi_k (F)| \leq \lim \nu_k (\xi_k ; R) + \varepsilon \]
for every \( F \in H^{\ast} \), \( |F| \leq R \), and
\[ \nu_k (\xi ; R) \leq \lim \nu_k (\xi_k ; R) + \varepsilon. \]
But \( \varepsilon > 0 \) was an arbitrary number and so we proved the first of inequalities in question. Now, assume \( \varepsilon \leq \liminf \nu_n (\xi_n) \). Then there exists a subsequence \( \{ \xi_n \} \) of \( \{ \xi_n \} \) such that \( \nu_n (\xi_n) < \varepsilon \) for \( k = 1, 2, \ldots \). This implies that \( \nu_k (\xi_k ; 1/\varepsilon) < 1 \) for \( k = 1, 2, \ldots \). Application of the first inequality now yields \( \nu_k (\xi_k ; 1/\varepsilon) < 1 \), and \( \nu_n (\xi) < \varepsilon \). This accomplishes the proof.

6.1. Let us now consider linear functionals on \( H^{\ast} \). Similarly as for \( H^{\ast} \) we can define classes: \( (H^{\ast})^\# \) of norm continuous functionals on \( H^{\ast} \), \( (H^{\ast})^\# \) of modular continuous functionals and \( (H^{\ast})^\# \) of very weakly continuous functionals on \( H^{\ast} \) and others. In a similar fashion we define functionals \( \nu_n (\xi ; R), R > 0, \nu_n (\xi) \) with respect to \( H^{\ast} \). Let us observe here that all so far proved for linear functionals on \( H^{\ast} \) results with exception of 1.6, 2.3 and 4.5 hold also when \( H^{\ast} \) is replaced by \( H^{\ast} \). From
now on, to distinguish functionals relative to the space $H^*$ from those relative to the space $H^*$, the former will be marked with $c$, e.g. $v^c_\epsilon (\xi ; R)$ will denote $\sup \{ |\xi(F)| : F \in H^*, \|F\|_p \leq R \}$ for $\xi$ defined on $H^*$.

6.2. Let $\xi$ be a linear functional on $H^*$. Since $H^* \subset H^*$, then the range of $\xi$ can be restricted to $H^*$. Thus we can construct a linear functional $\xi^c$ on $H^*$ out of a linear one $\xi$ on $H^*$ in such a way that:

$$\xi^c(F) = \xi(F) \quad \text{for every } F \in H^*.$$ 

Clearly, for every $\xi \in (H^*)^*$, $\xi^c \in (H^*)^*$ and

$$v^c_\epsilon (\xi^c ; R) \leq v_\epsilon (\xi ; R) \quad \text{for every } R > 0$$

and then

$$v^c_\epsilon (\xi^c) \leq v_\epsilon (\xi).$$

6.3. For every functional $\xi \in (H^*)^*$ there exists a functional $\xi^c \in (H^*)^*$ such that

$$\xi^c(F) = \xi(F) \quad \text{for every } F \in H^*$$

and

$$v^c_\epsilon (\xi^c ; R) = v_\epsilon (\xi ; R) \quad \text{for every } R > 0.$$

Proof. If $\xi \in H^*$ then $\xi \in H^*$ for $0 \leq r < 1$. We define a nonnegative functional on $H^*$ by

$$p(F) = \sup \{ |\xi^c (T,F)| : 0 \leq r < 1 \}, \quad (F \in H^*).$$

Let $v^c_\epsilon (\xi^c ; R) \leq v_\epsilon (\xi ; R) \leq 0$ for $R > 0$. Then, by 2.1 and 3.6 of Section I for every $F \in H^*$, $\|F\|_p \leq R$ we have for every $0 \leq r < 1$

$$|\xi^c(T,F)| \leq R^{-1} v^c_\epsilon (\xi^c ; R) \|F\|_p \leq R^{-1} v_\epsilon (\xi ; R) \|F\|_p.$$ 

This implies that

$$p(F) \leq R^{-1} v^c_\epsilon (\xi^c ; R) \|F\|_p$$

for every $F \in H^*$, $\|F\|_p \leq R$. From this we conclude that $p(F) < \infty$ for every $F \in H^*$. This functional $p(\cdot)$ is, obviously, a homogeneous pseudonorm on $H^*$. By 3.6 of Section I for every $F \in H^*$, $\xi^c(T,F) \to \xi^c(F)$ for $r \to 1$ . From this we as well as definition of $p(\cdot)$ it follows that

$$|\xi^c(F)| \leq p(F) \quad \text{for every } F \in H^*.$$ 

By Hahn–Banach Theorem there exists a linear functional $\xi$ on $H^*$ such that

$$\xi(F) = \xi^c(F) \quad \text{for } F \in H^*$$

and

$$|\xi(F)| \leq p(F) \quad \text{for } F \in H^*.$$ 

The functional $\xi$ has the desired properties since for every $R > 0$ such that $v^c_\epsilon (\xi^c ; R) \leq \infty$ in view of $(\ast)$ we have

$$v^c_\epsilon (\xi^c ; R) \leq v_\epsilon (\xi^c ; R) \leq \sup (p(\cdot) : F \in H^*, \|F\|_p < R) \leq v^c_\epsilon (\xi^c ; R)$$

and for every $R > 0$ such that $v_\epsilon (\xi^c ; R) = \infty$, by the inequality $v^c_\epsilon (\xi^c ; R) \leq v_\epsilon (\xi^c ; R)$ also $v^c_\epsilon (\xi^c ; R) = \infty$.

6.4. For every $\xi \in (H^*)^*$ there exists a unique functional $\xi^c \in (H^*)^*$ such that

$$\xi(F) = \xi^c(F) \quad \text{for every } F \in H^*.$$ 

Moreover, for every $R > 0$,

$$v^c_\epsilon (\xi; R) = v_\epsilon (\xi; R).$$

Proof. In view of 6.3, for $\xi \in (H^*)^*$, there is a $\xi^c \in (H^*)^*$ such that

$$\xi(F) = \xi^c(F) \quad \text{for } F \in H^*$$

and $v^c_\epsilon (\xi^c ; R) = v_\epsilon (\xi^c ; R) \quad \text{for } R > 0$. By 2.4 $R^{-1} v^c_\epsilon (\xi^c ; E) \to 0$ as $E \to 0$ since $\xi^c \in (H^*)^*$. Thus also $R^{-1} v_\epsilon (\xi^c ; E) \to 0$ and by 2.4 $\xi^c \in (H^*)^*$. Now, let $\xi^c_1 \in (H^*)^*$ be another functional such that $\xi^c_1(F) = \xi^c(F)$ for every $F \in H^*$. Take an arbitrary $F \in H^*$. Since $T,F \in H^*$ for $0 \leq r < 1$, then $(\xi^c_1 - \xi)(F) = 0$. By 4.4 it follows that $(\xi^c_1 - \xi)(F) = 0$ and so $\xi_1 = \xi$.

6.5. For every $\xi \in (H^*)^*$ there exists a unique $\xi^c \in (H^*)^*$ such that

$$\xi(F) = \xi^c(F) \quad \text{for every } F \in H^*.$$ 

Moreover, for every $R > 0$,

$$v_\epsilon (\xi; R) = v^c_\epsilon (\xi; R).$$

Proof. By 6.4, in view of the inclusion $(H^*)^* \subset (H^*)^*$, for $\xi \in (H^*)^*$ there is a unique $\xi^c \in (H^*)^*$ such that $\xi(F) = \xi^c(F)$ for every $F \in H^*$. For this functional $v^c_\epsilon (\xi^c ; R) = v_\epsilon (\xi^c ; R)$ for every $R > 0$. We shall show that $\xi \in (H^*)^*$. Assume, on the contrary, that $\xi \in (H^*)^*$. Then there is an $a > 0$ and a sequence $\{F_n\} \subset H^*$ very weakly converging to 0 such that $|\xi(F_n)| \geq 2a$. Since $\xi \in (H^*)^*$ then by 4.4 $\lim_n |\xi(F_n)| = |\xi(F_n)|$ for $n = 1,2, \ldots$ It follows that for every $n$ there exists an $r_n$, $0 < r_n < 1$ such that

$$|\xi(T_n F_n) - \xi(F_n)| \leq \epsilon_n.$$

Thus for $n = 1,2, \ldots$

$$\epsilon_n \leq |\xi(F_n)| - |\xi(T_n F_n) - \xi(F_n)| \leq |\xi(T_n F_n)|.$$ 

Elements of $\{T_n F_n\}$ belong to $H^*$. Since $\{F_n\}$ very weakly converges to 0, by 3.6 of Section I

$$\sup_n \|T_n F_n\|_p \leq \sup_n \|F_n\|_p < \infty.$$
Applying now the Maximum Principle we get for any \( 0 \leq q < 1 \)
\[
\sup \{ \langle T_n F, a \rangle : |a| \leq q \} \leq \sup \{ \langle F, a \rangle : |a| \leq q \}.
\]
Thus \( \{ T_n F \} \) very weakly converges to \( 0 \). Now, since
\[
\lim_{n \to \infty} \langle T_n F, a \rangle = \langle T F, a \rangle = \ell(F),
\]
we get \( \ell \in (H^q_0)^\# \). This contradiction accomplishes the proof.

6.6. If \( \ell \in (H^q_0)^\# \), then for every \( F \in H^q \)
\[
\lim_{r \to \infty} \langle T_r F, a \rangle = \lim_{r \to \infty} \langle T_r F, a \rangle = \ell(F)
\]
and
\[
\lim_{r \to \infty} S_r \ell(F) = \lim_{r \to \infty} S_r \ell(F) = \ell(F).
\]
This is an immediate consequence of 3.6 of Section I.

6.7. A sequence of functionals \( \{ \ell_n \} \subset (H^q_0)^\# \) converges pointwise (on \( H^q_0 \)) if and only if \( \sup_{n \geq 1} \langle \ell_n, u \rangle < \infty \) and for \( m = 0, 1, 2, \ldots \) converges the sequence \( \{ \ell_n \cap U_m \} \) where \( U_m(\infty) = \infty \).

This theorem follows from 5.2 reformulated for \( H^q_0 \) and from the fact that polynomials form a dense subset of \( [H^q_0]^\# \).

7.1. If \( \ell \in (H^q_0)^\# \) then, as is known, \( H^q_0 = H^q \) and modular convergence and norm convergence are equivalent. Then we have
\[
(H^q_0)^\# = (H^q)^\# = (H^q)^{**} = (H^q_0)^{**}.
\]
Also
\[
(H^q_0)^\# = (H^q)^\# = (H^q)^{**} = (H^q_0)^{**}
\]
and
\[
(H^q_0)^\# = (H^q)^\# = (H^q)^{**} = (H^q_0)^{**}
\]
for every \( R > 0 \).

7.2. If \( \ell \in (H^q_0)^\# \) there exist non-trivial functionals \( \ell \in (H^q_0)^\# \) such that
\[
\ell(F) = 0
\]
for every \( F \in H^q_0 \).

Proof. The functional \( \ell(F) = 0 \) is by 3.5, of Section I a homogeneous pseudonorm satisfying conditions: \( \|F\| = 0 \) if and only if \( F \in H^q_0 \) and \( \|F\|_0 \leq \|F\|_0 \) for \( F \in H^q_0 \). If \( \ell \) does not satisfy \( \ell(F) = 0 \) then \( H^q_0 = H^q \). Take \( F_{\infty} \in H^q \setminus H^q_0 \) and put
\[
\ell(aF_{\infty}) = a(F_{\infty})_0
\]
for any number \( a \).

Further, by Hahn-Banach Theorem we get a linear functional \( \ell \) on \( H^q_0 \) such that \( \ell(F_{\infty}) = a(F_{\infty})_0 \) for any number \( a \) and such that \( |\ell(F)| \leq \|F\|_0 \) for every \( F \in H^q_0 \). Observe that \( \ell \) possesses the required properties.

7.3. If \( \ell \) does not satisfy \( \ell(F) = 0 \) then \( (H^q_0)^\# = (H^q_0)^{**} \).

Proof. By 7.2 there is then a non-trivial linear functional \( \ell \in (H^q_0)^\# \) such that \( \ell(F) = 0 \) for every \( F \in H^q_0 \). Suppose that \( (H^q_0)^\# = (H^q_0)^{**} \). Then \( \ell \in (H^q_0)^{**} \). By 4.5 we get now for every \( F \in H^q_0 \)
\[
\ell(F) = \lim_{r \to \infty} \langle T_r F, a \rangle = \lim_{r \to \infty} \langle T_r F, a \rangle = 0.
\]
This contradicts the assumption that \( \ell \) is non-trivial on \( H^q_0 \).

7.4. If \( \ell \in (H^q_0)^\# \) is a such functional that
\[
\ell(F) = 0
\]
for every \( F \in H^q_0 \) then \( \ell \in (H^q_0)^{**} \). More precisely there is a constant \( M \geq 0 \) such that
\[
|\ell(F)| \leq M \|F\|_0 \leq M \|F\|_0
\]
for every \( F \in H^q_0 \).

Proof. Since \( \ell \in (H^q_0)^\# \) then there exists a \( \delta > 0 \) such that for every \( F \in H^q_0 \), \( \|F\|_0 \leq \delta \) the condition \( |\ell(F)| \leq 1 \) holds. Now, let for \( F \in H^q_0 \) be \( \|F\|_0 \leq \frac{1}{\delta} \). Then, in view of 3.6 of Section I there is a \( \Theta \in H^q_0 \) such that \( \|\Theta - G\|_0 \leq \delta \). Thus \( |\ell(F)| = |\ell(\Theta - G)| \leq 1 \). So for every \( F \in H^q_0 \) such that \( \|F\|_0 \leq \frac{1}{\delta} \) is true that \( |\ell(F)| \leq 1 \). This yields that
\[
|\ell(F)| \leq \frac{1}{\delta} \|F\|_0 \leq \frac{1}{\delta} \|F\|_0
\]
for every \( F \in H^q_0 \).

7.5. \( (H^q_0)^\# \) will designate a class of all functionals \( \ell \in (H^q_0)^\# \) such that \( \ell(F) = 0 \) for every \( F \in H^q_0 \).

Clearly, if a sequence \( \{ \ell_n \} \subset (H^q_0)^\# \) pointwise converges on \( H^q \) to a \( \ell \in (H^q_0)^\# \) then \( \ell \in (H^q_0)^\# \). This, together with 7.4, implies that \( (H^q_0)^\# \) is a closed linear subspace of a Fréchet space \( (H^q_0)^\# \).

7.6. The space \( (H^q_0)^\# \) can be endowed with a homogeneous norm defined by
\[
\|F\|_0 = v_0(\ell(F) + 1) \sup \{ |\ell(F)| : F \in H^q_0, \|F\|_0 \leq \|F\|_0 \}, \|\ell(F)\|_0
\]
This norm is equivalent to \( (H^q_0)^\# \) with the norm \( \|F\|_0 \).

Proof. By 7.4 \( (H^q_0)^\# = (H^q_0)^\# \). From this we conclude that \( \|F\|_0 = v_0(\ell(F) + 1) \) is a homogeneous norm on \( (H^q_0)^\# \). Notice that \( (H^q_0)^\# \) is a closed linear subspace of a Banach space \( (H^q_0)^\# ; v_0(\ell(F) + 1) \). Thus, \( (H^q_0)^\# \) is a Fréchet space when equipped with any of the norms \( \|F\|_0 \) and \( \|\ell(F)\|_0 \). By 3.3, for a sequence \( \{ \ell_n \} \subset (H^q_0)^\# \), a convergence \( \|\ell_n\|_0 \to 0 \) implies con-
8.3. If \( \varphi \) satisfies condition (V), then \((H)^*\) = \((H^\prime)^\varphi\).

\textbf{Proof.} Let \( \xi \in (H^\prime)^\varphi \). By 2.2 there is then an \( R_0 > 0 \) such that
\[ v_\varphi(\xi; R_0) < \infty. \]
Let now \( R \) be any positive number. Since \( \varphi \) satisfies (V), then by 3.7 of Section I there is a \( c > 0 \) such that \( |aF|_\varphi \leq R \) for every \( F \in H^\varphi \), \( ||F||_\varphi \leq R \). From this follows that
\[ v_\varphi(\xi; R) = \sup \{|\xi(F)|: F \in H^\varphi, ||F||_\varphi \leq R\} \]
\[ = a^{-1} \sup \{|\xi(aF)|: F \in H^\varphi, ||F||_\varphi \leq R\} \]
\[ \leq a^{-1} \sup \{|\xi(F)|: F \in H^\varphi, ||F||_\varphi \leq R\} \]
\[ = a^{-1} v_\varphi(\xi; R_0) < \infty. \]
This implies that \( v_\varphi(\xi; R) < \infty \) for every \( R > 0 \) and, further, \( \xi \in (H^\prime)^\varphi \).
Thus \((H^\prime)^\varphi = (H^\prime)^\varphi\).

References