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# Smooth partitions of unity on some non-separable Banach spaces

by

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**Abstract.** It is shown that every Hilbert space, any space of the form $L_{2n}(X, \mu)$ and any space $c_0(A)$ admit $C^\infty$-partitions of unity. Moreover, all reflexive Banach spaces admit partitions of class $C^1$. The above results are obtained by verifying that a sufficient (and necessary) condition for a space to admit $C^k$-partitions of unity is satisfied in those cases; the condition is stated in Theorem 1.

Let $X$ be a metric space and let $S$ be a set of real functions on $X$. We say that $X$ admits $S$-partitions of unity if, given an open cover $\mathscr{U}$ of $X$, there is a locally finite partition of unity $(f_U)_{U\in\mathscr{U}}$ with $f_U|X-U=0$ and $f_U\in S$ for any $U\in\mathscr{U}$. It is of interest to know whether or not a given Banach space admits $C^k$-partitions of unity, $k=1,2,\ldots,\infty$. In the case of separable spaces Bonic and Frampton ([2], Th. 1), extending the method of Eells (cf. [7], pp. 28–30), proved:

(BF) *A separable Banach space $E$ admits $C^k$-partitions of unity iff there exists a non-constant function in $C^k(E)$ with bounded support.*

From this it follows that the separable Hilbert space and the spaces $c_0$, $l_{2n}$ and $L_{2n}(0,1)$, $n=1,2,\ldots$ admit $C^\infty$-partitions of unity (the $C^\infty$-partitions of unity on $l_2$ were first constructed by Eells, see [7]). Combining (BF) with the result of Kadec–Restrepo ([6] and [11]) one obtains also that any separable Banach space with a separable dual admits $C^1$-partitions of unity. For further discussion of smooth partitions of unity on separable Banach spaces we refer the reader to [2].

However, it is not known whether the statement (BF) remains true for non-separable Banach spaces, and only very recently Wells [13] showed that each Hilbert space admits partitions of class $C^1$. The aim of this paper is to prove that each of the spaces $c_0(A)$, $l_2(A)$, $L_{2n}(X,\mu)$ ($A$-an arbitrary set, $(X,\mu)$ — a positive measure space, $n\in N$) admits $C^\infty$-partitions of unity. We will also prove that any reflexive Banach space admits partitions of class $C^1$. Our approach is different from that of Wells and depends on the construction of some $\sigma$-locally finite base of open sets in $c_0(A)$.

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**1. Notation and lemmas.** We denote by $R$ the set of real numbers, by $N$ the set of positive integers, and we let $\bar{N} = N \cup \{\infty\}$. By $R^A$ we denote the product $\underset{a\in A}{\boldsymbol{P}} R_a$ ($R_a = R$ for each $a \in A$), with $p_a\colon R^A \to R_a$ the natural projections, $a \in A$. The points of $R^A$ will be denoted by $(x_a)$, where $x_\beta = p_\beta\big((x_a)\big)$. If $E$, $F$ are normed linear spaces and $U$ is an open subset of $E$, then $C^k(U, F)$ denotes the set of all mappings $f\colon U \to F$ having continuous $k$th *Fréchet* derivative; we also let $C^\infty(U, F) = \bigcap_{k\in N} C^k(U, F)$ and $C^k(U) = C^k(U, R)$. By "partition of unity" we always mean "locally finite partition of unity".

In what follows we shall consider pairs $(X, S)$ satisfying the following conditions

(i) *$X$ is a metric space and $S$ is a set of real-valued continuous functions on $X$.*

(ii) *If for some $f\colon X \to R$ there is an open cover $\mathcal{U}$ of $X$ and a set $\{g_U\colon U \in \mathcal{U}\} \subset S$ such that $g_U | U = f | U$ for all $U \in \mathcal{U}$, then $f \in S$.*

(iii) Given $n \in N$, $\varphi \in C^\infty(R^n)$ *and* $g_1, \ldots, g_n \in S$ *the composition* $\varphi \circ (g_1, \ldots, g_n)$ *belongs to* $S$.

For a pair $(X, S)$ satisfying (i) we shall denote by $\mathcal{U}_S$ the family $\{f^{-1}(0, \infty)\colon f \in S \text{ and image}(f) \subset [0,1]\}$. The following is easy to verify:

(I) *If a pair $(X, S)$ satisfies the conditions* (i)–(iii), *then the family $\mathcal{U}_S$ has the properties below:*

(iv) *For any $f \in S$ and $a \in R$ the sets $f^{-1}(a, \infty)$ and $f^{-1}(-\infty, a)$ are in $\mathcal{U}_S$.*

(v) *If $\mathcal{V} \subset \mathcal{U}_S$ is finite and $\mathcal{W} \subset \mathcal{U}_S$ is locally finite, then $\bigcap_{V\in\mathcal{V}} V$ and $\bigcup_{W\in\mathcal{W}} W$ are in $\mathcal{U}_S$.*

We shall use the following lemma of Bonic–Frampton (Th. 1 in [2], cf. also [7] p. 29); for the sake of completeness we include the proof here.

(II) *Let $(X, S)$ fulfil the conditions* (i), (iv) *and* (v). *Then for every family $\{U_n\}_{n\in N} \subset \mathcal{U}_S$ which is a cover of $X$ there exists a locally finite cover $\{V_n\}_{n\in N} \subset \mathcal{U}_S$ of $X$ such that $V_n \subset U_n$ for any $n$.*

Proof. Let $f_n \in S$ $(n = 1, 2, \ldots)$ be such nonnegative functions that $U_n = f_n^{-1}(0, \infty)$. We set $V_n = U_n \cap \bigcap_{i<n} f_i^{-1}(-\infty, 1/n)$. The family $\{V_n\}_{n\in N}$ is locally finite (for if $x$ is in $U_i$ then the set $W_x = f_i^{-1}(f_i(x)/2, \infty)$ is an open neighbourhood of $x$ intersecting only the $V_n$'s with $n \le \max\big(2/f_i(x), i\big)$) and we have $\bigcup_{n\in N} V_n = X$ (observe that $V_n \supset U_n \setminus \bigcup_{i<n} U_i$ for any $n \in N$).

Using the statement (II) we establish

LEMMA 1. *For a pair $(X, S)$ satisfying* (i)–(iii) *the following conditions are equivalent:*

(a) *$X$ admits $S$-partitions of unity.*





(a′) *For any closed set $A \subset X$ and an open neighbourhood $W$ of $A$ there is some $U \in \mathcal{U}_S$ with $A \subset U \subset W$.*

(b) *$\mathcal{U}_S$ contains a $\sigma$-locally finite base of the topology of $X$* [1]

Proof. The equivalence (a) $\Leftrightarrow$ (a′) and the implication (a) $\Rightarrow$ (b) can be proved in a standard way, using the paracompactness of $X$.

(b) $\Rightarrow$ (a′). Let $A$ and $W$ be as in (a′) and let $\mathcal{V} \subset \mathcal{U}_S$ be a base which can be expressed as $\mathcal{V} = \bigcup_{n\in N} \mathcal{V}_n$ with $\mathcal{V}_n$ locally finite for all $n$. We set $\mathcal{U}_n = \{V \in \mathcal{V}_n\colon V \subset W\}$, $\mathcal{W}_n = \{V \in \mathcal{V}_n\colon V \cap A = \emptyset\}$, $\tilde{U}_n = \bigcup_{V\in\mathcal{U}_n} V$ and $\tilde{W}_n = \bigcup_{V\in\mathcal{W}_n} V$; because of (v) the $\tilde{U}_n$'s and the $\tilde{W}_n$'s are in $\mathcal{U}_S$. Since $\mathcal{V}$ is a base of the topology of $X$, $\{\tilde{U}_n\}_{n\in N} \cup \{\tilde{W}_n\}_{n\in N}$ is a cover of $X$ and therefore, by (II), one can find $U_n \subset \tilde{U}_n$ and $W_n \subset \tilde{W}_n$ such that $\{U_n\}_{n\in N} \cup \{W_n\}_{n\in N}$ is a locally finite cover of $X$ and $U_n \in \mathcal{U}_S$ for $n \in N$. The set $U = \bigcup_{n\in N} U_n$ satisfies $A \subset U \subset W$ and $U \in \mathcal{U}_S$ (use (v) again).

Let us recall that, for a given set $A$, $c_0(A)$ is the linear space consisting of all $x = (x_a) \in R^A$ with $\{a \in A\colon |x_a| > 1/n\}$ finite for any $n \in N$; $c_0(A)$ is regarded as a Banach space under the norm $\|(x_a)\| = \sup\{|x_a|\colon a \in A\}$. Let $S_0$ be the set of all functions in $C^\infty\big(c_0(A)\big)$ which locally depend only on finitely many coordinates (i. e. $S_0$ consists of the functions $f\colon c_0(A) \to R$ such that, given any $y \in c_0(A)$, there are $n \in N$, $a_1, \ldots, a_n \in A$ and $\varphi \in C^\infty(R^n)$ with $f\big((x_a)\big) = \varphi(x_{a_1}, \ldots, x_{a_n})$ for all $(x_a)$ in some neighbourhood of $y$).

Our main lemma is:

LEMMA 2. *There is a $\sigma$-locally finite base $\mathcal{W}$ of the topology of $c_0(A)$ with $\mathcal{W} \subset \mathcal{U}_{S_0}$.*

Proof. It was Bonic and Frampton ([1], p. 395) who observed that for any $r > 0$ the ball $B_r = \{y \in c_0(A)\colon \|y\| < r\}$ belongs to $\mathcal{U}_{S_0}$. Their argument was as follows: if $\varphi_r \in C^\infty(R, [0,1])$ satisfies $\varphi_r(t) = 1$ for $t < r/2$ and $\varphi_r(t) = 0$ iff $t \geqslant r$, then letting $f_r\big((x_a)\big) = \prod_{a\in A} \varphi_r(x_a)$ one gets $f_r \in S_0$ with $B_r = f_r^{-1}(0, \infty)$.

For any $n \in N$ let us now denote by $H_n$ the set of all injections of $\{1, \ldots, n\}$ into $A$ and by $K_n$ the set $\big\{(a, r) \in Q^n \times Q\colon \inf\{|p_i(a)|\}_{i\le n} > r > 0\big\}$; $Q$ stands here for the set of rational numbers. For $h \in H_n$ we define the

[1] A family of subsets is said to be $\sigma$-locally finite if it is a union of a countable number of locally finite subfamilies. The equivalence (a) $\Leftrightarrow$ (b) can be viewed as a natural extension to non-separable spaces of the result (BF) cited in the introduction: for a separable space $X$, (b) is equivalent to the condition that $\mathcal{U}_S$ is a base of the topology of $X$, and if $X$ is a normed linear space and $\mathcal{U}_S$ is invariant under the homotheties and translations of $X$ then the last condition is satisfied iff $\mathcal{U}_S$ contains a non-empty bounded set.

linear operator $T_h\colon R^n \to c_0(A)$ by

$$p_a \circ T_h(a) = \begin{cases} 0 & \text{for } \alpha \notin \text{image } (h), \\ p_i(a) & \text{for } \alpha = h(i). \end{cases}$$

The desired family of sets is

$$\mathscr{W} = \{B_{1/n}\}_{n\in N} \cup \bigcup_{n\in N} \{T_h(a)+B_r\colon (a, r)\in K_n, h\in H_n\}.$$

Only the conditions (j) and (jj) below need to be proved

(j) *For any $x = (x_a)\in c_0(A)$ and $\varepsilon > 0$ there is some $W\in \mathscr{W}$ with $x\in W$ and* diam $W = \sup\{\|y-z\|\colon y, z\in W\} < 2\varepsilon$.

To establish this observe that since $\mathscr{W}$ contains $\{B_{1/n}\}_{n\in N}$ one can assume $(x_a) \neq 0$ and $\|(x_a)\| > \varepsilon$. Since the set $\{|x_a|\colon \alpha\in A\} \cup \{0\}$ is compact and countable there is a $r\in (0, \varepsilon)\setminus\{|x_a|\colon \alpha\in A\}$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the (non-empty) set $\{\alpha\in A\colon |x_a| > \varepsilon\}$, let $a = (a_1, \ldots, a_n)\in Q^n$ satisfy $|a_i| > r$ and $|a_i - x_{a_i}| < r$ for $1 \leqslant i \leqslant n$, and let $h\colon \{1, \ldots, n\} \to A$ be the map $h(i) = \alpha_i$ (we assume $\alpha_i \neq \alpha_j$ if $i \neq j$). Obviously $(a, r)\in K_n$ and $\|T_h(a) - x\| < r$; hence the set $W = T_h(a)+B_r$ satisfies $x\in W\in \mathscr{W}$ and diam $W = 2r < 2\varepsilon$.

(jj) *$\mathscr{W}$ is $\sigma$-locally finite.*

Since $\{B_{1/n}\}_{n\in N}$ and $\bigcup_{n\in N} K_n$ are countable, it is enough to show that, for every $n\in N$ and $(a, r)\in K_n$, the family $\{T_h(a)+B_r\colon h\in H_n\}$ is locally finite. Fix $n\in N$ and $(a, r)\in K_n$ and let $x = (x_a)$ be a point in $c_0(A)$. Let $\varepsilon = 2^{-1}\cdot\inf\{|p_i(a)| - r\}_{i\leqslant n}$ and let $\{\alpha_1, \ldots, \alpha_m\} = \{\alpha\in A\colon |x_a| > \varepsilon\}$. For any $y = (y_a)$ in $x+B_\varepsilon$ we have $|y_\beta| < 2\varepsilon$ if $\beta\notin \{\alpha_1, \ldots, \alpha_m\}$ while for any $z = (z_a)\in T_h(a)+B_r$ the inequality $|z_\gamma| > 2\varepsilon$ holds true for all $\gamma\in$ image$(h)$. Thus the ball $x+B_\varepsilon$ intersects only the sets $T_h(a)+B_r$ with image $(h) \subset \{\alpha_1, \ldots, \alpha_m\}$. The number of such sets is finite.

**2. Smooth partitions of unity.** Now we prove

THEOREM 1. *The following conditions are equivalent for a pair $(X, S)$ satisfying* (i), (ii), *and* (iii):

(a) *$X$ admits $S$-partitions of unity.*

(b) $\mathscr{U}_S$ contains *a $\sigma$-locally finite base of the topology of $X$.*

(c) *There is a set $A$ and a homeomorphic embedding $u\colon X \to c_0(A)$ with $p_a\circ u\in S$ for any $\alpha\in A$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) is asserted in Lemma 1, whereas the implication (c) $\Rightarrow$ (b) follows from Lemma 2: if $\mathscr{W}$ is as in the Lemma, then $\mathscr{V} = u^{-1}(\mathscr{W})$ is a $\sigma$-locally finite base of the topology of $X$ and $\mathscr{V}\subset \mathscr{U}_S$ (since $\{f\circ u\colon f\in S_0\} \subset S$, use (ii) and (iii)).

(b) $\Rightarrow$ (c). Let $\mathscr{V}\subset \mathscr{U}_S$ be a base of open sets in $X$ such that $\mathscr{V} = \bigcup_{n\in N} \mathscr{V}_n$ where the $\mathscr{V}_n$'s are locally finite and pair-wise disjoint. For each $V\in\mathscr{V}$ let $f_V\colon X\to [0,1]$ be a function in $S$ with $V = f_V^{-1}(0, \infty)$. We define $u\colon X\to c_0(\mathscr{V})$ by $p_V\circ u(x) = \frac{1}{n} f_V(x)$ if $V\in\mathscr{V}_n$. Then $u$ is continuous, because for any $n\in N$ and $x\in X$ the inequality $\|u(y)-u(x)\| \leqslant 2n^{-1}$ is valid for all $y$ in $W \cap \bigcap_{i\leqslant l} \{z\colon f_{V_i}(z) \in (f_{V_i}(x)-n^{-1}, f_{V_i}(x)+n^{-1})\}$; $W$ is here an open set for which $\{V\in \mathscr{V}_1 \cup \ldots \cup \mathscr{V}_n\colon V\cap W \neq \emptyset\} = \{V_1, \ldots, V_l\}$ is finite. Moreover, we have $\|u(x)-u(y)\| \geqslant n^{-1} f_V(x)$ when $x\in V\in\mathscr{V}_n$ and $y\notin V$, and therefore $u\colon X\to u(X)$ is a homeomorphism.

COROLLARY 1. *Let $E$, $F$ be normed linear spaces with $F$ admitting $C^k$-partitions of unity ($k\in \overline{N}$). If there exists some $u\in C^k(E, F)$ which is a homeomorphic embedding, then $E$ admits $C^k$-partitions of unity. In particular, every (not necessarily closed) linear subspace of $F$ admits $C^k$-partitions of unity.*

Proof. Apply the equivalence (a) $\Leftrightarrow$ (c) to $(E, C^k(E))$ and $(F, C^k(F))$.

COROLLARY 2. *If the normed linear spaces $E_1$ and $E_2$ admit $C^k$-partitions of unity ($k\in\overline{N}$), then so does $E_1\oplus E_2$.*

Proof. Let $u_1\colon E_1\to c_0(A_1)$ and $u_2\colon E_2\to c_0(A_2)$ be homeomorphic embeddings such that $A_1\cap A_2 = \emptyset$ and $p_a\circ u_i\in C^k(E_i)$ for all $\alpha\in A_i$, $i = 1, 2$. If $L\colon c_0(A_1)\times c_0(A_2)\to c_0(A_1\cup A_2)$ is the natural isomorphism then $u\colon E_1\oplus E_2\to c_0(A_1\cup A_2)$ defined by $u(e_1, e_2) = L(u_1(e_1), u_2(e_2))$ satisfies (c) for $(E_1\oplus E_2, C^k(E_1\oplus E_2))$.

Using Theorem 1 with $X = c_0(A)$ and $u =$ the identity, we obtain

THEOREM 2. *The Banach space $c_0(A)$ admits $C^\infty$-partitions of unity ($A$ — an arbitrary set).*

From Theorem 1 we also have

THEOREM 3. *Every Hilbert space $H$ admits $C^\infty$-partitions of unity.*

Proof. We may assume that $H = l_2(A)$ where $1\notin A$. It is well known (see [5], p. 14) that the map $u\colon l_2(A)\to c_0(1\cup A)$ defined by

$$(*)\qquad p_\beta\circ u((x_a)) = \begin{cases} \|(x_a)\|^2 & \text{for } \beta = 1 \\ x_\beta & \text{for } \beta\in A \end{cases}$$

is a homeomorphic embedding. Thus the result follows from Theorem 1.

REMARK 1. *In fact, $c_0(A)$ admits $S_0$-partitions of unity ($S_0$ is as in Lemma 2) and $l_2(A)$ admits $S$-partitions of unity, where $S$ is the set of all functions on $l_2(A)$ which locally are of the form $(x_a)\to \varphi(\|(x_a)\|^2, x_{a_1}, \ldots, x_{a_n})$; $n\in N$, $\alpha_1, \ldots, \alpha_n\in A$, $\varphi\in C^\infty(R^{n+1})$.*

Proof. Apply the equivalence (a) $\Leftrightarrow$ (c) to $(c_0(A), S_0)$ and $(l_2(A), S)$ respectively ($u$ is the identity or is defined by (*)).

Now we shall study smooth partitions of unity on reflexive Banach spaces; our arguments will use some advanced theorems due to Lindenstrauss and Troyanski. First let us observe the following:

(III) *Let $E$ be a reflexive Banach space whose norm $|||\ |||$ is locally uniformly convex*(2) *and let $L: E \to c_0(A)$ be a continuous linear operator with* $\ker(L) = \{0\}$ *and $A \cap N = \emptyset$. If $\varphi_n \in C^\infty(R)$ $(n = 1, 2, \ldots)$ are nondecreasing functions with $\varphi_n(t) = t$ for $t > 1/n$ and $\varphi_n(t) = 0$ for $t < 1/2n$, then the map $u: E \to c_0(N \cup A)$ defined by*

$$p_\beta \circ u(x) = \begin{cases} \varphi_\beta(|||x|||)/\beta & \text{for } \beta \in N, \\ p_\beta \circ L(x) & \text{for } \beta \in A, \end{cases}$$

*is a homeomorphic embedding.*

Proof. It is easy to see that the set $\{p_a \circ L:\ a \in A\}$ is linearly dense in $E$. Otherwise there would exist $0 \neq x \in E^{**} \subset E$ such that $\langle x, p_a \circ L\rangle = 0$ for any $a \in A$; this is impossible because $\ker(L) = \{0\}$.

Now let $(x_n)$ be a sequence of points of $E$ such that $\lim_{n\to\infty} |||u(x_n) - u(x_1)||| = 0$. We then have $\lim_{n\to\infty} |||x_n||| = |||x_1|||$ and $\lim_{n\to\infty} (p_a \circ L)(x_n) = (p_a \circ L)(x_1)$ for all $a \in A$, whence $(x_n)$ is weakly convergent to $x_1$. Since the norm $|||\ |||$ is locally uniformly convex we get(2) $\lim_{n\to\infty} |||x_n - x_1||| = 0$.

Therefore the map $u: E \to u(E)$ is closed; the continuity of $u$ is obvious.

Recall that a norm $\|\ \|$ on a linear space $E$ is said to be of class $C^k$, if the function $x \to \|x\|$, $x \neq 0$, is in $C^k(E \setminus \{0\})$. By Lindenstrauss [6] for every reflexive Banach space $E$ there exists a continuous linear operator $L: E \to c_0(A)$ with $\ker(L) = \{0\}$. Thus from (III) and Theorem 1 we obtain

Theorem 4. *Let $E$ be a reflexive Banach space. If there exists on $E$ an (equivalent) locally uniformly convex norm of class $C^k$, then $E$ admits $C^k$-partitions of unity ($k \in \overline{N}$).*

The following corollary generalizes Theorem 3:

Corollary 3. *Let $(X, \mu)$ be a positive measure space. Then for any $n \in N$ the Banach space $L_{2n}(X, \mu)$ admits $C^\infty$-partitions of unity, and for any $p \in R$ and $k \in N$ with $k < p$ the space $L_p(X, \mu)$ admits $C^k$-partitions of unity.*

(2) i. e. for any sequence $(y_n)$ of points of $E$ with $|||y_n||| = 1$, $n \in N$, the convergence $\lim_{n\to\infty} |||y_n + y_1||| = 2$ implies $\lim_{n\to\infty} |||y_n - y_1||| = 0$ ([9], p. 226). It is known and easy to verify that if the norm $|||\ |||$ on $E$ is locally uniformly convex, then for any sequence $(x_n)$ in $E$ with $\lim_{n\to\infty} |||x_n||| = |||x_1|||$ we have $\lim_{n\to\infty} |||x_n - x_1||| = 0$ iff $(x_n)$ is weakly convergent to $x_1$ (cf. the proof of Theorem 2.2 in [9]).



Proof. For any $p > 1$ the space $L_p(X, \mu)$ is reflexive and its standard norm is locally uniformly convex (Clarkson [3]). Moreover it can easily be shown (cf. [2], p. 887) that the (standard) norm on $L_p(X, \mu)$ is of class $C^k$ for all $k \in N$ with $k < p$ and is of class $C^\infty$ when $p$ is an even positive integer. Thus the assertion follows from Theorem 4. (3)

By Troyanski ([12], proof of Corollary 6) every reflexive space admits an equivalent locally uniformly convex norm of class $C^1$. Combining this with Theorem 4 we get

Theorem 5. *Every reflexive Banach space admits $C^1$-partitions of unity.*

Let us recall that if a normed linear space $E$ admits $C^k$-partitions of unity then every paracompact $C^k$-manifold $M$ modelled on $E$ also admits such partitions (this can be established e. g. by checking that if the topology of $E$ has a $\sigma$-locally finite base $\mathscr{V} \subset \mathscr{U}_{C^k(E)}$ then the topology of $M$ has some $\sigma$-locally finite base $\mathscr{W} \subset \mathscr{U}_{C^k(M)}$). Consequently any paracompact Hilbertian manifold of class $C^k$ admits $C^k$-partitions of unity and any (paracompact) diferentiable manifold modelled on a reflexive space admits $C^1$-partitions of unity.

**3. Approximation of continuous functions.** Let $(E, |||\ |||)$ be a normed linear space. For any linear operator $L: E \to c_0(A)$ and any normed linear space $F$ we define $S_L(E, F)$ as the set of functions $u: E \to F$ which are locally of the form

$$x \to \varphi(|||x|||, p_{a_1} \circ L(x), \ldots, p_{a_n} \circ L(x)),$$

where $n \in N$, $a_1, \ldots, a_n \in A$, and $\varphi \in C^\infty(R^{n+1}, F)$ satisfies $\frac{\partial \varphi}{\partial x_0}(x_0, \ldots, x_n) = 0$ for all $(x_0, \ldots, x_n) \in R^{n+1}$ with sufficiently small $x_0$.

Proposition 1. *Let $E$ be a reflexive Banach space whose norm $|||\ |||$ is locally uniformly convex and let $L: E \to c_0(A)$ be a linear operator with* $\ker(L) = \{0\}$. *Then for every normed linear space $(F, \|\ \|)$ and for any continuous mappings $u: E \to F$ and $\varepsilon: E \to (0, 1)$ there is a $v \in S_L(E, F)$* with $\|u(x) - v(x)\| < \varepsilon(x)$ *for all $x \in E$.*

Proof. From (III) and Theorem 1 we infer that $E$ admits $S_L(E, R)$-partitions of unity. Thus it remains to apply the following general fact, which results from the method of C. Kuratowski (see [0], pp. 366-367):

(K) *If the space $X$ admits $S$-partitions of unity, then for any normed linear space $(F, \|\ \|)$ and any continuous maps $u: X \to F$, $\varepsilon: X \to (0, 1)$*

(3) If $L_p(X, \mu)$ is infinite-dimensional, then it contains a subspace isomorphic to $l_p$. Therefore, by a result of Kurzweil (Studia Math. 14, p. 227), the space $L_p(X, \mu)$ admits $C^k$-partitions of unity for some $k \geq p$ only if it is finite-dimensional or $p$ is an even integer. Let us note also that Proposition 5 of [2], Theorem 1 and (III) combine to show that $L_p(X, \mu)$ admits partitions of class $A^p$ in the terminology of [2].

*there is a $S$-partition of unity $(f_i)_{i\in I}$ and a family $(w_i)_{i\in I}$ of points of $F$ such that*

$$\Big\| u(x) - \sum_{i\in I} f_i(x)\cdot w_i \Big\| < \varepsilon(x) \quad \text{for all } x \in X.$$

In the special case $E = l_p(A)$, $p > 1$, taking for $|||\ |||$ the standard norm and for $L$ the formal identity operator, we conclude that each continuous $u\colon l_p(A) \to F$ can be arbitrarily close approximated by functions which locally are of the form $(x_a) \to \varphi(|||(x_a)|||^p, x_{a_1}, \ldots, x_{a_n})$ $(n \in N, a_1, \ldots, a_n \in A, \varphi \in C^\infty(R^{n+1}))$.

Similarly it follows from Remark 2 that any continuous $u\colon c_0(A) \to F$ can be approximated by such functions $g \in C^\infty(c_0(A), F)$ which locally depend only on finitely many coordinates.

By this method one can also prove the following fact

PROPOSITION 2. *Let $X = \mathop{\boldsymbol{P}}_{n\in N} X_n$, where $X_n$ are metric spaces, and for any metric space $Y$ denote by $S(X, Y)$ the set of those continuous functions $v\colon X \to Y$ which locally depend only on finitely many coordinates. If $(Y, \varrho)$ is a metric* ANR, *then for any two continuous maps $u\colon X \to Y$ and $\varepsilon\colon X \to (0, 1)$ there is a $v \in S(X, Y)$ with $\varrho(u(x), v(x)) < \varepsilon(x)$ for all $x \in X$.*

Proof. Let us begin with the case where $Y = F$ is a normed linear space. By (K) it is then enough to show that $X$ admits $S(X, R)$-partitions of unity. To this aim observe that each of the spaces $X_i$ may be embedded into the unit ball of some $c_0(A_i)$ (use Theorem 1 and the paracompactness of metric spaces). Let $h_i\colon X_i \to c_0(A_i)$ be the embeddings, $i \in N$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and let $L\colon \mathop{\boldsymbol{P}}_{i\in N} c_0(A_i) \to c_0(\bigcup_{i\in N} A_i)$ be the map such that $p_a \circ L((x_i)) = n^{-1}\cdot p_a(x_n)$ for any $(n, a)$ with $a \in A_n$. It is easy to see that $L$ is a homeomorphic embedding and that $w\colon X \to c_0(\bigcup_{i\in N} A_i)$ defined by $w((x_i)) = L((w_i(x_i)))$ satisfies the condition (c) for the pair $(X, S(X, R))$. Therefore the assertion follows from the equivalence (a) $\Leftrightarrow$ (c).

The general case can now be proved in the standard way, using a closed isometric embedding of $Y$ into a normed linear space $F$ (cf. [10]) and a retraction onto $Y$ of some open set $U \subset F$.

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