continuous with respect to $\sigma(A'', A')$ and $\sigma(B'', B')$, Lemma 2.1 implies that T maps the identity of A'' onto that of B''. If T is bipositive (resp. isometric), so is T^{**}, as was noted above. Thus Theorem 1.1 may be applied to show that (iv) combined with either (ii) or (iii) implies (i).

Note. As Kadison observes in [5], p. 502, his generalized Schwarz inequality may be used to show independently of the corresponding result for C^*-algebras with identity that in the above theorem (ii) and (iii) together imply (i).

For any C^*-algebra A, let H_A denote the real Banach space of the self-adjoint elements of A.

THEOREM 3.2. *Let A and B be C^*-algebras and $T: A \to B$ a vector space isomorphism. If T maps H_A isometrically onto H_B, then T is isometric.*

Proof. By Lemma 2.4 T is bounded, so we have the bounded maps $T^*: B' \to A'$ and $T^{**}: A'' \to B''$. The real Banach space $H_{A'}$ of the continuous Hermitian linear forms on A may be identified with the Banach space dual of H_A (see [1], p. 5). Similarly, $(H_{A'})'$ identifies with $H_{A''}$. This follows form Corollary 12.1.3 (iii) in [1] and the fact that for any two vectors ξ and η in the Hilbert space underlying A'' the linear form $x \mapsto (x\xi, \eta)$ belongs to the predual of A''. The argument used in [1] 1.2.6, p. 5 may be adapted to show that this identification preserves norms. Similar statements hold for B. We have $\|T \mid H_A\| = \|T^* \mid H_{B'}\| = \|T^{**}|H_{A''}\|$, and applying this result also to T^{-1} we see that T^{**} is isometric on $H_{A''}$. Theorem 2 in [5] combined with Theorem 5 in [4] then shows that T^{**}, hence T, is everywhere isometric.

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Some more Banach spaces which contain l^1

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Abstract. Let X^* be a conjugate Banach space containing a subspace isomorphic to $L^1(\mu)$. Sufficient conditions on the measure μ are given which insure that X contains a subspace isomorphic to l^1.

Introduction. The purpose of this paper is the extension of the results of Pełczyński [11] concerning the embedding of $L^1(\mu)$ spaces into conjugate Banach spaces. The main result is the following:

THEOREM 1. *Let X be a Banach space. Assume that either*

(I) *X^* contains a (closed) subspace isomorphic to $L^1(\mu)$ where μ is a non purely atomic measure; or*

(II) *X^* contains a (closed) subspace isomorphic to $l^1(\Gamma)$ and the dimension of X is less than the cardinality of Γ.*

Then X contains a subspace isomorphic to l^1.

It is an immediate consequence of this theorem and results of Rosenthal [13] that if X is a separable Banach space with X^* non-separable and X is either an $\mathscr{L}_\infty$ space or a quotient space of $C[0, 1]$, then X contains a subspace isomorphic to l^1. (For the definition and properties of $\mathscr{L}_p$ spaces, see [9] and [10].) It also follows from Theorem 1 and results in [11] that if X is separable and X^* satisfies either (I) or (II) of Theorem 1, then $C[0, 1]$ is isomorphic to a quotient space of X.

The proof of Theorem 1 involves a modification of methods introduced by Pełczyński in [11] (except in (II) in the case where X is not separable). Pełczyński proved Theorem 1 under the added assumptions that the subspace of X^* isomorphic to $L^1(\mu)$ or $l^1(\Gamma)$ is a "seminorming" subspace of X^*, and, in case (II), that X is separable. (For the definition of seminorming, see [11], p. 232.) Delbaen [2] idependently proved Theorem 1 (I) and 1 (II) in the case where X is separable (using essentially the same idea as in Proposition 2 and the remark which follows it). Johnson and Rosenthal [6] have recently given a different proof of Theorem 1 (I) using weak-* basic sequences.

The author wishes to express his appreciation to Professor Rosenthal for suggesting this problem and for many helpful conversations concerning it.

Preliminaries. All Banach spaces will be real Banach spaces, and will be denoted by B, X, and Z. B will be said to be a subspace of X if B is a closed linear submanifold in X. S_X denotes the unit ball of X, i. e., $S_X = \{x \in X\colon \|x\| \leqslant 1\}$. We will refer to a bounded linear operator $T\colon X \to B$ as an operator. An operator $T\colon X \to B$ is an isomorphism if it is one-one with closed range. If T is an isomorphism from X onto B, then X and B are said to be isomorphic. A sequence (x_n) in X is a basic sequence if given any x in the closed linear span of the x_n's, there exists a unique sequence of scalars (a_n) such that $x = \sum_{n=1}^{\infty} a_n x_n$. If (x_n) and (z_n) are basic sequences in Banach spaces X and Z, then (x_n) and (z_n) are equivalent if given a sequence (a_n) of scalars, then $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n z_n$ converges If (x_n) and (z_n) are equivalent, then it follows from the closed graph theorem that their closed linear spans are isomorphic.

If X is a Banach space, then the conjugate or dual space of X is denoted by X^*. If $T\colon X \to Y$ is an operator, then $T^*\colon Y^* \to X^*$ denotes the adjoint operator to T. By the weak-* topology on X^* we mean the X topology on X^*. (See for example [3], p. 420.) If Y is a subset of X^*, then $\mathrm{cl}^*(Y)$ denotes the weak-* closure of Y in X^*.

A subset Y of X^* is said to be *norming* if there exist $\delta, K > 0$ such that $\delta\|x\| \leqslant \sup\{|y(x)|\colon y \in Y\} \leqslant K\|x\|$ for all $x \in X$, i. e., if $\|x\|' = \sup\{|y(x)|\colon y \in Y\}$ defines a norm equivalent to the usual norm on X.

Remark. It follows easily from the Hahn–Banach Theorem that if Y is a bounded convex subset of X^*, then Y is norming if and only if there exists $\delta > 0$ such that $\mathrm{cl}^*(Y) \supset \delta S_{X^*}$.

Let Γ be a set. Then $l^1(\Gamma)$ ($l^\infty(\Gamma)$ respectively) is the Banach space of real valued functions $f\colon \Gamma \to \boldsymbol{R}$ such that

$$\|f\|_1 = \sum\{|f(\gamma)|\colon \gamma \in \Gamma\} < \infty \ (\|f\|_\infty = \sup\{|f(\gamma)|\colon \gamma \in \Gamma\} < \infty \text{ respectively}).$$

If $\mathrm{card}(\Gamma) = \aleph_0$ ($\mathrm{card}(\Gamma)$ denotes the cardinality of Γ), then we write $l^1 = l^1(\Gamma)$ and $l^\infty = l_\infty(\Gamma)$. The usual basis for l^1 is the basis (e_n) where

$$e_n(m) = \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases} \quad \text{for all integers } m.$$

If Ω is a set, Σ a σ-algebra of subsets of Ω and μ a (positive) measure, then by $L^1(\mu) = L^1(\Omega, \Sigma, \mu)$ we mean the Banach space of equivalence classes of μ-measurable functions on Ω such that $\|f\|_1 = \int |f|\,d\mu < \infty$. If $\Omega = [0, 1]$, Σ the σ-algebra of Borel subsets of $[0, 1]$, and μ Lebesgue measure, we denote $L^1(\mu)$ by L^1.

If X is a Banach space, the dimension of X, denoted $\dim(X)$, is the least cardinal number $\mathfrak{m}$ such that there exists a set $\mathscr{A}$ in X of cardinality $\mathfrak{m}$ whose closed linear span is all of X. Similarly, the density character of a topological space K is the least cardinal number $\mathfrak{m}$ such that there exists a dense set in K of cardinality $\mathfrak{m}$.

Results. The first proposition and the remark which follows give a sufficient condition for a Banach space X to contain a subspace isomorphic to l^1. In what follows, let $\chi\colon X \to X^{**}$ denote the natural embedding of X into its second conjugate X^{**}.

PROPOSITION 2. *Let $T\colon Z \to X^*$ be an isomorphism. Assume that $Y = T^*(\chi S_X)$ contains a basic sequence equivalent to the usual basis in l^1. Then X contains a subspace isomorphic to l^1.*

Proof. Let (y_n) be a basic sequence in Y equivalent to the usual basis in l^1. For each n, pick $x_n \in S_X$ such that $T^*(\chi x_n) = y_n$. It is easily verified that (x_n) is a basic sequence in X equivalent to the usual basis in l^1.

Remark. In Proposition 2, the set Y is a bounded convex subset of Z^* such that $\mathrm{cl}^*(Y) \supset (1/\|T^{-1}\|)S_{Z^*}$. (For

$$\begin{aligned} \|z\| \leqslant \|T^{-1}\|\,\|Tz\| &= \|T^{-1}\| \sup\{|\chi(x)(Tz)|\colon x \in S_X\} \\ &= \|T^{-1}\| \sup\{|T^*(\chi(x))(z)|\colon x \in S_X\} \\ &= \|T^{-1}\| \sup\{|y(z)|\colon y \in Y\} \end{aligned}$$

and the fact that $\mathrm{cl}^*(Y) \supset (1/\|T^{-1}\|)S_{Z^*}$ now follows from the remark on norming sets in the preliminaries.)

The next lemma provides the principal means in this paper of showing that certain Banach spaces contain a subspace isomorphic to l^1. In it, we examine certain bounded norming convex subsets of Z^* for the spaces $Z = L^1(\mu)$ (μ non purely atomic) and $Z = l^1(\Gamma)$.

LEMMA 3. (i) *Let μ be a non purely atomic measure. If Y is a bounded convex norming subset of $[L^1(\mu)]^*$, then Y contains a basic sequence (y_n) equivalent to the usual basis in l^1.*

(ii) *Let Γ be an infinite set and Y a bounded convex norming subset of $l^\infty(\Gamma)$ with the norm density character of $Y < \mathrm{card}(\Gamma)$. Then Y contains a basic sequence (y_n) equivalent to the usual basis in l^1.*

The proof of (i) is identical to those of Propositions 2.2 and 2.3 of [11] together with the following observations: In the proofs of these propositions, Y is a closed subspace rather than a bounded convex set. However, the assumption that Y be closed is used there only to conclude that Y contains a subspace isomorphic to l^1, rather than elements (y_n) equivalent to the usual basis in l^1. Moreover, a linear manifold Y norms $[L^1(\mu)]^*$ if for some $\delta > 0$, $\mathrm{cl}^*(S_Y) \supset \delta S_{[L^1(\mu)]^*}$. The set S_Y is what we have called "Y".

The proof of (ii) is an immediate consequence of the next technical lemma, which generalizes Proposition 2.4 of [11]. In what follows, let $\mathfrak{m}$

denote an infinite cardinal number and $\mathfrak{m}^+$ the successor cardinal to $\mathfrak{m}$. We use the fact that $\mathfrak{m}^+$ is not the sum of $\leqslant \mathfrak{m}$ cardinals each $\leqslant \mathfrak{m}$.

LEMMA 4. *Let Γ be a set of cardinality $\geqslant \mathfrak{m}^+$, and Y a bounded convex subset of $l^\infty(\Gamma)$ of norm density character $\leqslant \mathfrak{m}$ such that* $\mathrm{cl}^*(Y) \supset S_{l^\infty(\Gamma)}$.

Let $\Gamma_1, \ldots, \Gamma_n, \Gamma_{n+1}, \ldots, \Gamma_{n+m}$ be pairwise disjoint subsets of Γ with $\mathrm{card}(\Gamma_i) \geqslant \mathfrak{m}^+$ *for all* $1 \leqslant i \leqslant n+m$.

Then there exists $y \in Y$ such that

$$\mathrm{card}\{\gamma \in \Gamma_i\colon\ y(\gamma) \geqslant 1/2\} \geqslant \mathfrak{m}^+ \qquad \text{for } 1 \leqslant i \leqslant n$$

and

$$\mathrm{card}\{\gamma \in \Gamma_i\colon\ y(\gamma) \leqslant -1/2\} \geqslant \mathfrak{m}^+ \qquad \text{for } n+1 \leqslant i \leqslant n+m.$$

Proof. Let $\mathscr{A}$ be a set of cardinality $\leqslant \mathfrak{m}$ and $\{y_\alpha\colon \alpha \in \mathscr{A}\}$ a dense set in Y. Put $\Delta = \Gamma_1 \times \ldots \times \Gamma_n \times \Gamma_{n+1} \times \ldots \times \Gamma_{n+m}$. We claim that for each $(\gamma_1, \ldots, \gamma_{n+m}) \in \Delta$, there exists an $\alpha \in \mathscr{A}$ such that

$$(*) \qquad \begin{aligned} &y_\alpha(\gamma_i) \geqslant 1/2 && \text{for } 1 \leqslant i \leqslant n \quad \text{and} \\ &y_\alpha(\gamma_i) \leqslant -1/2 && \text{for } n+1 \leqslant i \leqslant n+m. \end{aligned}$$

To prove the claim, define $r \in l^\infty(\Gamma)$ by

$$r(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \bigcup_{i=1}^{n} \Gamma_i, \\ -1 & \text{if } \gamma \in \bigcup_{i=n+1}^{n+m} \Gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

Also define (for $1 \leqslant i \leqslant n+m$) $f_i \in l^1(\Gamma)$ by

$$f_i(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then since $\mathrm{cl}^*(Y) \supset S_{l^\infty(\Gamma)}$ and $\{y_\alpha\colon \alpha \in \mathscr{A}\}$ is norm dense in Y, there exists an $\alpha \in \mathscr{A}$ such that

$$(1) \qquad \Big|\sum_{\gamma\in\Gamma} r(\gamma) f_i(\gamma) - \sum_{\gamma\in\Gamma} y_\alpha(\gamma) f_i(\gamma)\Big| < 1/2 \qquad \text{for } 1 \leqslant i \leqslant n+m.$$

Computing the left hand side of (1), we see that

$$(2) \qquad \begin{aligned} &|1 - y_\alpha(\gamma_i)| < 1/2 && \text{for } 1 \leqslant i \leqslant n \quad \text{and} \\ &|-1 - y_\alpha(\gamma_i)| < 1/2 && \text{for } n+1 \leqslant i \leqslant n+m. \end{aligned}$$

The inequalities (*) are immediate from (2), proving the claim.

Now let Γ' be a set of cardinality $\mathfrak{m}^+$, and for each i, $1 \leqslant i \leqslant n+m$, let $\varphi_i\colon \Gamma' \to \Gamma_i$ be an injection. Then the subset $\Delta' \subset \Delta$ defined by $\Delta' = \{(\varphi_1(\gamma'), \ldots, \varphi_{n+m}(\gamma'))\colon \gamma' \in \Gamma'\}$ is a set of cardinality $\mathfrak{m}^+$ with the property that if $(\gamma_1, \ldots, \gamma_{n+m}), (\delta_1, \ldots, \delta_{n+m}) \in \Delta'$ and $\gamma_i = \delta_i$ for some i, then $\gamma_i = \delta_i$ for all i.

It follows from the claim that for each $\delta = (\gamma_1, \ldots, \gamma_{n+m}) \in \Delta'$, we can choose an $\alpha = \alpha(\delta) \in \mathscr{A}$ such that y_α satisfies (*) for δ. Thus the function $\psi\colon \Delta' \to \mathscr{A}$ defined by $\psi(\delta) = \alpha(\delta)$ is well defined, and so, $\Delta' = \bigcup\{\psi^{-1}(\alpha)\colon \alpha \in \psi(\Delta')\}$. Therefore, since $\psi^{-1}(\alpha) \cap \psi^{-1}(\beta) = \emptyset$ if $\alpha \neq \beta$, $\mathfrak{m}^+ = \mathrm{card}(\Delta') = \Sigma\{\mathrm{card}(\psi^{-1}(\alpha))\colon \alpha \in \psi(\Delta')\}$. But now, since $\mathrm{card}(\psi(\Delta')) \leqslant \mathrm{card}(\mathscr{A}) \leqslant \mathfrak{m}$, it follows from the remark preceding this lemma that for some $\alpha \in \mathscr{A}$, $\mathrm{card}(\psi^{-1}(\alpha)) = \mathfrak{m}^+$.

We claim that this α satisfies the conclusion of lemma. To see this, let $\Pi_i\colon \Delta \to \Gamma_i$ be the projection of Δ onto Γ_i. Then, by the definition of Δ', $\mathrm{card}(\Pi_i(\psi^{-1}(\alpha))) = \mathfrak{m}^+$ for all $1 \leqslant i \leqslant n+m$. Moreover, $y_\alpha(\gamma) \geqslant 1/2$ for all $\gamma \in \Pi_i(\psi^{-1}(\alpha))$, $1 \leqslant i \leqslant n$, and $y_\alpha(\gamma) \leqslant -1/2$ for all $\gamma \in \Pi_i(\psi^{-1}(\alpha))$, $n+1 \leqslant i \leqslant n+m$. This completes the proof of Lemma 4.

Proof of Lemma 3 (ii). By considering if necessary the set rY instead of the set Y (for some sufficiently large $r > 0$), we may assume $\mathrm{cl}^*(Y) \supset S_{l^\infty(\Gamma)}$.

By Proposition 2.2 of [11], it suffices to show the following:

(**) Given an integer m and a finite collection $\Gamma_0, \ldots, \Gamma_{2^m-1}$ of pairwise disjoint subsets of Γ each of cardinality $\geqslant \mathfrak{m}^+$, there exists $y \in Y$ such that $\mathrm{card}\{\gamma \in \Gamma_i\colon (-1)^i y(\gamma) \geqslant 1/2\} \geqslant \mathfrak{m}^+$ for $0 \leqslant i \leqslant 2^m - 1$.

But defining $\Gamma_i' = \Gamma_{2i}$ for $0 \leqslant i \leqslant 2^{m-1}-1$ and $\Gamma'_{2^{m-1}+i} = \Gamma_{2i+1}$ for $0 \leqslant i \leqslant 2^{m-1}-1$, we can select (by Lemma 4) a $y \in Y$ such that (**′) $\mathrm{card}\{\gamma \in \Gamma_i'\colon y(\gamma) \geqslant 1/2\} \geqslant \mathfrak{m}^+$ for $0 \leqslant i \leqslant 2^{m-1}-1$ and $\mathrm{card}\{\gamma \in \Gamma_i'\colon y(\gamma) \leqslant -1/2\} \geqslant \mathfrak{m}^+$ for $2^{m-1} \leqslant i \leqslant 2^m - 1$.

This is equivalent to (**), and completes the proof.

Theorem 1 now follows immediately from Proposition 2 and Lemma 3. For if $T\colon L^1(\mu) \to X^*$ ($T\colon l^1(\Gamma) \to X^*$ respectively) is an isomorphism and μ, X, and Γ are as in the statement of the theorem, then $Y = T^*(\chi S_X)$ is a bounded convex norming subset of $[L^1(\mu)]^*$ (of $l^\infty(\Gamma)$) and the density character of $Y \leqslant \dim X < \mathrm{card}(\Gamma)$, respectively. (This last is true since the density character of $S_X = \dim X$ and Y is a continuous image of S_X. Hence the density character of $Y \leqslant$ density character of S_X.) In either case, Y contains a basic sequence (y_n) equivalent to the usual basis in l^1, and the theorem follows from the remark following Proposition 2.

We remark that Theorem 3.4 of [11] can now be stated without the assumption that the subspace isomorphic to $L^1(\mu)$ or $l^1(\Gamma)$ is semi-norming. We state some of these equivalences for the sake of completeness.

THEOREM 5. *Let X be a separable Banach space. Then the following are equivalent:*

(i) *X contains a subspace isomorphic to l^1;*

(ii) $C[0,1]$ *is isomorphic to a quotient space of* X. (If K is a compact Hausdorff space, then $C(K)$ is the Banach space of continuous functions on K with $\|f\| = \sup\{|f(x)|: x \in K\}$ for $f \in C(K)$.)

(iii) X^* *contains a subspace isomorphic to* L^1.

(iv) X^* *contains a subspace isomorphic to* $l^1(\Gamma)$ *with* $\operatorname{card}(\Gamma) = \mathfrak{C}$.

(v) X^* *contains a subspace isomorphic to* $C[0,1]^*$.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (iv), and (v) were all proved in [11]. (iii) $\Rightarrow$ (i) is a restatement of Theorem 1 (I) (since if μ is non purely atomic then $L^1(\mu)$ contains a subspace isometric to L^1), and (iv) $\Rightarrow$ (i) is a special case of Theorem 1 (II). (v) $\Rightarrow$ (iii) is immediate since $C[0,1]^*$ contains a subspace isometric to L^1.

Remark. The result of Gelfand [5] that L^1 is not isomorphic to a subspace of a separable conjugate space now follows directly from the implication (iii) $\Rightarrow$ (v) in Theorem 5 since $C[0,1]^*$ is non separable.

We also have the following corollaries to Theorem 1.

Corollary 6. *Let X be either*

(i) *a separable $\mathscr{L}_\infty$ space with a non separable dual*; *or*

(ii) *isomorphic to a quotient space of $C[0,1]$ with a non separable dual.*

Then X contains a subspace isomorphic to l^1.

Proof. (i) The assumptions imply that X^* is a non separable space such that X^{**} is injective. (Cf. [10], p. 335.) It now follows from the remark following Theorem 2.3 of [13], p. 217, that X^* contains a subspace isomorphic to $l^1(\Gamma)$ for some uncountable set Γ. (Otherwise, it would follow from the remark that X^* is isomorphic to a subspace of L^1, which is impossible since X^* is non separable.) Hence, by Theorem 1 (II) X contains a subspace isomorphic to l^1.

(ii) Let $\varphi: C[0,1] \to X$ be an operator from $C[0,1]$ onto X. Then $\varphi^*: X^* \to C[0,1]^*$ is an isomorphism so (by assumption) $\varphi^*(X^*)$ is a non-separable subspace of $C[0,1]^*$. It now follows from [13], Lemma 1.3, that $\varphi^*(X^*)$ contains a subspace isomorphic to $l^1(\Gamma)$ for some uncountable set Γ. (Otherwise, it would follow from this lemma that there exists a positive $\mu \in C[0,1]^*$ (which we identify by the Riesz Representation Theorem ([3], p. 265) with $M[0,1]$, the Banach space of regular Borel measures on $[0,1]$) such that $\varphi^*(X^*)$ is a subspace of $L^1(\mu)$. But this is impossible, since for every $\mu \in M[0,1]$, $L^1(\mu)$ is separable). Hence X contains a subspace isomorphic to l^1. This completes the proof of the corollary.

Lewis and Stegall [7] have proved that if X is a separable $\mathscr{L}_\infty$ space and X^* is separable, then X^* is isomorphic to l^1. Combining this result with Corollary 6 (i), we have the following:

Corollary 7. *Let X be a separable $\mathscr{L}_\infty$ space. Then either X^* is isomorphic to l^1 or X contains a subspace isomorphic to l^1.*

A final corollary to Theorem 1 is a topological result which depends heavily on work of Pełczyński and Semadeni [12] on spaces of continuous functions. Recall that a topological space K is dispersed if it contains no non-empty perfect subset. The weight of a topological space K ($\operatorname{wt}(K)$) is the least cardinal number $\mathfrak{m}$ such that there exists a base for the topology of K having cardinality $\mathfrak{m}$.

Corollary 8. *Let K be a dispersed compact Hausdorff space. Then* $\operatorname{wt}(K) = \operatorname{card}(K)$.

Proof. It is well known that $\operatorname{wt}(K) = \dim C(K)$, that $\operatorname{wt}(K) \leqslant \operatorname{card}(K)$ ([4], p. 105), and that $C(K)^*$ contains a subspace isometric to $l^1(\Gamma)$ with $\operatorname{card}(\Gamma) = \operatorname{card}(K)$. If $\operatorname{wt}(K) < \operatorname{card}(K)$, then by Theorem 1 (II), $C(K)$ contains a subspace isomorphic to l^1. But by the Main Theorem of [12], p. 214, K is not dispersed if $C(K)$ contains a subspace isomorphic to l^1. Therefore $\operatorname{wt}(K) = \operatorname{card}(K)$.

Final Remarks. It can be shown that Theorem 1 (II) cannot be improved in the following sense: There exist Banach spaces X with $\dim X > \aleph_0$ such that X^* contains a subspace isomorphic to $l^1(\Gamma)$ (even as a complemented subspace) and such that $\dim(X) < \operatorname{card}(\Gamma)$ but such that X does not contain a subspace isomorphic to $l^1(\Omega)$ with $\operatorname{card}(\Omega) > \aleph_0$.

To see this, let $\mathfrak{m}$ be an uncountable cardinal number such that $\mathfrak{m}^{\aleph_0} = 2^{\mathfrak{m}}$, (cf. [14], pp. 153–154), and let Δ be a set of cardinality $\mathfrak{m}$. Let Δ^* be the one point compactification of the (discrete) set Δ. Then, by Propositions 3.1 and 3.2 of [8], the space $K = \Delta^* \times \Delta^* \times \Delta^* \times \ldots$ is homeomorphic to a weakly compact set in a Banach space. Thus by Proposition 1 of [1], $C(K)$ is a weakly compactly generated (WCG) Banach space. It is easily seen that $\dim C(K) = \mathfrak{m}$ and $C(K)^* = M(K)$ contains a complemented subspace isometric to $l^1(\Gamma)$ with $\operatorname{card}(\Gamma) = 2^{\mathfrak{m}}$. However, it is well known (cf. [13], Remark 2, p. 214) that no WCG Banach space contains a subspace isomorphic to $l^1(\Omega)$ with $\operatorname{card}(\Omega) > \aleph_0$. (For another example, see [13], p. 236.)

A corresponding problem for $L^1(\mu)$ spaces is the following: If X^* contains a subspace isomorphic to $L^1(\mu)$ where μ is a finite homogeneous measure, does X contain a subspace isomorphic to $l^1(\Gamma)$ with $\operatorname{card}(\Gamma) = \dim(L^1(\mu))$? This was conjectured in [11] (under the additional assumption that $L^1(\mu)$ is embedded as a seminorming subspace of X^*). This problem appears still to be unsolved.

Finally, we mention that Theorem 1 and the corollaries which follow provide a positive solution in special cases to the question of Lindenstrauss: If X is separable and X^* is not, then does X contain a subspace isomorphic to l^1?

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Smooth partitions of unity on some non-separable Banach spaces

by

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Abstract. It is shown that every Hilbert space, any space of the form $L_{2n}(X, \mu)$ and any space $c_0(A)$ admit C^∞-partitions of unity. Moreover, all reflexive Banach spaces admit partitions of class C^1. The above results are obtained by verifying that a sufficient (and necessary) condition for a space to admit C^k-partitions of unity is satisfied in those cases; the condition is stated in Theorem 1.

Let X be a metric space and let S be a set of real functions on X. We say that X admits S-partitions of unity if, given an open cover $\mathcal{U}$ of X, there is a locally finite partition of unity $(f_U)_{U\in\mathcal{U}}$ with $f_U|X-U=0$ and $f_U\in S$ for any $U\in\mathcal{U}$. It is of interest to know whether or not a given Banach space admits C^k-partitions of unity, $k=1,2,\ldots,\infty$. In the case of separable spaces Bonic and Frampton ([2], Th. 1), extending the method of Eells (cf. [7], pp. 28–30), proved:

(BF) *A separable Banach space E admits C^k-partitions of unity iff there exists a non-constant function in $C^k(E)$ with bounded support.*

From this it follows that the separable Hilbert space and the spaces c_0, l_{2n} and $L_{2n}(0,1)$, $n=1,2,\ldots$ admit C^∞-partitions of unity (the C^∞-partitions of unity on l_2 were first constructed by Eells, see [7]). Combining (BF) with the result of Kadec–Restrepo ([6] and [11]) one obtains also that any separable Banach space with a separable dual admits C^1-partitions of unity. For further discussion of smooth partitions of unity on separable Banach spaces we refer the reader to [2].

However, it is not known whether the statement (BF) remains true for non-separable Banach spaces, and only very recently Wells [13] showed that each Hilbert space admits partitions of class C^1. The aim of this paper is to prove that each of the spaces $c_0(A)$, $l_2(A)$, $L_{2n}(X,\mu)$ (A-an arbitrary set, (X,μ) — a positive measure space, $n\in N$) admits C^∞-partitions of unity. We will also prove that any reflexive Banach space admits partitions of class C^1. Our approach is different from that of Wells and depends on the construction of some σ-locally finite base of open sets in $c_0(A)$.

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