References

[1] N. Jacobson, *Lectures in Abstract Algebra*. Vol. III.
[2] G. H. Meisters, *A formally real ring of distributions* Notices Amer. Math. Soc. 17 (1970), p. 940. Abstract No. 70T — A184.
[3] J. Mikusiński, *Operational Calculus*. 1959.
[4] — *Convolution of functions of several variables*, Studia Math. 20 (1961), pp. 301–312.
[5] — and C. Ryll-Nardzewski, *A theorem on bounded moments*. Studia Math. 13 (1953), pp. 51–55.
[6] L. Schwartz. *Theorie des distributions*, Paris 1966.
[7] M. Ahuja, Ph. D. thesis, Uni. of Colorado (1971).

Received December 6, 1971 (495)

STUDIA MATHEMATICA, T. XLVI. (1973)

Vector space isomorphisms of C^*-algebras

by

KARI YLINEN (Helsinki)

Abstract. For a vector space isomorphism of two C^*-algebras, connections existing between the properties of being a C^*-isomorphism, isometric, bipositive, or preserving an approximate identity, are indicated.

1. Introduction. This paper is concerned with extending to the non-unit situation some results obtained by Kadison [4], [5], in the course of characterizing the linear isometries between C^*-algebras with identity or between their real linear subspaces of self-adjoint elements. Following Kadison we call a linear isomorphism between two C^*-algebras a *quantum mechanical isomorphism* or a *C^*-isomorphism* if $T(x^*) = (Tx)^*$ and $T(a^n) = (Ta)^n$ for each self-adjoint element a and natural number n. For two C^*-algebras A and B, a linear map $T: A \to B$ is *positive*, if Ta is positive for each positive $a \in A$. If T is a vector space isomorphism and both T and T^{-1} are positive, we call T *bipositive*. In this terminology some of Kadison's results may be stated in the following form (see [4], Theorem 5, its proof, Theorem 7, and [5] Corollary 5):

THEOREM 1.1. (Kadison) *Let A and B be C^*-algebras with identities $e_1 \in A$ and $e_2 \in B$ and $T: A \to B$ a vector space isomorphism. If T is a C^*-isomorphism, T is isometric and bipositive, and $Te_1 = e_2$. Conversely, any two of the latter three properties together imply that T is a C^*-isomorphism.*

In Section 3 we extend this theorem to cover the case of linear isomorphisms between general C^*-algebras by replacing the identity with an approximate identity. Kadison's results are also applied to show that the natural extension of a real linear isometric isomorphism between the subspaces of self-adjoint elements of two C^*-algebras is also isometric. Our main tool is the Sherman–Takeda–Grothendieck theory (see [3], [6] and [7]) yielding the structure of a von Neumann algebra in the bidual of a C^*-algebra. For the basic theory of C^*-algebras we refer to [1].

2. Auxiliary results. Let A be a C^*-algebra. We identify its bidual A'' with the enveloping von Neumann algebra of A (cf. [1], p. 237). In this identification the weak operator topology of A'' coincides with $\sigma(A'', A')$ and the structure of A'' extends that of A via the canonical embedding $x \mapsto \dot{x}$. We use the term 'approximate identity' in the sense of [1], p. 359.

LEMMA 2.1. *If $(u_j)_{j\in J}$ is an approximate identity in the C^*-algebra A, the net $(u_j)_{j\in J}$ converges with respect to $\sigma(A'', A')$ to the identity e of A''.*

Proof. By [1], 1.1.10 and 2.6.4, any $f\in A'$ is a linear combination of positive linear forms. It is therefore sufficient to show that $\lim_j f(u_j) = \langle f, e\rangle$ for each positive linear form f on A. But if $f\in A'$ is positive, its canonical image $\tilde{f}\in A'''$ is a positive linear form on A'' (cf. [1] Corollary 12.1.3). Therefore, by Proposition 2.1.9 and Proposition 2.1.5 (v) in [1], we have

$$\lim_j f(u_j) = \|f\| = \|\tilde{f}\| = \langle \tilde{f}, e\rangle.$$

COROLLARY. *Let $(u_j)_{j\in J}$ be an approximate identity in the C^*-algebra A. Then $f\in A'$ is a positive linear form if and only if*

$$\lim_j f(u_j) = \|f\|. \tag{1}$$

Proof. The necessity of (1) is given in [1], Proposition 2.1.5 (v). Suppose, conversely, that (1) holds for some $f\in A'$. For the canonical image $\tilde{f}\in A'''$ of f the above lemma implies that $\|\tilde{f}\| = \|f\| = \langle f, e\rangle = \langle e, \tilde{f}\rangle$. Therefore $\tilde{f}$, hence f, is a positive linear form by Proposition 2.1.9 in [1].

In the next three lemmas A and B are C^*-algebras.

LEMMA 2.2. *Let $\tilde{A}$ (resp. $\tilde{B}$) be the C^*-algebra obtained by adjoining the identity e_1 to A (resp. e_2 to B). If $T\colon A\to B$ is a C^*-isomorphism, the natural extension $\tilde{T}\colon \tilde{A}\to\tilde{B}$, defined by $\tilde{T}(x+\lambda e_1) = Tx+\lambda e_2$, is isometric.*

Proof. Since $\tilde{T}$ is obviously a C^*-isomorphism, the lemma is a consequence of Theorem 1.1.

LEMMA 2.3. *Let $T\colon A\to B$ be a C^*-isomorphism. There is an approximate identity $(u_j)_{j\in J}$ in A such that $(Tu_j)_{j\in J}$ is an approximate identity for B.*

Proof. We modify slightly the construction used in the proof of Proposition 1.7.2 in [1], p. 15. Let J be the directed set consisting of all finite sets of self-adjoint elements of A, ordered by inclusion. If $j = \{a_1, \ldots, a_n\}\in J$, set $v_j = a_1^2+\ldots+a_n^2$. With the notation of the preceding lemma, the extension $\tilde{T}$ of T is a C^*-isomorphism. As v_j is positive, it is self-adjoint, and so is Tv_j. Let A_j (resp. B_j) denote the commutative C^*-subalgebra of $\tilde{A}$ generated by v_j and the identity e_1 (resp. of $\tilde{B}$ generated by Tv_j and e_2).Since A_j (resp B_j) consists of polynomials in v_j (resp. Tv_j) and their uniform limits, it follows from the definition of a C^*-isomorphism and the fact that $\tilde{T}$ is isometric (Lemma 2.2) that $\tilde{T}$ defines a C^*-algebra isomorphism $T_j\colon A_j\to B_j$. In particular, if we define $u_j = v_j(n^{-1}e_1+v_j)^{-1}\in A$, we have $Tu_j = Tv_j(n^{-1}e_2+Tv_j)^{-1}$. As in the proof of Proposition 1.7.2 in [1] it is seen that $\|u_j\|\leqslant 1$ and $\lim_j u_j a = \lim_j a u_j = a$ for any self-adjoint $a\in A$. Similary, $\|Tu_j\|\leqslant 1$, and as $Tv_j = (Ta_1)^2+\ldots+(Ta_n)^2$ and T defines a bijection between the sets of self-adjoint elements of A and B, $\lim_j (Tu_j)b = \lim_j bTu_j = b$ for any self-adjoint $b\in B$. Since every element of A and B is a linear combination of self-adjoint elements, it follows that $(u_j)_{j\in J}$ is an approximate identity for A and $(Tu_j)_{j\in J}$ for B.

LEMMA 2.4. *Let $T\colon A\to B$ be a linear map which preserves self-adjointness. T is bounded if its restriction to the real Banach space of the self-adjoint elements of A is bounded.*

Proof. By virtue of the principle of uniform boundedness (cf. [2], p. 66) and the fact that each $f\in B'$ is a linear combination of two Hermitian linear forms, it suffices to show that $\sup\{|f(Tx)|\,|x\in A, \|x\|\leqslant 1\}$ is finite for any continuous Hermitian linear form f on B. By hypothesis, $f\circ T$ is also Hermitian and continuous on the space of the self-adjoint elements of A. Thus $f\circ T$ is continuous everywhere (see the argument in [1], 1.2.6), and the assertion follows.

3. The main theorems.

THEOREM 3.1. *Let A and B be C^*-algebras and $T\colon A\to B$ a vector space isomorphism. Consider the following four statements:*

(i) *T is a C^*-isomorphism,*

(ii) *T is bipositive,*

(iii) *T is isometric,*

(iv) *T maps some approximate identity of A onto an approximate identity of B.*

Statement (i) *implies each one of* (ii) *to* (iv), *and any two of the statements* (ii) *to* (iv) *together imply* (i).

Proof. Suppose T is a C^*-isomorphism. Lemma 2.2 shows T to be isometric. The proof of bipositivity may be given using Kadison's original argument in [4] p. 329, since it does not depend on the existence of an identity. Statement (iv) is proved in Lemma 2.3. Suppose next that T is bipositive. Since any continuous linear functional on a C^*-algebra is a linear combination of positive linear forms and each positive linear form on a C^*-algebra is bounded, the uniform boundedness principle may be used in a manner analogous to the proof of Lemma 2.4 to show that any positive linear map between C^*-algebras is bounded. In particular, T has a second transpose $T^{**}\colon A''\to B''$. As T^* maps the positive cone of B' onto that of A', and an element of A'' (resp. B'') is positive as an operator if and only if it is non-negative on the positive linear forms on A (resp. B) (see [1] Corollary 12.1.3 (iii) and note that each vector ξ in the Hilbert space underlying A'' defines a normal positive form $x\mapsto(x\xi, \xi)$ on A''), the isomorphism T^{**} is bipositive. If T is also isometric, so is T^{**}. Then Theorem 1.1 shows that T^{**}, hence T, is a C^*-isomorphism. Suppose now that T is bounded and (iv) holds. As $T^{**}\colon A''\to B''$ is continuous with respect to $\sigma(A'', A')$ and $\sigma(B'', B')$, Lemma 2.1 implies that T maps the identity of A'' onto that of B''. If T is bipositive (resp. isometric), so is T^{**}, as was noted above. Thus Theorem 1.1 may be applied to show that (iv) combined with either (ii) or (iii) implies (i).

Note. As Kadison observes in [5], p. 502, his generalized Schwarz inequality may be used to show independently of the corresponding result for C^*-algebras with identity that in the above theorem (ii) and (iii) together imply (i).

For any C^*-algebra A, let H_A denote the real Banach space of the self-adjoint elements of A.

THEOREM 3.2. *Let A and B be C^*-algebras and $T: A \to B$ a vector space isomorphism. If T maps H_A isometrically onto H_B, then T is isometric.*

Proof. By Lemma 2.4 T is bounded, so we have the bounded maps $T^*: B' \to A'$ and $T^{**}: A'' \to B''$. The real Banach space $H_{A'}$ of the continuous Hermitian linear forms on A may be identified with the Banach space dual of H_A (see [1], p. 5). Similarly, $(H_{A'})'$ identifies with $H_{A''}$. This follows form Corollary 12.1.3 (iii) in [1] and the fact that for any two vectors ξ and η in the Hilbert space underlying A'' the linear form $x \mapsto (x\xi, \eta)$ belongs to the predual of A''. The argument used in [1] 1.2.6, p. 5 may be adapted to show that this identification preserves norms. Similar statements hold for B. We have $\|T \mid H_A\| = \|T^* \mid H_{B'}\| = \|T^{**}|H_{A''}\|$, and applying this result also to T^{-1} we see that T^{**} is isometric on $H_{A''}$. Theorem 2 in [5] combined with Theorem 5 in [4] then shows that T^{**}, hence T, is everywhere isometric.

References

[1] J. Dixmier, *Les C^*-algèbres et leurs représentations*, (Cahiers Scientifiques 29), Paris 1964.

[2] N. Dunford, J. Schwartz, *Linear operators*, Part I: *General theory*. New York 1958.

[3] A. Grothendieck, *Un résultat sur le dual d'une C^*-algèbre*. J. Math. Pures Appl. 36 (1957), pp. 97–108.

[4] R. V. Kadison, *Isometries of operator algebras*, Ann. Math. 54 (1951), pp. 325–338.

[5] — *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. Math. 56 (1952), pp. 494–503.

[6] S. Sherman, *The second adjoint of a C^*-algebra*, Proc. Intern. Congr. Math. Cambridge 1 (1950), p. 470.

[7] Z. Takeda, *Conjugate spaces of operator algebras*. Proc. Japan Acad. 30 (1954), pp. 90–95.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI

Received December 10, 1971 (453)

TUDIA MATHEMATICA, T. XLVI. (1973)

Some more Banach spaces which contain l^1

by

JAMES HAGLER (Berkeley, Cal.)

Abstract. Let X^* be a conjugate Banach space containing a subspace isomorphic to $L^1(\mu)$. Sufficient conditions on the measure μ are given which insure that X contains a subspace isomorphic to l^1.

Introduction. The purpose of this paper is the extension of the results of Pełczyński [11] concerning the embedding of $L^1(\mu)$ spaces into conjugate Banach spaces. The main result is the following:

THEOREM 1. *Let X be a Banach space. Assume that either*

(I) *X^* contains a (closed) subspace isomorphic to $L^1(\mu)$ where μ is a non purely atomic measure; or*

(II) *X^* contains a (closed) subspace isomorphic to $l^1(\Gamma)$ and the dimension of X is less than the cardinality of Γ.*

Then X contains a subspace isomorphic to l^1.

It is an immediate consequence of this theorem and results of Rosenthal [13] that if X is a separable Banach space with X^* non-separable and X is either an $\mathscr{L}_\infty$ space or a quotient space of $C[0,1]$, then X contains a subspace isomorphic to l^1. (For the definition and properties of $\mathscr{L}_p$ spaces, see [9] and [10].) It also follows from Theorem 1 and results in [11] that if X is separable and X^* satisfies either (I) or (II) of Theorem 1, then $C[0,1]$ is isomorphic to a quotient space of X.

The proof of Theorem 1 involves a modification of methods introduced by Pełczyński in [11] (except in (II) in the case where X is not separable). Pełczyński proved Theorem 1 under the added assumptions that the subspace of X^* isomorphic to $L^1(\mu)$ or $l^1(\Gamma)$ is a "seminorming" subspace of X^*, and, in case (II), that X is separable. (For the definition of seminorming, see [11], p. 232.) Delbaen [2] idependently proved Theorem 1 (I) and 1 (II) in the case where X is separable (using essentially the same idea as in Proposition 2 and the remark which follows it). Johnson and Rosenthal [6] have recently given a different proof of Theorem 1 (I) using weak-* basic sequences.

The author wishes to express his appreciation to Professor Rosenthal for suggesting this problem and for many helpful conversations concerning it.