Corollary 2. (Nullstellensatz for Principal ideals). *If* $g \in \mathcal{O}_0(E)$ *is irreducible and* $f \in \mathcal{O}_0(E)$ *is identically zero on* $V(g)$ *(the zero set of* g*), then there exists* $h \in \mathcal{O}_0(E)$ *such that* $f = g \cdot h$.

Proof. Just a question of obtaining a factorization of f and g on suitably large finite dimensional subspaces of E, applying the classical result and dividing to obtain $h \in \mathcal{O}_0(E)$.

Corollary 3. *If* X *is an analytic subset of a complex Banach manifold* U *then: If for all* $x \in X$ *the germ* X_x *does not contain a principal germ* ([6]), *the pair* $(U-X, U)$ *possesses the property of extension* ([6]).

Proof. From [6] all we must prove is the special case where U is an open ball in E, $X = V(f_1, f_2)$, where $f_1, f_2\colon U \to C$ and $h\colon U-X \to C$ is analytic. Using the theorem we can reproduce the situation on sufficiently large finite dimensional subspaces of E and apply the classical extension theorem to obtain a function $\bar{h}\colon U \to C$ which is analytic on $U-X$ and also analytic on all finite dimensional (affine) subspaces of U. The result follows immediately from work in [1] and the fact that $U-X$ is open, connected and non-empty.

Problem. Localise Theorem 1.

References

[1] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1971), pp. 77–112.

[2] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings*, Studia Math. 39 (1971), pp. 59–76.

[3] E. Hille and E. G. Philips, *Functional analysis and semigroups*, Colloq. Amer. Math. Soc. 1957.

[4] S. J. Greenfield and N. R. Wallach, *The Hilbert ball and bi-ball are holomorphically inequivalent*, Bull. Amer. Math. Soc. 77, March 1971.

[5] Narasimhan, *Introduction to the theory of analytic spaces*, 1966.

[6] J. P. Ramis, *Sous-ensembles Analytiques d'une variété Banachique Complexe*, 1970.

[7] *Seminaire de Géométrie Analytique*. Publications Mathematique D'Orsay, 1968–69.

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Formally real rings of distributions

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Let $\mathcal{D}$ denote the set of test functions, and its dual $\mathcal{D}'$ denote the set of Schwartz distributions [6]. Let $\mathcal{D}'_+$ denote the set of those elements of $\mathcal{D}'$, which have support in the positive cone $\boldsymbol{R}^n_+$, where

$$\boldsymbol{R}^n_+ = \{(t_1, t_2, \ldots, t_n)\colon t_i \in R, t_i \geqslant 0 \text{ for } i = 1, 2, \ldots, n\}.$$

It is well known that the set $\mathcal{D}'_+$ is a commutative ring under the operations addition, $+$, and convolution $*$. Moreover the ring $\mathcal{D}'_+$ has no zero divisors ([6], p. 173) and hence can be embedded into a quotient field M. In the one dimensional case, where $n = 1$, M is the quotient field of Mikusinski operators [3].

Let $(\mathcal{D}'_+)_r$ denote the set of all T in $\mathcal{D}'_+$, for which $T(\varphi)$ is a real number, whenever φ is a real valued test function. The aim of this paper is to show that, whereas $\mathcal{D}'_+$ and M cannot be (linearly) ordered, the ring $(\mathcal{D}'_+)_r$ and its quotient field M_r are both formally real and hence can be (linearly) ordered.

1. Let $\mathcal{D}_r$ denote the subset of $\mathcal{D}$ consisting of the real valued test functions, and let $\mathcal{D}'_r$ denote its real dual, i. e. the set of real valued continuous linear functionals on $\mathcal{D}_r$. Let $(\mathcal{D}'_r)_+ = \{T \in \mathcal{D}'_r\colon \text{support } T \subset \boldsymbol{R}^n_+\}$.

The relation between $(\mathcal{D}'_+)_r$ and $(\mathcal{D}'_r)_+$ is far from superficial.

Theorem I. $(\mathcal{D}'_+)_r$ *and* $(\mathcal{D}'_r)_+$ *are isomorphic as convolution algebras over the reals.*

Proof. Let $T \in \mathcal{D}'_+$ and $\varphi \in \mathcal{D}$. Let $T = T_1 + iT_2$, and $\Phi(x) = \alpha(x) + i\beta(x)$ be their decompositions into real and imaginary parts. Then $T(\Phi) = (T_1+iT_2)(\alpha+i\beta)$. It follows that if $T \in (\mathcal{D}'_+)_r$, then

$$T(\Phi) = T_1(\alpha) + iT_1(\beta).$$

Let $\tilde{T}$ denote the restriction of T_1 to $\mathcal{D}_r$. Then $\theta\colon T \to \tilde{T}$ furnishes the desired isomorphism. ■

* These results are taken from the author's doctoral dissertation [7] at the University of Colorado, written under the direction of Prof. G. H. Meisters.

2. From Artin and Schreier's theory of formally real integral fields ([1], p. 269), we know that a field or a ring is orderable (linearly) if and only if it is formally real.

DEFINITION. A ring R is *formally real* if

$$a_i \in R, \sum_{i=1}^{n} a_i^2 = 0 \Rightarrow a_i = 0 \quad \text{for each } i.$$

The ring $\mathscr{D}'_+$ cannot be ordered because $T*T+(iT)*(iT) = T^2-T^2 = 0$ for any $T \in \mathscr{D}'_+$. Theorem I suggests that $(\mathscr{D}'_+)_r$ is orderable iff $(\mathscr{D}'_r)_+$ is. For that matter there are some more rings which are formally real if and only if $(\mathscr{D}'_+)_r$ is.

Let C_+ denote the set of real valued continuous functions on $\mathbf{R}^n_+$, and C^∞_+ denote its subset consisting of all infinitely differentiable functions. $L_n = L^1_{\mathrm{loc}}(\mathbf{R}^n_+)$ denote the set of all real valued, locally integrable functions on $\mathbf{R}^n_+$. Under the operations of $+$ and $*$, C_+, C^∞_+, L_n are all commutative rings. Each one is also a real vector space, and hence a convolution algebra over reals. There is an obvious embedding

$$C^\infty_+ \to C_+ \to L_n \to (\mathscr{D}'_r)_+.$$

THEOREM II. *C^∞_+ is formally real if and only if $(\mathscr{D}'_r)_+$ is.*

Proof. Assume C^∞_+ is formally real. Suppose $T_i \in (\mathscr{D}'_r)_+$ and $\sum_{i=1}^{m} T_i * T_i = 0$. Select any $\varphi \in \mathscr{D}_r$. Then $\varphi*\varphi \in \mathscr{D}_r$.

$$0 = \sum_{i=1}^{m} T_i*T_i \Rightarrow \left(\sum T_i*T_i\right)*(\varphi*\varphi) = 0 \Rightarrow \sum_{i=1}^{m} (T_i*\varphi)*(T_i*\varphi) = 0.$$

But $T_i*\varphi \in C^\infty_+$, which is formally real.

$\therefore T_i*\varphi = 0$ for each i. This is true for any such φ. Therefore $T_i = 0$ for each i, and proves $(\mathscr{D}'_r)_+$ is formally real. The converse is obvious from the embedding. ∎

COROLLARY. *C_+, C^∞_+, L_n, $(\mathscr{D}'_r)_+$, $(\mathscr{D}'_+)_r$ are either all formally real, or none is.*

3. The task of proving $(\mathscr{D}'_+)_r$ formally real is thus simplified to proving C^∞_+ or $C_+ = C(\mathbf{R}^n_+)$ formally real. The proof for $C(\mathbf{R}^n_+)$, when $n=1$, is shown below.

THEOREM III. *$C(\mathbf{R}^1_+)$ is formally real.*

Proof. Let $f_i \in C(\mathbf{R}^1_+)$ and $\sum_{i=1}^{m} f_i*f_i = 0$. We will prove that $f_i = 0$, for each i.

Select $T > 0$ and a positive integer n.

$$\sum_{i=1}^{m} f_i*f_i = 0 \Rightarrow \int_{t=0}^{2T} e^{n(2T-t)} \sum_{i=1}^{m} f_i*f_i(t)\,dt = 0.$$

Thus

$$0 = \int_{t=0}^{2T} e^{n(2T-t)} \sum_{i=1}^{m} \int_{u=0}^{t} f_i(t-u)f_i(u)\,du\,dt = \int_{t=0}^{2T}\int_{u=0}^{t} e^{n(2T-t)} \sum_{i=1}^{m} f_i(t-u)f_i(u)\,du\,dt.$$

We change variables from u, t to v and w by the formula

$$u = T-v,$$
$$t = 2T-v-w.$$

We have

$$0 = \iint_{\Delta} = \iint_{\Delta} e^{n(v+w)} \sum_{i=1}^{m} f_i(T-v)f_i(T-w)\,dv\,dw,$$

where $\Delta = \{(v,w)\colon v+w \geqslant 0;\ v \leqslant T, w \leqslant T\}$.
Let $\Delta' = \{(v,w)\colon v+w \leqslant 0;\ v \geqslant -T;\ w \geqslant -T\}$.

$$\iint_{\Delta'} = \iint_{\Delta} + \iint_{\Delta'} = \iint_{\Delta\cup\Delta'} e^{n(v+w)} \sum_{i=1}^{m} f_i(T-v)f_i(T-w)\,dv\,dw$$
$$= \sum_{i=1}^{m} \int_{v=-T}^{T}\int_{w=-T}^{T} e^{nv}e^{nw} f_i(T-v)f_i(T-w)\,dv\,dw$$
$$= \sum_{i=1}^{m} \int_{v=-T}^{T} e^{nv} f_i(T-v)\,dv \int_{-T}^{T} e^{nw} f_i(T-w)\,dw$$
$$= \sum_{i=1}^{m} \Big(\int_{-T}^{T} e^{nv} f_i(T-v)\,dv\Big)^2.$$

Each of these summands is a non-negative real number, and so is $\iint_{\Delta'}$.

$$\iint_{\Delta'} = \Big|\iint_{\Delta'}\Big| = \Big|\iint_{\Delta'} e^{n(v+w)} \sum_{i=1}^{m} f_i(T-v)f_i(T-w)\,dv\,dw\Big|$$
$$\leqslant \sum_{i=1}^{m} \iint_{\Delta'} |e^{n(v+w)}|\,|f_i(T-v)|\,|f_i(T-w)|\,dv\,dw.$$

In Δ', since $v+w \leqslant 0$, $|e^{n(v+w)}| \leqslant 1$.

$\therefore \iint_{\Delta'} \leqslant \sum_{i=1}^{m} \iint_{\Delta'} |f_i(T-v)|\,|f_i(T-w)|\,dv\,dw \leqslant m\,2T^2A^2$, where $A = \max_{1\leqslant i\leqslant m} \{\sup_{t\in[0,2T]} |f_i(t)|\}$.

Thus, $\sum_{i=1}^{m} \Big(\int_{-T}^{T} e^{nv} f_i(T-v)\,dv\Big)^2 = \iint_{\Delta'} \leqslant 2mA^2T$.

$\therefore$ For any fixed i, $\Big|\int_{-T}^{T} e^{nv} f_i(T-v)\,dv\Big|^2 \leqslant 2mA^2T^2$ and $\Big|\int_{-T}^{T} e^{nv} f_i(T-v)\,dv\Big| \leqslant \sqrt{2m}\,AT$.

Using triangle equality

$$\Big|\int_0^T e^{nv} f_i(T-v)\,dv\Big| \leqslant \sqrt{2m}\,AT + \Big|\int_{-T}^0 e^{nv} f_i(T-v)\,dv\Big|$$

$$\leqslant \sqrt{2m}\,AT + \int_{-T}^0 |e^{nv}|\,|f_i(T-v)|\,dv$$

$$\leqslant \sqrt{2m}\,AT + AT.$$

For every positive integer n, $|\int_0^T e^{nv} f_i(T-v)\,dv| \leqslant (\sqrt{2m}+1)AT$, which is independent of n.

Using Moments theorem [5] $f_i(T-v) = 0$ for $0 \leqslant v \leqslant T$, i. e. $f_i(t) = 0$ for $0 \leqslant t \leqslant T$. Since T was arbitrarily chosen, $f_i = 0$. This is true for each i. ∎

4. To prove $C(\boldsymbol{R}^n_+)$, $n > 1$, is formally real; we closely follow the paper 'Convolution of several variables' by J. Mikusiński [4]. This paper should be referred to for definitions and details.

Let $\mathscr{A}$ be a commutative Banach algebra over the reals, and $\mathscr{A}_1$ denote its least extension with unity. For $t \geqslant 0$, let $E(t)$ be an 'exponential operator' on $\mathscr{A}$ i. e., it satisfies

(1) $E(0) = 1$.

(2) $E(t)xy = (E(t)x)y$.

(3) $t \to E(t)\,x$ is a continuous map for each x.

(4) There exists $l \in A_1$, a non-zero divisor, satisfying

$$\frac{d}{dt}\big(E(t)lx\big) = E(t)x \quad \text{for every } x \text{ in } \mathscr{A}.$$

From (1) through (4) it follows that

(5) $E(t_1+t_2) = E(t_1)\cdot E(t_2)$.

For $t \geqslant 0$, let $f(t)$, $g(t)$ denote $\mathscr{A}$-valued functions (Bochner) integrable on $\boldsymbol{R}^1_+$.

Select $T > 0$.

Let

$$\tilde f = \int_0^T E(t)f(t)\,dt,$$

$$(f*g)(t) = \int_0^t f(t-u)g(u)\,du, \quad \text{and}$$

$$K(f,g)(T-t) = \int_{u=0}^{T-t} f(t+u)g(T-u)\,du.$$

Lemma. $\tilde f\cdot \tilde g = \widetilde{f*g} + E(T)\tilde K(f,g)$.

Proof. See [4].

Corollary 1. $(\tilde f)^2 = \tilde f * \tilde f = \widetilde{f*f} + E(T)\tilde K(f,f)$.

Corollary 2. *For a real number $s \geqslant 1$, let $\tilde f(s)$ denote $\int_0^T E(st)f(t)\,dt$. Then*

$$\tilde f(s)\cdot\tilde g(s) = \widetilde{f*g}(s) + E(st)\tilde K(f,g)(s).$$

Proof. See [4].

Theorem IV. *Let $\mathscr{A}$ denote a Banach algebra over $\mathbf{R}$, and $\mathscr{B}$ denote the set of all $\mathscr{A}$-valued functions locally (Bochner) integrable on $[0,\infty)$. For a $T > 0$, let $\mathscr{B}_T$ denote the set of all $\mathscr{A}$-valued functions, defined and locally integrable on $[0, T]$. Then*

(a) *$\mathscr{B}$ and $\mathscr{B}_T$ are convolution algebras over $\mathbf{R}$, and $\mathscr{B}_T$ is a Banach algebra under the norm $\|f\|_1 = \int_0^T |f(t)|\,dt$ where $|\ |$ denoted the norm on $\mathscr{A}$.*

(b) *If $\mathscr{A}$ satisfies $|a^2| = |a|^2$ for every a in $\mathscr{A}$, then $\mathscr{A}$ has no zero divisors.*

(c) *If $\mathscr{A}$ satisfies $|\sum_{i=1}^m a_i^2| = \sum_{i=1}^m |a_i|^2$, then $\mathscr{B}$ is a formally real ring.*

Proof. Mikusiński [4] has proved (a), and (b). We show proof of (c), which is a slight modification of proof for (b).

Let $f_i \in \mathscr{B}$ and $\sum_{i=1}^m f_i * f_i = 0$. We will show $f_i = 0$, for each i.

From the lemma above, $\tilde f_i\cdot\tilde f_i = \widetilde{f_i*f_i} + E(T)\tilde K(f_i,f_i)$. Therefore,

$$\sum_{i=1}^m(\tilde f_i)^2 = \sum_{i=1}^m \widetilde{f_i*f_i} + E(T)\sum_{i=1}^m \tilde K(f_i,f_i)$$

$$= \widetilde{\sum f_i*f_i} + E(T)\sum_{i=1}^m \tilde K(f_i,f_i)$$

$$= 0 + E(T)\sum_{i=1}^m \tilde K(f_i,f_i).$$

Let n be a positive integer. We choose $E(t) = e^{-nt}$ as the exponential operator.

We have, $\sum_{i=1}^m (\tilde f_i)^2 = e^{-nT}\sum_{i=1}^m \tilde K(f_i,f_i)$, and

$$\Big|\sum_{i=1}^m (\tilde f_i)^2\Big| = |e^{-nT}|\,\Big|\sum_{i=1}^m \tilde K(f_i,f_i)\Big|.$$

Using hypothesis on $\mathscr{A}$,

$$\sum_{i=1}^{m} |\tilde{f}_i|^2 = e^{-nT}\Big|\sum_{i=1}^{m}\tilde{K}(f_i, f_i)\Big| \leqslant e^{-nT}\sum_{i=1}^{m}|\tilde{K}(f_i, f_i)|.$$

As shown in Theorem II of [4], $|\tilde{K}(f_i, f_i)| \leqslant M_i$, where M_i depends on f_i, T, but not on n. Thus, $\sum_{i=1}^{m} |\tilde{f}_i|^2 \leqslant e^{-nT} \sum_{i=1}^{m} M_i$.

For any fixed i, $|\tilde{f}_i|^2 \leqslant e^{-nT}\sum_{i=1}^{m} M_i$, and $|\tilde{f}_i| \leqslant e^{-n\frac{T}{2}}\sqrt{\sum_{i=1}^{m} M_i}$.

$\therefore\ e^{n\frac{T}{2}}|\tilde{f}_i| \leqslant M$, where $M = \sqrt{\sum_{i=1}^{m} M_i}$.

$$e^{n\frac{T}{2}}|\tilde{f}_i| = e^{n\frac{T}{2}}\Big|\int_0^T e^{-nt}f_i(t)\,dt\Big| = \Big|\int_0^T e^{n\left(\frac{T}{2}-t\right)}f_i(t)\,dt\Big| \leqslant M.$$

Using triangle inequality,

$$\begin{aligned}\Big|\int_0^{T/2} e^{n\left(\frac{T}{2}-t\right)} f_i(t)\,dt\Big| &\leqslant M + \Big|\int_{T/2}^{T} e^{n\left(\frac{T}{2}-t\right)} f_i(t)\,dt\Big| \\ &\leqslant M + \int_{T/2}^{T} |e^{n\left(\frac{T}{2}-t\right)}|\,|f_i(t)|\,dt \\ &\leqslant M + \int_{T/2}^{T} |f_i(t)|\,dt = L \quad \text{say,}\end{aligned}$$

where L is independent of n.

For every positive integer n,

$$\Big|\int_0^{T/2} e^{n\left(\frac{T}{2}-t\right)} f_i(t)\,dt\Big| = \Big|\int_0^{T/2} e^{nt} f_i\Big(\frac{T}{2}-t\Big)\,dt\Big| \leqslant L \quad \text{(independent of } n).$$

Using Moments theorem, $f_i(t) = 0$, for $0 \leqslant t \leqslant \frac{T}{2}$. But T was arbitrarily chosen, so $f_i = 0$. This is true for each i. ■

Corollary. *$L_1 = L^1_{\rm loc}(\boldsymbol{R}^1_+)$ is formally real. More precisely, if $\sum_{i=1}^{m} f_i * f_i(t) = 0$ in $0 \leqslant t \leqslant T$, then, $f_i(t) = 0$ in $0 \leqslant t \leqslant \frac{T}{2}$ for each i.*

5. We recall that $\tilde{f}(s) = \int_0^T E(st)f(t)\,dt.$, and

$$\tilde{f}(s)\cdot\tilde{g}(s) = \widetilde{f*g}(s) + E(sT)\tilde{K}(f, g)(s), \quad \text{for } s \geqslant 1.$$

Theorem V. *Let X be a commutative Banach algebra over $\mathbf{R}$ and Y be the set of all X-valued functions integrable on $0 \leqslant u \leqslant U$, for some $U > 0$.*

For $u \geqslant 0$, let $E(u)$ be an exponential operator on X, satisfying

(i) $E(U) = 0$.

(ii) *There exists $u = u_0$ satisfying $x_i \in X$, and*

$$\sum_{i=1}^{m} x_i^2 = 0 \Rightarrow E(u_0)x_i = 0 \quad \textit{for each } i.$$

Let $f_i \in Y$, and $E(u)\sum_{i=1}^{m} f_i * f_i\,(u) = 0$ for $0 \leqslant u \leqslant U$. Then $E(u+u_0)f_i(u) = 0$ for $0 \leqslant u \leqslant U$, and for each i.

Proof. Let $s \geqslant 1$.

$$E(u)\sum_{i=1}^{m} f_i * f_i(u) = 0 \text{ in } [0, U] \Rightarrow E(su)\sum f_i * f_i(u) = 0 \text{ in } [0, U] \Rightarrow$$

$$\int_{u=0}^{U} E(su)\sum f_i * f_i(u)\,du$$

$$= \sum_{i=1}^{m}\int_{u=0}^{U} E(su)(f_i * f_i)(u)\,du = \sum_{i=1}^{m}\widetilde{f_i * f_i}(s) = 0.$$

By making use of Corollary 2, p. 25, we have $\Sigma\,(\tilde{f}_i(s))^2 = \Sigma\widetilde{f_i * f_i}(s) + E(sU)\,\Sigma\,\tilde{K}(f_i, f_i)(s)$. But $\sum_{i=1}^{m}\widetilde{f_i * f_i}(s) = 0$, and since $E(U) = 0$, $E(sU) = 0$. Thus $\sum_{i=1}^{m}(\tilde{f}_i(s))^2 = 0$.

Using hypothesis (ii) on X, we have

$$E(u_0)\tilde{f}_i(s) = 0, \quad \text{for each } i, \text{ i.e.,}$$

$$E(u_0)\int_{u=0}^{U} E(su)f_i(u)\,du = \int_{u=0}^{U} E(su)E(u_0)f_i(u)\,du = 0$$

for each i, and $s \geqslant 1$.

Using theorem III of ([4], p. 304),

$$E(su)E(u_0)f_i(u) = 0, \quad \text{for } 0 \leqslant u \leqslant U \text{ and } s \geqslant 1.$$

In particular, for $s = 1$, $E(u+u_0)f_i(u) = 0$, for each i. ■

Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{B}_T$ be as in Theorem IV. Let $U \geqslant 0$ be arbitrarily chosen. For $u \geqslant 0$, let $E(u)$ be the exponential operator on $\mathscr{B}_T$ defined by

$$(E(u)f)(t) = \begin{cases} f\left(t - \frac{T}{U}u\right) & \text{if } \frac{t}{T} \geqslant \frac{u}{U}, \\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathscr{S}$ be the set of all $\mathscr{B}_T$ valued functions integrable on $0 \leqslant u \leqslant U$. We notice that

(i) $E(U) = 0$.

(ii) Suppose $f_i \in \mathscr{B}_T$ and $\sum_{i=1}^{m} f_i * f_i = 0$. The proof of Theorem IV shows that $f_i(t) = 0$, for $0 \leqslant t \leqslant \frac{T}{2}$, for each i, or, in other words, $E\left(\frac{U}{2}\right) f_i = 0$, for each i.

Corollary. *If $f_i \in \mathscr{S}$, and $E(u) \sum_{i=1}^{m} f_i * f_i(u) = 0$, for $0 \leqslant u \leqslant U$, then $E\left(u + \frac{U}{2}\right) f_i(u) = 0$ for $0 \leqslant u \leqslant U$ for each i.*

6. We recall that $L_n = L^1_{\mathrm{loc}}(\boldsymbol{R}^n_+)$ = set of all real valued functions locally integrable on $\boldsymbol{R}^n_+$. Select $T_1, T_2, \ldots, T_n$ strictly positive numbers. Let $\Delta^n(r)$ denote the simplex

$$\Delta^n(r) = \left\{(t_1, t_2, \ldots, t_n):\ t_i \in \boldsymbol{R}^1_+;\ \frac{t_1}{T_1} + \frac{t_2}{T_2} + \ldots + \frac{t_n}{T_n} \leqslant r\right\},$$

where $r > 0$.

Let $\mathscr{B}_n$ denote the set of all real valued functions defined and integrable on $\Delta^n(1)$. For $f \in \mathscr{B}_n$, let $\|f\|_n = \int_{t \in \Delta^n(1)} |f(t)|\, dt$. It is clear that for every positive integer n, $\mathscr{B}_n$ is a convolution algebra over reals, and a Banach algebra under the norm defined above. Corollary to theorem IV may now be worded as: if $f_i \in \mathscr{B}_1$ and $\sum_{i=1}^{m} f_i * f_i = 0$, then $f_i = 0$ on the set $\Delta^1(1/2)$ for each i.

The above statement can be generalized to n dimensions.

Theorem VI. *If $f_i \in \mathscr{B}_n$ and $\sum_{i=1}^{m} f_i * f_i = 0$, then for each i, $f_i = 0$ on the set $\Delta^n(1/2)$.*

Proof. We use induction on n. As already noticed, the theorem is true for $n = 1$. Now suppose the theorem is true for $n = k-1$. Assume $f_i \in \mathscr{B}_k$ and $\sum_{i=1}^{m} f_i * f_i = 0$. We may look at $\mathscr{B}_k$ as the set of $\mathscr{B}_{k-1}$ valued functions integrable on $0 \leqslant t_k \leqslant T_k$.

For $f \in \mathscr{B}_k$, and $0 \leqslant t_k \leqslant T_k$, let $f(t_k)$ denote the element of $\mathscr{B}_{k-1}$ defined by,

$$f(t_k)(t_1, t_2, \ldots, t_{k-1}) = f(t_1, t_2, \ldots, t_k).$$

We also define an exponential operator on $\mathscr{B}_{k-1}$. For $0 \leqslant t_k \leqslant T_k$, and $f \in \mathscr{B}_{k-1}$, let

$$E(t_k) f = \begin{cases} f\left(t_1 - \frac{T_1}{T_k} t_k, t_2 - \frac{T_2}{T_k} t_k, \ldots, t_{k-1} - \frac{T_{k-1}}{T_k} t_k\right) & \text{if all the coordinates are positive,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

(i) $E(T_k) = 0$,

(ii) if $x_i \in \mathscr{B}_{k-1}$ and $\sum_{i=1}^{m} x_i * x_i = 0$, i. e., $\Sigma\, x_i * x_i = 0$ on the set $\Delta^{k-1}(1)$, then by induction hypothesis, $x_i = 0$ for each i, on $\Delta^{k-1}(1/2)$. In other words, $E\left(\frac{T_k}{2}\right) x_i = 0$ for each i.

We see that $\mathscr{B}_{k-1}$ satisfies all the conditions on X in Theorem V. We have $f_i \in \mathscr{B}_k$ and $\sum_{i=1}^{m} f_i * f_i = 0$. In the language of exponential operator, this means,

$$\mathrm{E}(t_k) \sum_{i=1}^{m} f_i * f_i(t_k) = 0 \quad \text{for } 0 \leqslant t_k \leqslant T_k.$$

From Theorem V, we conclude $E\left(t_k + \frac{T_k}{2}\right) f_i(t_k) = 0$ for $0 \leqslant t_k \leqslant T_k$, for each i.

This means that for each i, $f_i = 0$, on $\Delta^k(1/2)$. Thus, the theorem is true for $n = k$ as well. ■

Corollary. *L_n is formally real for all n.*

Proof. Suppose $f_i \in L_n$, and $\sum_{i=1}^{m} f_i * f_i = 0$. Select $T_1, T_2, \ldots, T_n$ strictly positive numbers. Let f_i also denote the restriction of f_i on $\Delta^n(1)$. Then $f_i \in \mathscr{B}_n$ and $\sum_{i=1}^{m} f_i * f_i = 0$. From Theorem VI, we conclude $f_i = 0$ on $\Delta^n(1/2)$, for each i. Since T_i were all arbitrarily chosen, we conclude $f_i = 0$ on $\boldsymbol{R}^n_+$, for each i. This proves that L_n is formally real.

Corollary. (i) *C_+, C^∞_+, L_n, $(\mathscr{D}'_r)_+$, $(\mathscr{D}'_+)_r$ are all formally real.*

(ii) *If M_r denotes the quotient field of $(\mathscr{D}'_r)_+$, then M_r is formally real.*

(iii) *The quotient field of Mikusiński operators is formally real.*

References

[1] N. Jacobson, *Lectures in Abstract Algebra*. Vol. III.
[2] G. H. Meisters, *A formally real ring of distributions* Notices Amer. Math. Soc. 17 (1970), p. 940. Abstract No. 70T — A184.
[3] J. Mikusiński, *Operational Calculus*. 1959.
[4] — *Convolution of functions of several variables*, Studia Math. 20 (1961), pp. 301–312.
[5] — and C. Ryll-Nardzewski, *A theorem on bounded moments*. Studia Math. 13 (1953), pp. 51–55.
[6] L. Schwartz. *Theorie des distributions*, Paris 1966.
[7] M. Ahuja, Ph. D. thesis, Uni. of Colorado (1971).

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Vector space isomorphisms of C^*-algebras

by

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Abstract. For a vector space isomorphism of two C^*-algebras, connections existing between the properties of being a C^*-isomorphism, isometric, bipositive, or preserving an approximate identity, are indicated.

1. Introduction. This paper is concerned with extending to the non-unit situation some results obtained by Kadison [4], [5], in the course of characterizing the linear isometries between C^*-algebras with identity or between their real linear subspaces of self-adjoint elements. Following Kadison we call a linear isomorphism between two C^*-algebras a *quantum mechanical isomorphism* or a *C^*-isomorphism* if $T(x^*) = (Tx)^*$ and $T(a^n) = (Ta)^n$ for each self-adjoint element a and natural number n. For two C^*-algebras A and B, a linear map $T\colon A \to B$ is *positive*, if Ta is positive for each positive $a \in A$. If T is a vector space isomorphism and both T and T^{-1} are positive, we call T *bipositive*. In this terminology some of Kadison's results may be stated in the following form (see [4], Theorem 5, its proof, Theorem 7, and [5] Corollary 5):

THEOREM 1.1. (Kadison) *Let A and B be C^*-algebras with identities $e_1 \in A$ and $e_2 \in B$ and $T\colon A \to B$ a vector space isomorphism. If T is a C^*-isomorphism, T is isometric and bipositive, and $Te_1 = e_2$. Conversely, any two of the latter three properties together imply that T is a C^*-isomorphism.*

In Section 3 we extend this theorem to cover the case of linear isomorphisms between general C^*-algebras by replacing the identity with an approximate identity. Kadison's results are also applied to show that the natural extension of a real linear isometric isomorphism between the subspaces of self-adjoint elements of two C^*-algebras is also isometric. Our main tool is the Sherman–Takeda–Grothendieck theory (see [3], [6] and [7]) yielding the structure of a von Neumann algebra in the bidual of a C^*-algebra. For the basic theory of C^*-algebras we refer to [1].

2. Auxiliary results. Let A be a C^*-algebra. We identify its bidual A'' with the enveloping von Neumann algebra of A (cf. [1], p. 237). In this identification the weak operator topology of A'' coincides with $\sigma(A'', A')$ and the structure of A'' extends that of A via the canonical embedding $x \mapsto \dot{x}$. We use the term 'approximate identity' in the sense of [1], p. 359.