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Addendum to the paper "On singular integrals"

by

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Abstract. Let $x \in E^n$, $\tilde{f}_\varepsilon(x) = (K_\varepsilon * f)(x)$, where $K_\varepsilon(x) = \Lambda(x)|x|^{-n}$ for $|x| > \varepsilon$ and is 0 otherwise, and $\Lambda(x)$ is a homogeneous function of degree 0 whose mean value over the unit sphere $(\Sigma)\,|x| = 1$ is 0. Let $\omega_1(\delta)$ be the "rotational" modulus of continuity of Λ in the metric L^1: $\omega_1(\delta) = \sup_{|\varrho|<\delta} \int_\Sigma |\Lambda(\varrho x) - \Lambda(x)|\,d\sigma$, where ϱ is an arbitrary rotation on Σ, $|\varrho|$ its magnitude, and $d\sigma$ the element of surface area. The note clarifies a proof of the following theorem: If $\omega_1(\delta)$ satisfies the Dini condition, i.e., $\int_0^1 \delta^{-1}\omega_1(\delta)\,d\delta < \infty$, then the operation $\sup_\varepsilon |\tilde{f}_\varepsilon(x)| = Tf$ is of weak type (1, 1).

The authors of this note have received a number of inquiries about section 13(a) of the paper "On the existence of singular integrals" written jointly with the late Mary Weiss and published in vol. 10 of the "Proceedings of Symposia in Pure Mathematics" of AMS. The result stated there has since been generalized (see N. M. Rivière, "Singular Integrals and Multiplier Operators", Arkiv för Matematik, vol. 2 (1971), pp. 243–278). In spite of this we feel that a clarification along the lines of our paper might still be useful.

We acknowledge and are grateful for suggestions of Dr. Calixto P. Calderón which we have incorporated in this note.

We wish to estimate the function $\tilde{f}_\varepsilon(x)$ in the statement of Theorem 2, but with ε depending on x in an arbitrary fashion. As in § 7, this is done by estimating $\tilde{h}_\varepsilon(x)$ and $\tilde{g}_\varepsilon(x)$ separately. The estimation of $\tilde{g}_\varepsilon(x)$ requires no further explanation. As for $\tilde{h}_\varepsilon(x)$ one proceeds as follows. First one chooses λ (see § 7) so that if $\bar{Q}_j$ is the cube Q_j expanded λ times then the distance between Q_j and the complement $C\bar{Q}_j$ of $\bar{Q}_j$ is twice the diameter of Q_j. Next one considers the function

$$I_1(x) = \sum \int_{Q_j} [|h(z)| + 2^{n+2}y]\,|K(x-z) - K(x-z_j)|\,dz$$

where y is the same as in (7.1). This function is similar to the $I(x)$ in § 7 and also has the property that

$$\int_{C\bar{S}} I_1(x)\,dx \leqslant c\,\|f\|_1, \qquad \bar{S} = \bigcup \bar{Q}_j. \tag{1}$$

In order to estimate $\tilde{h}_\varepsilon(x)$ one sets

$$\tilde{h}_\varepsilon(x) = \Sigma_1 \int_{Q_j} h(z)K(x-z)\,dz + \Sigma_2 \int_{Q_j^*} h(z)K(x-z)\,dz,$$

where Σ_1 is extended over all Q_j entirely contained in $\{z \mid |z-x| > \varepsilon\}$, Σ_2 is extended over the remaining Q_j and $Q_j^* = Q_j \cap \{z \mid |z-x| > \varepsilon\}$. By an argument analogous to the one employed in § 7, one sees readily that if $x \in C\bar{S}$, $\bar{S} = \bigcup \bar{Q}_j$, then $|\Sigma_1| \leqslant I_1(x)$. On the other hand, as will be shown below, $|\Sigma_2| \leqslant c[y + I_1(x)]$, and consequently the set of points of $C\bar{S}$ where $|\tilde{h}_\varepsilon(x)| > 2cy$ is contained in the set where $(1+c)I_1(x) > cy$. From this and (1), it follows that the set of points where $|\tilde{h}_\varepsilon(x)| > 2cy$ has measure not exceeding $c\|f\|_1/y + |\bar{S}| \leqslant c\|f\|_1/y$, which is the desired conclusion.

To see that $|\Sigma_2| \leqslant c[y + I_1(x)]$, one observes that if $|Q_j^*| > \frac{1}{2}|Q_j|$ the mean value m_j of h on Q_j^* does not exceed $2^{n+2}y$ in absolute value. Thus

$$\int_{Q_j^*} h(z)K(x-z)\,dx = \int_{Q_j^*} [h(z)-m_j]K(x-z)\,dz + m_j \int_{Q_j^*} K(x-z)\,dz$$

and

$$(2)\qquad \Big|\int_{Q_j^*} h(z)K(x-z)\,dz\Big| \leqslant \int_{Q_j} |h(z)-m_j|\,|K(x-z)-K(x-z_j)|\,dz + 2^{n+2}y \int_{Q_j} |K(x-z)|\,dz.$$

If $|Q_j^*| \leqslant \frac{1}{2}|Q_j|$ then the mean value m_j of h on $Q_j - Q_j^*$ does not exceed $2^{n+2}y$ in absolute value and

$$\int_{Q_j^*} h(z)K(x-z)\,dz = \int_{Q_j^*} h(z)K(x-z)\,dz + \int_{Q_j-Q_j^*} [h(z)-m_j]K(x-z)\,dz + m_j \int_{Q_j-Q_j^*} K(x-z)\,dz,$$

whence it follows that

$$(3)\qquad \Big|\int_{Q_j^*} h(z)K(x-z)\,dz\Big| \leqslant 2\int_{Q_j} [|h(z)| + |m_j|]\,|K(x-z)-K(x-z_j)|\,dz + 2^{n+2}y \int_{Q_j} |K(x-z)|\,dz.$$

Combining (2) and (3) one obtains

$$|\Sigma_2| \leqslant 2I_1(x) + 2^{n+2}y\Sigma_2 \int_{Q_j} |K(x-z)|\,dz,$$

where the sum on the right is extended over all Q_j intersecting the spherical surface $|z-x| = \varepsilon$. But then, since $x \in C\bar{S}$, i.e., $x \in C\bar{Q}_j$, each such Q_j is contained in $\{z \mid |z-x| \geqslant \varepsilon/2\} \cap \{z \mid |z-x| \leqslant 3\varepsilon/2\}$ and consequently, since $K(x)$ is positively homogeneous of degree $-n$,

$$\Sigma_2 \int_{Q_j} |K(x-z)|\,dz \leqslant \int_{1/2\varepsilon<|x|<3/2\varepsilon} |K(x)|\,dx = c,$$

whence the desired result follows.

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