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On linear functionals in Hardy-Orlicz spaces, II

by

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Abstract. This paper is the second part of a paper under the same title which is also published in Studia Mathematica. The paper contains two sections: III and IV. In the Section III we give the representation of linear functionals on a Hardy–Orlicz space $H^{*\varphi}$ ($H^{\circ\varphi}$) for the general case, where φ is a log-convex φ-function (Theorem III. 3.1–3.4). Section IV contains a more precise representation of linear functionals on a Hardy–Orlicz space $H^{*\psi}$ ($H^{\circ\psi}$) for convex φ-function ψ (Theorems IV.1.1–3.4) and the representation of linear functionals on the dual space for $H^{*\psi}$ (Theorems IV.4.1–4.6). Throughout the whole paper the investigations concern three types of linear functionals on a Hardy-Orlicz space: norm continuous, modular continuous and very weakly continuous ones.

This paper is a continuation of paper [6]. We adopt the notation and continue the section numbering of paper I. We cite the results of both parts, I and II, writing the number of the section and the number the result in the section; within the same section the section number is omitted.

III. REPRESENTATION OF LINEAR FUNCTIONALS

1.1. Let F_1 and F_2 be two analytic functions in the circle D and let

$$F_1(z) = \sum_{n=0}^{\infty} \gamma_n(F_1) z^n \quad \text{and} \quad F_2(z) = \sum_{n=0}^{\infty} \gamma_n(F_2) z^n \quad \text{for } z \in D.$$

The radius of convergence of both these power series is not less than 1, and so the radius of convergence of the series

$$(F_1 * F_2)(z) = \sum_{n=0}^{\infty} \gamma_n(F_1)\gamma_n(F_2) z^n$$

is also not less than 1. The function $F_1 * F_2$ will be called a *convolution of the functions* F_1 *and* F_2.

It is easy to verify that convolution has the following properties on the space of all analytic functions in D:

1° $F_1 * F_2 = F_2 * F_1$ (convolution is commutative),

2° $(F_1 * F_2) * F_3 = F_1 * (F_2 * F_3)$ (convolution is associative),

3° $(F_1 + F_2) * F_3 = F_1 * F_3 + F_2 * F_3$ (distributivity of convolution with respect to addition),

4° $(aF_1) * F_2 = F_1 * (aF_2) = a(F_1 * F_2)$ for any number a,

5° The function

$$I(z) = (1-z)^{-1} = \sum_{n=0}^{\infty} z^n \quad (z \in D)$$

is a convolution unity (i.e. $I * F = F$ for every function F analytic in D).

1.2. For functions F_1 and F_2 analytic in D the following is true

$$(F_1 * F_2)(z) = \frac{1}{2\pi} \int_0^{2\pi} F_1(z_1 e^{it}) F_2(z_2 e^{-it})\, dt \quad \text{for } z \in D,$$

where z_1, z_2 are number from the circle D such that $z = z_1 z_2$ ([11]).

1.3. *For $z \in D$ and $k = 1, 2, \ldots$ let*

$$I_k(z) = z^k I^{k+1}(z) = z^k (1-z)^{-k-1}.$$

Then for any function F analytic in D

$$(I_k * F)(z) = \frac{1}{k!} F^{(k)}(z) z^k$$

for $z \in D$ and $k = 1, 2, \ldots$

Proof. Let $z \in D$. We have, by 1.2, for an r such that $|z| < r < 1$,

$$(I_k * F)(z) = \frac{1}{2\pi} \int_0^{2\pi} I_k\left(\frac{z}{r} e^{-it}\right) F(re^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{z^k re^{it}}{(re^{it} - z)^{k+1}} F(re^{it})\, dt$$

$$= z^k \frac{1}{2\pi i} \int_{C_r} \frac{F(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta = z^k \frac{1}{k!} F^{(k)}(z),$$

where $C_r = \{\zeta\colon |\zeta| = r\}$ with the positive orientation.

1.4. *For any functions F_1 and F_2 analytic in D the following two relations hold:*

$$(T_r F_1) * F_2 = F_1 * (T_r F_2) = T_r(F_1 * F_2) \quad \textit{for } 0 \leqslant r \leqslant 1$$

and

$$(S_h F_1) * F_2 = F_1 * (S_h F_2) = S_h(F_1 * F_2) \quad \textit{for real } h.$$

The easy proof of this lemma is omitted.

1.5. Let f be an integrable function on $[0, 2\pi)$. An analytic function in D defined by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(t)}{1 - ze^{-it}}\, dt \quad (z \in D)$$

is called a *Cauchy integral of function f*.

It is evident that the Cauchy integral of a function f can be represented in the form of the following power series

$$F(z) = \sum_{n=0}^{\infty} \gamma_n(F) z^n \quad (z \in D),$$

whose coefficients are given by

$$\gamma_n(F) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt \quad \textit{for } n = 0, 1, 2, \ldots$$

1.6. *If F is an analytic function in D and G is the Cauchy integral of a function g integrable on $[0, 2\pi)$, then*

$$(F * G)(z) = \frac{1}{2\pi} \int_0^{2\pi} F(ze^{-it}) g(t)\, dt \quad \textit{for } z \in D.$$

Proof. From 1.5 we have for $z \in D$

$$\frac{1}{2\pi} \int_0^{2\pi} F(ze^{-it}) g(t)\, dt = \sum_{n=0}^{\infty} \gamma_n(F) z^n \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-int}\, dt$$

$$= \sum_{n=0}^{\infty} \gamma_n(F) \gamma_n(G) z^n = (F * G)(z).$$

1.7. *If f is an integrable function on $[0, 2\pi)$ such that*

$$\int_0^{2\pi} f(t) e^{int}\, dt = 0 \quad \textit{for } n = 1, 2, \ldots,$$

then a function F being the Cauchy integral of f is the Poisson integral of f. Hence in this case the Cauchy integral F of f belongs to the Hardy class H^1 ([15], Chap. VII. § 9).

1.8. *If for some constants $d > 0$ and $u_0 \geqslant 0$ we have the inequality*

$$u \leqslant d\varphi(u) \quad \textit{for } u \geqslant u_0,$$

(then $L^{*\varphi} \subset L^1$) and if a function f belonging to $L^{*\varphi}$ (resp. to L^φ, $L^{\circ\varphi}$) satisfies the condition

$$\int_0^{2\pi} f(t)e^{int}dt = 0 \quad \text{for } n = 1, 2, \ldots,$$

then the Cauchy integral F of f belongs to $H^{*\varphi}$ (resp. to H^φ, $H^{\circ\varphi}$).

Proof. Let $f \in L^{*\varphi}$ satisfy the condition $\int_0^{2\pi} f(t)e^{int}dt = 0$ for $n = 1, 2, \ldots$ Then $f \in L^1$. In virtue of 1.7 the Cauchy integral F of f belongs to H^1 and $F(e^{it}) = f(t)$ for almost every $t \in [0, 2\pi)$. Thus $F \in N'$ and $F(e^{i\cdot}) \in L^{*\varphi}(L^\varphi, L^{\circ\varphi})$. This implies, in view of I. 3.3, that $F \in H^{*\varphi}$ (resp. H^φ, $H^{\circ\varphi}$).

2.1. Let G be an analytic function in D. We shall designate

$$\nu'_\varphi(z; G; R) = \sup\{2\pi|(F*G)(z)|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant R\}$$

fot $z \in D$ and $R > 0$, and

$$\nu'_\varphi(G; R) = \sup\{\nu'_\varphi(z; G; R):\ z \in D\} \quad \text{for } R > 0.$$

For any function G analytic in D and for arbitrary $R > 0$ the following relations hold:

1° $\nu'_\varphi(z; G; R) = \nu'_\varphi(|z|; G; R)$ *for* $z \in D$,

2° $\nu'_\varphi(r; G; R)$ *is a non-decreasing function of r in $[0, 1)$. Thus*

$$\nu'_\varphi(G; R) = \lim_{r\to 1-} \nu'_\varphi(r; G; R).$$

Proof. Let $z = re^{it}$. Since for every $F \in H^{*\varphi}$ we have $\|S_t F\|_\varphi = \|F\|_\varphi$, by 1.4 it follows that

$$\begin{aligned}\nu'_\varphi(z; G; R) &= \sup\{2\pi|(F*G)(re^{it})|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant R\}\\ &= \sup\{2\pi|(S_t F*G)(r)|:\ F \in H^{*\varphi},\ \|S_t F\|_\varphi \leqslant R\} = \nu'_\varphi(r; G; R).\end{aligned}$$

Let now $0 \leqslant r_1 < r_2 < 1$. In virtue of the Maximum Principle for every $F \in H^{*\varphi}$ such that $\|F\|_\varphi \leqslant R$ there is a z such that $|z| = r_2$ and

$$2\pi|(F*G)(r_1)| \leqslant 2\pi|(F*G)(z)| \leqslant \nu'_\varphi(z; G; R) = \nu'_\varphi(r_2; G; R).$$

This yields $\nu'_\varphi(r_1; G; R) \leqslant \nu'_\varphi(r_2; G; R)$.

2.2. $(H^{*\varphi})'$ will denote the class of all functions G analytic in D for which $\nu'_\varphi(G; R) < \infty$ for some number $R > 0$, and $(H^{*\varphi})'_0$ — a class of all functions G analytic in D for which $\nu'_\varphi(G; R) < \infty$ for all $R > 0$. Besides, for $R > 0$ we introduce the class $(H^{*\varphi})'_R$ of all functions G analytic in D for which $\nu'_\varphi(G; R) < \infty$.

We observe that

$$(H^{*\varphi})' = \bigcup_{n=1}^{\infty} (H^{*\varphi})'_{1/n} \quad \text{and} \quad (H^{*\varphi})'_0 = \bigcap_{n=1}^{\infty} (H^{*\varphi})'_n.$$

2.3. *For any function G analytic in D*

$$\nu'_\varphi(z; G; R) = \sup\{2\pi|(F*G)(z)|:\ F \in H^{\circ\varphi},\ \|F\|_\varphi \leqslant R\}$$

for arbitrary $z \in D$ and $R > 0$.

On this account there is no need to introduce analogous classes to those in 2.2 for $H^{\circ\varphi}$.

Proof. Clearly, for arbitrary $z \in D$ and $R > 0$

$$\sup\{2\pi|(F*G)(z)|:\ F \in H^{\circ\varphi},\ \|F\|_\varphi \leqslant R\} \leqslant \nu'_\varphi(z; G; R).$$

Let $F \in H^{*\varphi}$ be any function such that $\|F\|_\varphi \leqslant R$. Then $T_r F \in H^{\circ\varphi}$ and $\|T_r F\|_\varphi \leqslant R$ for every $0 \leqslant r < 1$. Hence for every $0 \leqslant r < 1$ we get

$$|(F*G)(rz)| = |(T_r F*G)(z)| \leqslant \sup\{|(F*G)(z)|:\ F \in H^{\circ\varphi},\ \|F\|_\varphi \leqslant R\}.$$

Passing to the limit with $r \to 1-$ we obtain

$$|(F*G)(z)| \leqslant \sup\{|(F*G)(z)|:\ F \in H^{\circ\varphi},\ \|F\|_\varphi \leqslant R\}.$$

From this follows the required inequality

$$\nu'_\varphi(z; G; R) \leqslant \sup\{2\pi|(F*G)(z)|:\ F \in H^{\circ\varphi},\ \|F\|_\varphi \leqslant R\}.$$

2.4. *If $G \in (H^{*\varphi})'_R$, where $R > 0$, then for any function $F \in H^{*\varphi}$ such that $\|F\|_\varphi \leqslant R$ and arbitrary $z \in D$ the inequality*

$$2\pi|(F*G)(z)| \leqslant R^{-1}\nu'_\varphi(G; R)\|F\|_\varphi$$

holds.

Proof. If $F = 0$ then also $F*G = 0$ and so in this case our inequality is satisfied. Let us assume then that $0 < \|F\|_\varphi \leqslant R$. I.3.4 implies that also $\|RF/\|F\|_\varphi\|_\varphi \leqslant R$. Thus for arbitrary $z \in D$ we get

$$2\pi|((RF/\|F\|_\varphi)*G)(z)| \leqslant \nu'_\varphi(z; G; R) \leqslant \nu'_\varphi(G; R).$$

The desired inequality now follows.

2.5. *For any function G analytic in D the term $R^{-1}\nu'_\varphi(G; R)$ is a non-decreasing function for $R > 0$. More precisely*

$$R^{-1}\nu'_\varphi(G; R) = \sup\{2\pi|(F*G)(z)|:\ F \in H^{*\varphi},\ \mu_\varphi(F) \leqslant R,\ z \in D\} \quad \text{for } R > 0.$$

Proof. This result is obtained from the following verification:

$$\begin{aligned}R^{-1}\nu'_\varphi(G; R) &= \sup\{2\pi|(R^{-1}F*G)(z)|:\ F \in H^{*\varphi},\ \|F\|_\varphi \leqslant R,\ z \in D\}\\ &= \sup\{2\pi|(F*G)(z)|:\ F \in H^{*\varphi},\ \|RF\|_\varphi \leqslant R,\ z \in D\}\\ &= \sup\{2\pi|(F*G)(z)|:\ F \in H^{*\varphi},\ \mu_\varphi(F) \leqslant R,\ z \in D\}.\end{aligned}$$

2.6. $(H_m^{*\varphi})'$ will denote the class of all function G analytic in D for which $R^{-1}\nu'_\varphi(G;R)\to 0$ as $R\to 0$.

Since $R^{-1}\nu'_\varphi(G;R)\to 0$ as $R\to 0$ implies $\nu'_\varphi(G;R)<\infty$ for some $R>0$, we have

$$(H_m^{*\varphi})' \subset (H^{*\varphi})'.$$

Now the following notations will be introduced.

$$(H_m^{*\varphi})'_0 = (H_m^{*\varphi})'\cap(H^{*\varphi})'_0 \text{ and } (H_m^{*\varphi})'_R = (H_m^{*\varphi})'\cap(H^{*\varphi})'_R \quad \text{for } R>0.$$

2.7. *For any $G\in(H^{*\varphi})'$ and any $F\in H^{*\varphi}$ the function $F*G$ is bounded in D.*

Proof. Since $G\in(H^{*\varphi})'$, it means that there is a number $R>0$ such that $\nu'_\varphi(G;R)<\infty$. For $F\in H^{*\varphi}$ we take a number $a>0$ such that $\|aF\|_\varphi\leqslant R$. Then we have

$$2\pi|(aF*G)(z)|\leqslant\nu'_\varphi(G;R)$$

and

$$|(F*G)(z)|\leqslant(2\pi a)^{-1}\nu'_\varphi(G;R)$$

for every $z\in D$. This proves that $F*G$ is bounded in D.

2.8. *For any $G\in(H^{*\varphi})'$ and any $F\in H^{\circ\varphi}$ the function $F*G$ has the radial limits*

$$\lim_{r\to 1-}(F*G)(re^{it}) = (F*G)(e^{it})$$

everywhere on the circumference $\{z\colon |z|=1\}$; the function $F*G$ completed with these limits is continuous in the circle $\bar D=\{z\colon |z|\leqslant 1\}$.

Proof. Since $G\in(H^{*\varphi})$, there is a number $R>0$ such that $\nu'_\varphi(G;R)<\infty$. In view of 3.6 of Section I $\|S_hF-F\|_\varphi\to 0$ as $h\to 0$ for $F\in H^{\circ\varphi}$ and $\|S_hF\|_\varphi=\|F\|_\varphi$ for every real h. This implies that for every $0<\varepsilon\leqslant R$ there is a $\delta>0$ such that for $|h_1-h_2|\leqslant\delta$ it is true that $\|S_{h_1-h_2}F-F\|_\varphi\leqslant\varepsilon$. Then also

$$\|S_{h_1}F-S_{h_2}F\|_\varphi = \|S_{h_2}(S_{h_1-h_2}F-F)\|_\varphi = \|S_{h_1-h_2}F-F\|_\varphi\leqslant\varepsilon.$$

By 2.4 we get for $|h_1-h_2|\leqslant\delta$ and arbitrary $0\leqslant r<1$

$$\begin{aligned}2\pi|(F*G)(re^{ih_1})-(F*G)(re^{ih_2})| &= 2\pi\big|\big((S_{h_1}F-S_{h_2}F)*G\big)(r)\big|\\ &\leqslant R^{-1}\nu'_\varphi(G;R)\|S_{h_1}F-S_{h_2}F\|_\varphi\leqslant R^{-1}\nu'_\varphi(G;R)\,\varepsilon.\end{aligned}$$

Thus the functions $f_r(t)=(F*G)(re^{it})$ are equicontinuous for $0\leqslant r<1$ with respect to t. From 2.7 we also deduce that these functions are uniformly bounded for $0\leqslant r<1$. Since $F*G$ is bounded in D, it follows from Fatou's theorem that for almost every t there exists a limit

$$\lim_{r\to 1-}f_r(t) = \lim_{r\to 1-}(F*G)(re^{it}) = (F*G)(e^{it}).$$

Applying Arzela's theorem, we conclude that the sequence $\{f_r\}$ converges uniformly as $r\to 1-$. Hence the limits $(F*G)(e^{it})$ exist for all t and the function $F*G$ completed with these limits is continuous in $\bar D$.

2.9. *For any $G\in(H_m^{*\varphi})'$ and any $F\in H^{*\varphi}$ the function $F*G$ has the radial limits*

$$\lim_{r\to 1-}(F*G)(re^{it}) = (F*G)(e^{it})$$

everywhere on the circumference $\{z\colon |z|=1\}$, and its completion with these limits is continuous in the circle $\bar D$.

Proof. Let $F\in H^{*\varphi}$. Then, for some constant $a>0$, $\mu_\varphi(aF)<\infty$ and by 1.3.6 $\mu_\varphi\big(\tfrac12 a(S_hF-F)\big)\to 0$ as $h\to 0$. Hence for every $\varepsilon_1>0$ there is a $\delta>0$ such that $\mu_\varphi\big(\tfrac12 a(S_{h_1-h_2}F-F)\big)\leqslant\varepsilon_1$ for $|h_1-h_2|\leqslant\delta$. Thus

$$\begin{aligned}\mu_\varphi\big(\tfrac12 a(S_{h_1}F-S_{h_2}F)\big) &= \mu_\varphi\Big(S_{h_2}\big(\tfrac12 a(S_{h_1-h_2}F-F)\big)\Big)\\ &= \mu_\varphi\big(\tfrac12 a(S_{h_1-h_2}F-F)\big)\leqslant\varepsilon_1.\end{aligned}$$

Now, since $G\in(H_m^{*\varphi})'$ and in view of 2.5 we conclude that for every $\varepsilon>0$ there is an $\varepsilon_1>0$ such that if $F\in H^{*\varphi}$ is such that $\mu_\varphi(F)\leqslant\varepsilon_1$ then $|(F*G)(z)|\leqslant\tfrac12 a\varepsilon$ for every $z\in D$. Thus, for $|h_1-h_2|\leqslant\delta$ and arbitrary $0\leqslant r<1$, we get for our function

$$|(F*G)(re^{ih_1})-(F*G)(re^{ih_2})| = 2a^{-1}\Big|\big(\tfrac12 a(S_{h_1}F-S_{h_2}F)*G\big)(r)\Big|\leqslant\varepsilon.$$

Application of a procedure similar to that in the proof of 2.8 yields the desired result.

3.1. *The functional defined as*

$$(+)\qquad \xi^\circ(F) = 2\pi(F*G)(1) = \lim_{r\to 1-}2\pi(F*G)(r) \quad \textit{for } F\in H^{\circ\varphi}$$

belongs for every $G\in(H^{*\varphi})'$ to $(H^{\circ\varphi})^\#$. Furthermore, for every $R>0$

$$\nu^\circ_\varphi(\xi^\circ;R) = \nu'_\varphi(G;R).$$

Proof. That this functional is a linear one is evident in view of 2.8 and 1.1. By 2.1 and 2.3 for any $R>0$ we get

$$\begin{aligned}\nu^\circ_\varphi(\xi^\circ;R) &= \sup\{2\pi|(F*G)(1)|\colon F\in H^{\circ\varphi},\ \|F\|_\varphi\leqslant R\}\\ &= \sup\{2\pi|(F*G)(r)|\colon F\in H^{*\varphi},\ \|F\|_\varphi\leqslant R,\ 0\leqslant r<1\}\\ &= \nu'_\varphi(G;R).\end{aligned}$$

This implies that $\xi^\circ\in(H^{\circ\varphi})^\#$.

3.2. *For every functional $\xi^\circ\in(H^{\circ\varphi})^\#$ there is a unique function G analytic in D such that $(+)$ is satisfied. This function belongs to $(H^{*\varphi})'$ and* *is of the form*

$$(++)\qquad G(z) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \xi^\circ(U_n)z^n \quad (z\in D),$$

where $U_n(z) = z^n$ *for* $z\in D$ *and* $n = 0, 1, 2, \ldots$

Proof. We shall first show that the function G defined by $(++)$ is analytic in D. To this end let us observe that for arbitrary $a > 0$ and $n = 0, 1, 2, \ldots$

$$\|aU_n\|_\varphi = \inf\{\varepsilon > 0\colon 2\pi\varphi(a/\varepsilon) \leqslant \varepsilon\}.$$

Let $\nu_\varphi^\circ(\xi^\circ; R) < \infty$ for $R > 0$. We choose such an $a_0 > 0$ that $\|a_0 U_n\|_\varphi \leqslant R$ for $n = 0, 1, 2, \ldots$ Then

$$|\xi^\circ(U_n)| \leqslant a_0^{-1}\nu_\varphi^\circ(\xi^\circ; R) \quad \text{for } n = 0, 1, 2, \ldots$$

This means that the coefficients of G are uniformly bounded. This allows us to conclude that G is analytic in D.

Let $F\in H^{\circ\varphi}$. For any $0 \leqslant r < 1$ the polynomial sequence $\{\sum_{k=0}^{n}\gamma_k(F)r^kU_k\}$ converges uniformly in the circle $\bar{D}$ to the function T_rF, and thus is norm convergent to T_rF. Hence

$$2\pi(F*G)(r) = \sum_{k=0}^{\infty}\gamma_k(F)\,\xi^\circ(U_k)r^k = \lim_{n\to\infty}\xi^\circ\Big(\sum_{k=0}^{n}\gamma_k(F)r^kU_k\Big) = \xi^\circ(T_rF).$$

From this, in view of II.6.6, we deduce that $\lim\limits_{r\to 1-}(F*G)(r) = (F*G)(1)$ exists and $(+)$ holds.

Next we shall show that the function G expressed in $(++)$ is the only analytic function in D satisfying $(+)$. Let G_1 be an analytic function in D for which $(+)$ holds. Since the functions U_n, $(n = 0, 1, 2, \ldots)$, belong to $H^{\circ\varphi}$ it follows that

$$\frac{1}{2\pi}\xi^\circ(U_n) = \lim_{r\to 1-}(U_n*G_1)(r) = \lim_{r\to 1-}\gamma_n(G_1)r^n = \gamma_n(G_1)$$

for $n = 0, 1, 2, \ldots$ Hence $G_1 = G$.

It remains to prove that $G\in(H^{*\varphi})'$. Taking into account 2.3 we have for $0\leqslant r<1$

$$\begin{aligned}\nu'_\varphi(r; G; R) &= \sup\{|\xi^\circ(T_rF)|\colon F\in H^{\circ\varphi},\ \|F\|_\varphi\leqslant R\}\\ &\leqslant \sup\{|\xi^\circ(F)|\colon F\in H^{\circ\varphi},\ \|F\|_\varphi\leqslant R\} = \nu_\varphi^\circ(\xi^\circ; R).\end{aligned}$$

This implies that $\nu'_\varphi(G; R)\leqslant \nu_\varphi^\circ(\xi^\circ; R)$ and so $G\in(H^{*\varphi})'$.

3.3. *The functional* ξ *defined as*

$$(+)\qquad \xi(F) = 2\pi(F*G)(1) = \lim_{r\to 1-}2\pi(F*G)(r) \quad \textit{for } F\in H^{*\varphi}$$

belongs to $(H_m^{*\varphi})^{\#}$ *for every function* $G\in(H_m^{*\varphi})'$. *Besides, for every* $R > 0$,

$$\nu_\varphi(\xi; R) = \nu'_\varphi(G; R).$$

Proof. That the functional so defined is a linear one follows immediately from 2.9 and 1.1. In view of 2.1 we get for arbitrary $R > 0$

$$\begin{aligned}\nu_\varphi(\xi; R) &= \sup\{2\pi|(F*G)(1)|\colon F\in H^{*\varphi},\ \|F\|_\varphi\leqslant R\}\\ &= \sup\{2\pi|(F*G)(r)|\colon F\in H^{*\varphi},\ \|F\|_\varphi\leqslant R,\ 0\leqslant r<1\} = \nu'_\varphi(G; R).\end{aligned}$$

Hence, in virtue of II. 2.4, $\xi\in(H_m^{*\varphi})^{\#}$.

3.4. *For every functional* $\xi\in(H_m^{*\varphi})^{\#}$ *there exists a unique function* G *analytic in* D *and such that* $(+)$ *holds for* $F\in H^{*\varphi}$. *This function belongs to* $(H_m^{*\varphi})'$ *and is defined by* $(++)$.

Proof. Let us denote by ξ° a functional which is the restriction of ξ to $H^{\circ\varphi}$. Obviously $\xi^\circ\in(H_m^{\circ\varphi})^{\#}$. In view of 3.2, G defined by $(++)$ is the only analytic function in D for which $(+)$ is satisfied for $F\in H^{\circ\varphi}$. This function is an element of $(H^{*\varphi})'$. Now it follows from 3.1 that for this function the equation

$$\nu_\varphi^\circ(\xi^\circ; R) = \nu'_\varphi(G; R) \quad \text{for every } R > 0$$

holds. Thus, by II.2.4, $G\in(H_m^{*\varphi})'$. Taking into account 3.3 and II.6.4, we see that G is the only function for which $(+)$ is satisfied for $F\in H^{*\varphi}$.

3.5. *If* G *is an analytic function in* D *such that*

$$\lim_{r\to 1-}(F*G)(r) = (F*G)(1)$$

exists for every $F\in H^{*\varphi}$, *then* $G\in(H^{*\varphi})'$. *What is more, the functional* ξ *defined by* $(+)$ *for* $F\in H^{*\varphi}$ *belongs to* $(H^{*\varphi})^{\#}$ *and is such that*

$$\nu_\varphi(\xi; R) = \nu'_\varphi(G; R) \quad \textit{for every } R > 0.$$

A analogous statement holds for $H^{\circ\varphi}$.

Proof. Let us observe that for $0\leqslant r<1$

$$2\pi(F*G)(r) = 2\pi(T_rF*G)(1) = \xi(T_rF) = T_r^{\#}\xi(F)$$

for every $F\in H^{*\varphi}$. We shall demonstrate that the functionals $T_r^{\#}\xi$ belong to $(H^{*\varphi})^{\#}$ if $0\leqslant r<1$. Namely in view of II.1.2, we have

$$\begin{aligned}|T_r^{\#}\xi(F)| &= 2\pi|(F*G)(r)| = 2\pi\Big|\sum_{n=0}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big|\\ &\leqslant 2\pi\varphi_{-1}\Big(\frac{2\|F\|_\varphi}{\pi(1-r)}\Big)\|F\|_\varphi\sum_{n=0}^{\infty}|\gamma_n(G)|\Big(\frac{2r}{1+r}\Big)^n.\end{aligned}$$

The series on the right-hand side of the above inequality is convergent for $0 \leqslant r < 1$ since G is analytic in D and $0 \leqslant \frac{2r}{1+r} < 1$. This implies that $T_r^{\#}\xi \in (H^{*\varphi})^{\#}$ for $0 \leqslant r < 1$. ξ is a pointwise limit on $H^{*\varphi}$ of the functional sequence $\{T_r^{\#}\xi\}$ as $r \to 1-$, and so in virtue of II.5.1 it also belongs to $(H^{*\varphi})^{\#}$. As in the proof of 3.3, we get $\nu_\varphi(\xi; R) = \nu'_\varphi(G; R)$ for every $R > 0$. This, in view of II.2.2, implies that $G \in (H^{*\varphi})'$.

3.6. *For any functional $\xi \in (H^{*\varphi})^{\#}$ there is at most one function G analytic in D such that $(+)$ holds for $F \in H^{*\varphi}$. Whenever this function exists, it belongs to $(H^{*\varphi})'$ and is defined by $(++)$.*

Proof. Let us assume that such a function G exists. Then by 3.2 it belongs to $(H^{*\varphi})'$, is represented by $(++)$ and is the only analytic function for which $(+)$ holds for $F \in H^{*\varphi}$.

3.7. If φ satisfies the condition (Δ_2), then Theorems 3.1 and 3.2 and also Theorems 3.3 and 3.4 give a full representation of norm continuous linear functionals on $H^{*\varphi}$ since then $H^{*\varphi} = H^{\circ\varphi}$ and $(H^{*\varphi})^{\#} = (H^{*\varphi}_m)^{\#}$.

3.8. *If φ does not satisfy the condition (Δ_2), then there are functionals $\xi \in (H^{*\varphi})^{\#}_0$ for which there exists no function G analytic in D and satisfying $(+)$ for $F \in H^{*\varphi}$. All non-trivial functionals $\xi \in (\tilde{H}^{*\varphi})^{\#}$ are good examples of such a situation.*

Proof. Let ξ be a non-trivial functional from $(\tilde{H}^{*\varphi})^{\#}$. Suppose that there is a function G analytic in D satisfying $(+)$ for $F \in H^{*\varphi}$. In virtue of 3.6, G is represented by $(++)$. The functions $U_n(z) = z^n$, $z \in D$, $n = 0, 1, 2, \ldots$ all belong to $H^{\circ\varphi}$. Thus $\xi(U_n) = 0$ for $n = 0, 1, 2, \ldots$ Hence $G(z) = 0$ for all $z \in D$ and furthermore $\xi(F) = 2\pi(F*G)(1) = 0$ for every $F \in H^{*\varphi}$ in contradiction to the assumption made on ξ.

4.1. From 3.1 and 3.2 and II. 2.7, II 2.8 and II. 6.1 it immediately follows that the space $(H^{*\varphi})'_R$ for $R > 0$ is complete relative to the norm $\nu'_\varphi(\cdot; R)$ and that $(H^{*\varphi}_m)'_R$ is its closed linear subspace.

For $G \in (H^{*\varphi})'$ let us designate

$$\varkappa'_\varphi(G) = \inf\{\varepsilon > 0 \colon \nu'_\varphi(G; 1/\varepsilon) \leqslant 1\}.$$

We deduce from 3.1 and 3.2 that the properties of the functional $\varkappa'_\varphi$ on the space $(H^{*\varphi})'$ are analogous to those of $\varkappa^\circ_\varphi$ on the space $(H^{\circ\varphi})^{\#}$. Thus in view of II.6.1 and II. 3.3 we see that, for an arbitrary sequence $\{G_n\} \subset (H^{*\varphi})'$, $\varkappa'_\varphi(G_n) \to 0$ if and only if $\nu'_\varphi(G_n; R) \to 0$ for every $R > 0$. From II. 6.1, II. 3.4, II. 3.5, II. 3.6 and II. 3.7 we infer that the space $(H^{*\varphi})'$ is complete with respect to the metric $d(G_1, G_2) = \varkappa'_\varphi(G_1 - G_2)$ and that the spaces $(H^{*\varphi}_m)'$, $(H^{*\varphi})'_R$, $(H^{*\varphi}_m)'_R$ for every $R > 0$, $(H^{*\varphi})'_0$, $(H^{*\varphi}_m)'_0$ are its closed linear subspaces. Furthermore, we conclude from 6.1 and II.3.8 that $G \in (H^{*\varphi})'$

is an element of $(H^{*\varphi})'_0$ if and only if $\varkappa'_\varphi(aG) \to 0$ as $a \to 0$ and hence we infer that $[(H^{*\varphi})'_0, \varkappa'_\varphi]$ is a Fréchet space and $(H^{*\varphi}_m)'_0$ is a closed linear subspace.

4.2. We denote by $(H^{*\varphi}_{vw})'$ a class of all functions $G \in (H^{*\varphi}_m)'$ for which the functional ξ defined by $(+)$ for $F \in H^{*\varphi}$ belongs to $(H^{*\varphi}_{vw})^{\#}$.

From II.3.10 we infer that $(H^{*\varphi}_{vw})'$ is a closed linear subspace of $[(H^{*\varphi}_m)'_0, \varkappa'_\varphi]$.

4.3. *If G is an analytic function in D, then $T_r G \in (H^{*\varphi}_{vw})'$ for $0 \leqslant r < 1$.*

Proof. For a fixed r, $0 \leqslant r < 1$, we define a functional

$$\xi(F) = 2\pi(F*G)(r) = 2\pi(F*T_r G)(1) \quad \text{for } F \in H^{*\varphi}.$$

Let $\{F_m\} \subset H^{*\varphi}$ be a sequence very weakly converging to 0. Then, by II.1.5 $\sup\|F_m\|_{\varphi_m} = R < \infty$ and $\gamma_n(F_m) \to 0$ as $m \to \infty$ for $n = 0, 1, 2, \ldots$ Applying II.1.2, we get

$$\Big|\sum_{n=k}^{\infty}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \varphi_{-1}\Big(\frac{2R}{\pi(1-r)}\Big) R \sum_{n=k}^{\infty}|\gamma_n(G)|\Big(\frac{2r}{1+r}\Big)^n$$

for each m and k. The series on the right-hand side of the above inequality is convergent since G is analytic in D and $0 \leqslant \frac{2r}{1+r} < 1$. From this we conclude that for every $\varepsilon > 0$ there is a k such that

$$2\pi\Big|\sum_{n=k}^{\infty}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \frac{\varepsilon}{2} \quad \text{for } m = 1, 2, \ldots$$

Now, the fact that $\gamma_n(F_m) \to 0$ with $m \to \infty$ for $n = 0, 1, 2, \ldots$ implies that for an already fixed $\varepsilon > 0$ there exists an m_0 such that for $m \geqslant m_0$

$$2\pi\Big|\sum_{n=0}^{k-1}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \frac{\varepsilon}{2}.$$

Thus for $m \geqslant m_0$ we get

$$|\xi(F_m)| = 2\pi|(F_m*G)(r)| = 2\pi\Big|\sum_{n=0}^{\infty}\gamma_n(F_m)\gamma_n(G)r^n\Big| \leqslant \varepsilon.$$

This signifies that $\xi \in (H^{*\varphi}_{vw})^{\#}$. Hence $T_r G \in (H^{*\varphi}_{vw})'$.

4.4. *A function $G\in(H^{*\varphi})'$ belongs to $(H^{*\varphi}_{vw})'$ if and only if $\varkappa'_\varphi(T_rG-G)\to 0$ as $r\to 1-$.*

Proof. Let $G\in(H^{*\varphi}_{vw})'$. Let us consider a functional ξ defined in $(+)$ for $F\in H^{*\varphi}$. Then for every $0\leqslant r<1$

$$(T_r^{\#}\xi-\xi)(F)=\xi(T_rF)-\xi(F)=2\pi\big((T_rF*G)(1)-(F*G)(1)\big)$$
$$=2\pi\big((F*T_rG)(1)-(F*G)(1)\big)=2\pi\big(F*(T_rG-G)\big)(1).$$

In view of 3.3 we get for every $R>0$

$$\nu_\varphi(T_r^{\#}\xi-\xi;R)=\nu'_\varphi(T_rG-G;R).$$

Further, we have

$$\varkappa_\varphi(T_r^{\#}\xi-\xi)=\varkappa'_\varphi(T_rG-G)\quad\text{for } 0\leqslant r<1.$$

Now, in view of II.4.3, $\varkappa'_\varphi(T_rG-G)\to 0$ as $r\to 1-$.

Conversely, if $\varkappa'_\varphi(T_rG-G)\to 0$ as $r\to 1-$ for $G\in(H^{*\varphi})'$ then the application of 4.3, 4.2, 4.1 yields $G\in(H^{*\varphi}_{vw})'$.

4.5. *The space $[(H^{*\varphi}_{vw})',\varkappa'_\varphi]$ is separable. Polynomials with rational coefficients form a dense set in this space.*

Proof. Let $G\in(H^{*\varphi}_{vw})'$ and ε be an arbitrary positive number. By 4.4 there exists an r, $0\leqslant r<1$, such that $\varkappa'_\varphi(T_rG-G)\leqslant\frac{\varepsilon}{2}$. Further, II.1.2 implies that for every k and every $F\in H^{*\varphi}$ such that $\|F\|_\varphi\leqslant\frac{2}{\varepsilon}$ it is true that

$$\Big|\sum_{n=k}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big|\leqslant\varphi_{-1}\Big(\frac{4}{\varepsilon\pi(1-r)}\Big)\frac{2}{\varepsilon}\sum_{n=k}^{\infty}|\gamma_n(G)|\Big(\frac{2r}{1+r}\Big)^n.$$

Reasoning as in the proof of 4.3, we conclude that there is a k such that

$$2\pi\Big|\sum_{n=k}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big|\leqslant\tfrac12\quad\text{for } F\in H^{*\varphi}\text{ such that }\|F\|_\varphi\leqslant\frac{2}{\varepsilon}.$$

Let $M=\sup\{\nu'_\varphi(\gamma_n;2/\varepsilon):\ n=0,1,2,\ldots,k-1\}$. We take rational numbers a_n such that

$$|\gamma_n(G)r^n-a_n|\leqslant(4kM\pi)^{-1}\quad\text{for } n=0,1,2,\ldots,k-1$$

and construct a polynomial $Q(z)=\sum_{n=0}^{k-1}a_nz^n$. Now, for $F\in H^{*\varphi}$ such that $\|F\|_\varphi\leqslant 2/\varepsilon$ we get

$$2\pi\big|\big(F*(T_rG-Q)\big)(1)\big|$$
$$\leqslant 2\pi\sum_{n=0}^{k-1}|\gamma_n(F)|\,|\gamma_n(G)r^n-a_n|+2\pi\Big|\sum_{n=k}^{\infty}\gamma_n(F)\gamma_n(G)r^n\Big|\leqslant 1.$$

This means that $\nu'_\varphi(T_rG-Q;2/\varepsilon)\leqslant 1$. Hence $\varkappa'_\varphi(T_rG-Q)\leqslant\varepsilon/2$. It follows now that

$$\varkappa'_\varphi(G-Q)\leqslant\varkappa'_\varphi(G-T_rG)+\varkappa'_\varphi(T_rG-Q)\leqslant\varepsilon.$$

4.6. *For any functions $G\in(H^{*\varphi})'$ and $F\in H^{*\varphi}$ such that $\varkappa'_\varphi(G)\|F\|_\varphi\leqslant 1$ the inequality*

$$2\pi|(F*G)(z)|\leqslant\varkappa'_\varphi(G)\|F\|_\varphi\quad\textit{for } z\in D$$

holds.

Proof. For an arbitrary $\varepsilon>\varkappa'_\varphi(G)$ it is true that $\nu'_\varphi(G;1/\varepsilon)\leqslant 1$. Thus in view of 2.4 we obtain

$$2\pi|(F*G)(z)|\leqslant\varepsilon\|F\|_\varphi$$

for every $F\in H^{*\varphi}$ such that $\varepsilon\|F\|_\varphi\leqslant 1$ and every $z\in D$. Hence we get

$$2\pi|(F*G)(z)|\leqslant\varkappa'_\varphi(G)\|F\|_\varphi$$

for every $F\in H^{*\varphi}$ such that $\varkappa'_\varphi(G)\|F\|_\varphi<1$ and every $z\in D$. If now $F\in H^{*\varphi}$ is such that $\varkappa'_\varphi(G)\|F\|_\varphi=1$, then for $0<\alpha<1$, $\varkappa'_\varphi(G)\|\alpha F\|_\varphi<1$ and thus also

$$2\pi|(F*G)(z)|\leqslant\alpha^{-1}\varkappa'_\varphi(G)\|\alpha F\|_\varphi.$$

Passing with $\alpha\to 1$ we get the required part of the inequality.

5.1. We say that a sequence $\{G_n\}\subset(H^{*\varphi})'$ *converges very weakly to* $G\in(H^{*\varphi})'$ if $\sup_n\varkappa'_\varphi(G_n-G)<\infty$ and

$$\sup\{|G_n(z)-G(z)|:\ z\in E\}\to 0\quad\text{as}\quad n\to\infty$$

for every closed set $E\subset D$.

A sequence $\{G_n\}\subset(H^{*\varphi})'$ converges very weakly to $G\in(H^{*\varphi})'$ if and only if the corresponding (according to formulae $(+)$ and $(++)$) sequence of functionals $\{\xi_n^\circ\}\subset(H^{\circ\varphi})^{\#}$ converges pointwise to a functional $\xi^\circ\in(H^{\circ\varphi})^{\#}$ coorresponding to the function G.

Proof. Let a sequence $\{G_n\}\subset(H^{*\varphi})'$ converge very weakly to $G\in(H^{*\varphi})'$. The condition $\sup_n\varkappa'_\varphi(G_n-G)<\infty$ implies, in view of 4.1 that $\sup_n\varkappa_\varphi^\circ(\xi_n^\circ-\xi^\circ)<\infty$, where ξ_n° and ξ° are functionals corresponding to G_n and G, respectively. Since $\{G_n(z)\}$ converges uniformly to $G(z)$ on the circumference $\{z:\ |z|=r\}$, where $0<r<1$, it now follows on the application of the Cauchy formulae that $\gamma_k(G_n)\to\gamma_k(G)$ as $n\to\infty$ for $k=0,1,2,\ldots$ This implies that

$$\frac{1}{2\pi}\xi_n^\circ(U_k)=\gamma_k(G_n)\to\gamma_k(G)=\frac{1}{2\pi}\xi^\circ(U_k)\quad\text{as } n\to\infty$$

for $k=0,1,2,\ldots$ In virtue of II.6.7 we now see that $\{\xi_n^\circ\}$ converges pointwise on $H^{\circ\varphi}$ to ξ°.

Conversely, let a sequence $\{\xi_n^\circ\} \subset (H^{\circ\varphi})^{\#}$ be pointwise convergent to $\xi^\circ \in (H^{\circ\varphi})^{\#}$. Thus by II.5.2 we have $\sup_n \varkappa_\varphi^\circ(\xi_n^\circ - \xi^\circ) < \infty$. Further, 4.1 implies that $\sup_n \varkappa_\varphi'(G_n - G) < \infty$. Take $0 < r < 1$. Observe that the set $X = \{F_\zeta\}$, $|\zeta| \leqslant r$, where $F_\zeta(z) = (1-\zeta z)^{-1}$ for $z \in D$, is compact in the space of functions analytic in D and continuous in $\bar{D}$, and hence it is also compact in $[H^{\circ\varphi}, \|\cdot\|_\varphi]$. Since it is compact, it is bounded in $[H^{\circ\varphi}, \|\cdot\|_\varphi]$. Thus in view of $\sup_n \varkappa_\varphi'(G_n - G) < \infty$, by 4.6 it follows that the functions

$$\big(F_\zeta * (G_n - G)\big)(1) = G_n(\zeta) - G(\zeta)$$

are uniformly bounded on the circle $\{\zeta\colon |\zeta| \leqslant r\}$. Further, since the sequence $\{\xi_n^\circ\}$ converges pointwise to ξ°, we get

$$2\pi\big(G_n(\zeta) - G(\zeta)\big) = \xi_n^\circ(F_\zeta) - \xi^\circ(F_\zeta) \to 0 \quad \text{as } n \to \infty.$$

Applying now the Vitali theorem we have for $0 < \varrho < r$

$$\sup\{|G_n(z) - G(z)|\colon |z| \leqslant \varrho\} \to 0 \quad \text{as } n \to \infty.$$

Since r has been an arbitrary number such that $0 < r < 1$, we conclude that for every closed set $E \subset D$

$$\sup\{|G_n(z) - G(z)|\colon z \in E\} \to 0 \quad \text{as } n \to \infty.$$

Hence $\{G_n\}$ converges very weakly to G.

5.2. Certain theorems about $(H^{\circ\varphi})^{\#}$ can be transferred to $(H^{*\varphi})'$ on account of 5.1. Thus from II.6.1 and II.5.3 we get

If a sequence $\{G_n\} \subset (H^{*\varphi})'$ *converges very weakly to* $G \in (H^{*\varphi})'$, *then*

$$\nu_\varphi'(G; R) \leqslant \liminf_{n\to\infty} \nu_\varphi'(G_n; R) \quad \textit{for every } R > 0$$

and

$$\varkappa_\varphi'(G) \leqslant \liminf_{n\to\infty} \varkappa_\varphi'(G_n).$$

And from II.6.7 we get

A sequence $\{G_n\} \subset (H^{*\varphi})'$ *is very weakly convergent if and only if* $\sup_n \varkappa_\varphi'(G_n) < \infty$ *and a sequence* $\{\gamma_k(G_n)\}$ *is convergent for* $k = 0, 1, 2, \ldots$

We shall also prove that:

A sequence $\{G_n\} \subset (H^{*\varphi})'$ *is very weakly convergent if and only if* $\sup_n \varkappa_\varphi'(G_n) < \infty$ *and a sequence* $\{G_n(z)\}$ *converges on a set of points* $z \in D$ *having a cluster point in* D.

Proof. Let us take, as in the proof of 5.1, for $0 < r < 1$ a set of functions $X = \{F_\zeta\}$, $|\zeta| \leqslant r$, where $F_\zeta(z) = (1-\zeta z)^{-1}$ for $z \in D$ and in view of 4.6

$$(F_\zeta * G_n)(1) = G_n(\zeta)$$

are uniformly bounded in the circle $\{\zeta\colon |\zeta| \leqslant r\}$. Let us take r, $0 < r < 1$, such that the cluster point of the set of points $z \in D$ for which the sequence $\{G_n(z)\}$ is convergent is in the circle $\{\zeta\colon |\zeta| < r\}$. Applying Vitali's theorem, we see that $\{G_n(z)\}$ is then uniformly convergent on the circumference $\{\zeta\colon |\zeta| = \frac{1}{2}r\}$. By the Cauchy formulas we find that $\{\gamma_k(G_n)\}$ is convergent for $k = 0, 1, 2, \ldots$ Thus the sequence $\{G_n\}$ is very weakly convergent.

5.3. *For any* $R, M > 0$ *the set* $\{G \in (H^{*\varphi})'\colon \nu_\varphi'(G; R) \leqslant M\}$ *is sequentially very weakly compact.*

Proof. Let $\{G_n\} \subset (H^{*\varphi})'$ be such a sequence that $\nu_\varphi'(G_n; R) \leqslant M$ for $n = 1, 2, \ldots$ For $0 < r < 1$, as in the proof of 5.1 we consider the set $X = \{F_\zeta\}$, $|\zeta| \leqslant r$, where $F_\zeta(z) = (1-\zeta z)^{-1}$ for $z \in D$. Since X is bounded in $[H^{\circ\varphi}, \|\cdot\|_\varphi]$, we find by 2.4 that the functions $(F_\zeta * G_n)(1) = G_n(\zeta)$ are uniformly bounded in the circle $\{\zeta\colon |\zeta| \leqslant r\}$. We take a sequence of points $\{z_m\}$, each different from another and such that $|z_m| \leqslant \frac{1}{2}r$. This sequence $\{z_m\}$ clearly has a cluster point in D. Now, from $\{G_n\}$ we substract a subsequence $\{G_{n_1}\}$ converging in z_1, from $\{G_{n_1}\}$ a subsequence $\{G_{n_2}\}$ converging in z_2 and so on. The diagonal sequence $\{G_{n_n}\}$ is obviously convergent at every point of the set $\{z_m\}$. Since $\sup_n \varkappa_\varphi'(M^{-1}G_{n_n}) \leqslant R^{-1}$, $\{G_{n_n}\}$ is by 5.2 very weakly convergent.

5.4. *For any function* $G \in (H^{*\varphi})'$ *the following relations hold*

$$\nu_\varphi'(T_rG; R) = \nu_\varphi'(r; G; R) \leqslant \nu_\varphi'(G; R) \quad \textit{for } 0 \leqslant r < 1 \textit{ and } R > 0,$$

$$\nu_\varphi'(S_hG; R) = \nu_\varphi'(G; R) \quad \textit{for } h \textit{ real and } R > 0;$$

hence

$$\varkappa_\varphi'(T_rG) \leqslant \varkappa_\varphi'(G) \quad \textit{for } 0 \leqslant r < 1$$

and

$$\varkappa_\varphi'(S_hG) = \varkappa_\varphi'(G) \quad \textit{for } h \textit{ real},$$

and further

$$\nu_\varphi'(G; R) = \lim_{r\to 1-} \nu_\varphi'(T_rG; R) \quad \textit{for } R > 0$$

and

$$\varkappa_\varphi'(G) = \lim_{r\to 1-} \varkappa_\varphi'(T_rG).$$

These properties are immediate consequences of 2.1 and 5.2.

From this and 5.2 it easily follows that

A function G *analytic in* D *belongs to* $(H^{*\varphi})'$ *if and only if*

$$\sup\{\varkappa_\varphi'(T_rG)\colon 0 \leqslant r < 1\} < \infty.$$

6.1. *If* $(H^{*\varphi})' = (H^{*\varphi})_0'$, *then* $(H^{*\varphi})'$ *can be equipped with a homogeneous norm given by*

$$\|G\|_\varphi' = \nu_\varphi'(G; 1) = \sup\{2\pi|(F*G)(z)|\colon F \in H^{*\varphi}, \|F\|_\varphi \leqslant 1, z \in D\}$$

for $G\in(H^{*\varphi})'$. *This norm is then equivalent with the norm* $\varkappa'_\varphi$. *Moreover, for every* $Y\subset(H^{*\varphi})'$ *we have* $\sup\{\|G\|'_\varphi\colon G\in Y\}<\infty$ *if and only if* $\sup\{\varkappa'_\varphi(G)\colon G\in Y\}<\infty$.

This can easily be deduced from 4.1 and II.8.1. Hence we also infer that

$(H^{*\varphi})'=(H^{*\varphi})'_0$ *if and only if* $(H^{\circ\varphi})^{\#}=(H^{\circ\varphi})^{\#}_0$ *or if and only if* $(H^{*\varphi})^{\#}=(H^{*\varphi})^{\#}_0$.

6.2. $(H^{*\varphi})'=(H^{*\varphi})'_0$ *if and only if* $(H^{*\varphi}_m)'=(H^{*\varphi}_m)'_0$.

Proof. If $(H^{*\varphi})'=(H^{*\varphi})'_0$ then obviously $(H^{*\varphi}_m)'=(H^{*\varphi}_m)'_0$. Assume that $(H^{*\varphi}_m)'=(H^{*\varphi}_m)'_0$. Then for arbitrary $R_2>R_1>0$ we have $(H^{*\varphi}_m)'_{R_1}=(H^{*\varphi}_m)'_{R_2}=(H^{*\varphi}_m)'$. Hence, in view of 4.1, the space $(H^{*\varphi}_m)'$ is complete with respect to the two norms $\nu'_\varphi(\cdot;R_1)$ and $\nu'_\varphi(\cdot;R_2)$. Since we have also for every $G\in(H^{*\varphi}_m)'$, $\nu'_\varphi(G;R_1)\leqslant\nu'_\varphi(G;R_2)$ it follows from the Closed Graph Theorem applied to the identity operator that the norms $\nu'_\varphi(\cdot;R_1)$ and $\nu'_\varphi(\cdot;R_2)$ are equivalent on $(H^{*\varphi}_m)'$. Let now $G\in(H^{*\varphi})'$. Then for some $R_0>0$ we have $\nu'_\varphi(G;R_0)<\infty$. By 4.3, we then have $T_rG\in(H^{*\varphi}_m)'$ for $0\leqslant r<1$ and by 5.4 we get $\nu'_\varphi(T_rG;R_0)\leqslant\nu'_\varphi(G;R_0)$. Thus we see that the set $\{T_rG\}$, $0\leqslant r<1$, is bounded in the space $[(H^{*\varphi}_m)',\ \nu'_\varphi(\cdot;R_0)]$. Hence, in virtue of the first part of this proof, the set $\{T_rG\}$, $0\leqslant r<1$, is bounded in $[(H^{*\varphi}_m)',\ \nu'_\varphi(\cdot;R)]$ for any $R>0$. This means that for any $R>0$

$$\sup\{\nu'_\varphi(T_rG;R)\colon\ 0\leqslant r<1\}<\infty.$$

This leads us to the conclusion, in view of 5.4, that $\nu'_\varphi(G;R)<\infty$ for any $R>0$. Hence $G\in(H^{*\varphi})'_0$ and $(H^{*\varphi})'=(H^{*\varphi})'_0$.

6.3. *If* φ *satisfies condition* (V_2), *then* $(H^{*\varphi})'=(H^{*\varphi})'_0$.

This follows immediately from 8.3 of Section II.

7.1. *If, for a natural number m the integral*

$$J_m(\varphi)=\int_1^\infty u^{-1-(1/m)}\varphi(u)\,du$$

exists, then the functions

$$I_k(z)=z^kI^{k+1}(z)=\frac{z^k}{(1-z)^{k+1}}\qquad(z\in D)$$

for $k=0,1,\ldots,m-1$ *belong to* $H^{\circ\varphi}$.

Proof. For $\frac{1}{2}\leqslant r<1$ and $0<t<\pi$ we have

$$\begin{aligned}|1-re^{it}|^2&=1-2r\cos t+r^2\\&=(1-r)^2+2r(1-\cos t)\geqslant 2r(1-\cos t)\\&=4r\sin^2\tfrac12 t\geqslant\sin^2\tfrac12 t\geqslant(t/\pi)^2.\end{aligned}$$

Therefore, for $\frac12\leqslant r<1$, $a>0$ and $0\leqslant k\leqslant m-1$ we have

$$\mu_\varphi(r;aI_k)=\int_0^{2\pi}\varphi\left(a\left|\frac{re^{ikt}}{(1-re^{it})^{k+1}}\right|\right)dt\leqslant 2\int_0^\pi\varphi\left(\frac{a}{|1-re^{it}|^{k+1}}\right)dt$$

$$\leqslant 2\int_0^\pi\varphi\left(a\left(\frac{\pi}{t}\right)^{k+1}\right)dt\leqslant 2\int_0^\pi\varphi\left(a\left(\frac{\pi}{t}\right)^m\right)dt=\frac{2\pi}{m}a^{1/m}\int_a^\infty u^{-1-(1/m)}\varphi(u)\,du.$$

Hence 7.1 follows.

7.2. *If, for a natural number m, the integral* $J_m(\varphi)$ *exists, then every function* $G\in(H^{*\varphi})'$ *has derivatives of order* $k=0,1,\ldots,m-1$ *bounded in the circle D. Moreover, these derivatives completed by their radial limits*

$$\lim_{r\to1-}G^{(k)}(re^{it})=G^{(k)}(e^{it})$$

are continuous functions in $\bar D$.

Proof. By virtue of 7.1 the functions I_k for $k=0,1,\ldots,m-1$ belong to $H^{\circ\varphi}$. On account of 1.3, 2.7 and 2.8 we see that the functions

$$(I_k*G)(z)=\frac{1}{k!}G^{(k)}(z)\qquad(z\in D)$$

for $k=0,1,\ldots,m-1$ are bounded in D and, completed by their radial limits, they are continuous in $\bar D$.

7.3. *Let G be a function analytic in D and continuous in* $\bar D$. *If its derivative* $G^{(1)}$ *can be completed to a continuous function in* $\bar D$, *then the function* $G(e^{it})$ *of a real variable t has a derivative for every t*

$$\frac{d}{dt}G(e^{it})=ie^{it}G^{(1)}(e^{it}),\qquad\text{where } G^{(1)}(e^{it})=\lim_{r\to1-}G^{(1)}(re^{it}).$$

Proof. For arbitrary real t_0 and t we have

$$G(e^{it})-G(e^{it_0})=\lim_{r\to1-}\big(G(re^{it})-G(re^{it_0})\big)=\lim_{r\to1-}\int_{t_0}^tG^{(1)}(re^{i\tau})ire^{i\tau}\,d\tau.$$

Since $G^{(1)}$ can be completed to a continuous function in $\bar D$, we get

$$G(e^{it})-G(e^{it_0})=\int_{t_0}^tG^{(1)}(e^{i\tau})ie^{i\tau}\,d\tau.$$

The integrand is continuous, and so the function $G(e^{it})$ of a real variable t has a derivative for every t equal to $G^{(1)}(e^{it})ie^{it}$.

7.4. *If, for a natural number m, the integral $J_m(\varphi)$ exists, then for every function $G \in (H^{*\psi})'$ the function of a real variable*

$$G(e^{it}) = \lim_{r\to 1-} G(re^{it})$$

has continuous derivatives $\frac{d^k}{dt^k} G(e^{it})$ *for* $k = 0, 1, \ldots, m-1$.

This follows easily from 7.2 and 7.3.

Theorems 7.2 and 7.4 can be viewed as generalizations and improvements of certain results of Walters [14].

If $\varphi(u) = u^p$, $0 < p < 1/m$, then

$$\int_1^\infty u^{-1-(1/m)} u^p\, du = \big((1/m)-p\big)^{-1} < \infty.$$

Thus, Theorems 7.2 and 7.4 hold for the Hardy spaces H^p, $0 < p < 1/m$

IV. THE CASE OF SPACES $H^{*\psi}$ FOR CONVEX ψ.

1.1. In this section we shall deal with the problem of representation of linear functionals on the Hardy-Orlicz spaces $H^{*\psi}$, where ψ is a convex φ-function satisfying conditions (0_1) and (∞_1). In this case we shall use a homogeneous norm $\|\cdot\|_{1\psi}$ for the space $H^{*\psi}$. A convex φ-function satisfies condition (V_2) and so $(H^{*\psi})' = (H^{*\psi})'_0$, and for the space $(H^{*\psi})'$ we shall use the usual norm

$$\begin{aligned}\|G\|'_\psi &= \sup\{2\pi|(F*G)(z)|:\ F\in H^{*\psi},\ \|F\|_\psi \leqslant 1,\ z\in D\}\\ &= \sup\{2\pi|(F*G)(z)|:\ F\in H^{*\psi},\ \|F\|_{1\psi}\leqslant 1,\ z\in D\}, \qquad (G\in (H^{*\psi})').\end{aligned}$$

1.2. *For every function $G\in(H^{*\psi})'$ there exists a function $g\in L^{*\psi'}$ such that G is the Cauchy integral of g and $\|G\|'_\psi = \|g\|^*_{(\psi')}$.*

Proof. Let ξ° be a functional from $(H^{\circ\psi})^\#$ corresponding to $G\in(H^{*\psi})'$. Since the space $[H^{\circ\psi}, \|\cdot\|_{1\psi}]$ is, by boundary functions, isometric isomorphic with a linear subspace of $[L^{\circ\psi}, \|\cdot\|^*_{1\psi}]$, there exists, by virtue of the Hahn-Banach theorem, a functional $l\in(L^{\circ\psi})^\#$ such that $\xi^\circ(F) = l\big(F(e^{it})\big)$ for $F\in H^{\circ\psi}$ and $\|l\|^\#_\psi = \|\xi^\circ\|^\#_\psi$. It is known ([3], p. 128; see I. 2.5) that for a functional l there is a function $g\in L^{*\psi'}$ such that

$$l(f) = \int_0^{2\pi} f(t)g(2\pi-t)\,dt \quad \text{for } f\in L^{\circ\psi} \text{ and } \|g\|^*_{(\psi')} = \|l\|^\#_\psi.$$

Hence

$$\xi^\circ(F) = \int_0^{2\pi} F(e^{-it})g(t)\,dt \quad \text{for } F\in H^{\circ\psi} \text{ and } \|g\|^*_{(\psi')} = \|\xi^\circ\|^\#_\psi.$$

Let G_1 be the Cauchy integral of g. In view of III. 1.6 we get

$$(F*G_1)(r) = \frac{1}{2\pi}\int_0^{2\pi} F(re^{-it})g(t)\,dt \quad \text{for } 0\leqslant r<1.$$

Thus, for $0\leqslant r<1$ and $F\in H^{\circ\psi}$ we have

$$2\pi(F*G_1)(r) = \xi^\circ(T_rF) = 2\pi(F*G)(r).$$

This yields

$$2\pi(F*G_1)(1) = \xi^\circ(F) = 2\pi(F*G)(1)$$

for every $F\in H^{\circ\psi}$. Now, by III. 3.2, we get $G_1 = G$. Hence G is the Cauchy integral of $g\in L^{*\psi'}$ and $\|G\|'_\psi = \|\xi^\circ\|^\#_\psi = \|g\|^*_{(\psi')}$.

1.3. *For every $g\in L^{*\psi'}$, its Cauchy integral G belongs to $(H^{*\psi})'$. Moreover $\|G\|'_\psi \leqslant \|g\|^*_{(\psi')}$.*

Proof. Let G be the Cauchy integral of $g\in L^{*\psi'}$. Then, by III. 1.6, we get for $F\in H^{*\psi}$ and $0\leqslant r<1$,

$$(F*G)(r) = \frac{1}{2\pi}\int_0^{2\pi} F(re^{-it})g(t)\,dt.$$

This, together with the fact that $[H^{*\psi}, \|\cdot\|_{1\psi}]$ is isometric isomorphic to a linear subspace of $[L^{*\psi}, \|\cdot\|^*_{1\psi}]$, implies that for every $F\in H^{*\psi}$ such that $\|F\|_{1\psi}\leqslant 1$ and $0\leqslant r<1$

$$\begin{aligned}2\pi|(F*G)(r)| &\leqslant \sup\Big\{\Big|\int_0^{2\pi} f(t)g(2\pi-t)\,dt\Big| :\ f\in L^{*\psi},\ \|f\|^*_{1\psi}\leqslant 1\Big\}\\ &= \sup\Big\{\Big|\int_0^{2\pi} f(t)g(t)\,dt\Big| :\ \mathscr{I}_\psi(f)\leqslant 1\Big\} = \|g\|^*_{(\psi')}.\end{aligned}$$

Recalling now III. 2.1 we see that

$$\|G\|'_\psi = \nu'_\psi(G;1) \leqslant \|g\|^*_{(\psi')}.$$

This accomplishes the proof.

1.4. $L^{*\psi'}_+$ will denote the class of all functions $f\in L^{*\psi'}$ for which

$$\int_0^{2\pi} f(t)e^{-int}\,dt = 0 \quad \text{for } n = 0, 1, 2, \ldots$$

Clearly, $L^{*\psi'}_+$ is a closed linear subspace of $[L^{*\psi'}, \|\cdot\|^*_{(\psi')}]$. We define a relation "$\sim$" in $L^{*\psi'}$ in the following fashion: $f_1\sim f_2$ is equivalent to $f_1-f_2\in L^{*\psi'}_+$. This relation is equivalence in $L^{*\psi'}$. The quotient space $L^{*\psi'}/\!\sim\; = L^{*\psi'}/L^{*\psi'}_+$ will be denoted by $\tilde{L}^{*\psi}$. As usual, $g^\sim$ will denote an equivalence class determined by g. The space $\tilde{L}^{*\psi'}$ is a normed linear space with addition and multiplication defined as

$$\tilde{g}_1+\tilde{g}_2 = (g_1+g_2)^{\sim}, \quad a\tilde{g} = (ag)^{\sim}$$

and with the norm

$$\|\tilde{g}\|^{\sim}_{(\psi')} = \inf\{\|f+g\|^{*}_{(\psi')} : f\in L^{*\psi'}_{+}\}.$$

The space $[(H^{*\psi})', \|\cdot\|'_{\psi}]$ *is isometric isomorphic to a space* $[\tilde{L}^{*\psi'}, \|\cdot\|^{\sim}_{(\psi')}]$. *This isomorphism establishes a Cauchy integral.*

Proof. In view of 1.2 and 1.3 it suffices to show that the Cauchy integral G of $g\in L^{*\psi'}$ is identically equal to 0 in D if and only if $g\in L^{*\psi'}_{+}$. This in turn follows directly from III. 1.5.

1.5. $H^{*\psi'}\subset (H^{*\psi})'$ *and* $\|F\|'_{\psi}\leqslant \|F\|_{(\psi')}$ *for every* $F\in H^{*\psi'}$.

Proof. Let $F\in H^{*\psi'}$. Then $F(e^{i\cdot})\in L^{*\psi'}$ and

$$\int_0^{2\pi} F(e^{it})e^{int}dt = 0 \quad \text{for } n = 1, 2, \ldots$$

From this we deduce on account of III. 1.7 that F is the Cauchy integral of $F(e^{i\cdot})$. Hence, by 1.3, we find that $F\in (H^{*\psi})'$ and $\|F\|'_{\psi}\leqslant \|F(e^{i\cdot})\|^{*}_{(\psi')} = \|F\|_{(\psi')}$, (see also III. 1.8).

2.1. Let us define for $g\in L^{*\psi'}$

$$\mathscr{I}^{\sim}_{\psi'}(\tilde{g}) = \inf\{\mathscr{I}_{\psi'}(f+g): f\in L^{*\psi'}_{+}\}.$$

The functional $\mathscr{I}^{\sim}_{\psi'}(\cdot)$ *has the following properties on the space* $\tilde{L}^{*\psi'}$:

1° $\mathscr{I}^{\sim}_{\psi'}(\tilde{g}) = 0$ *if and only if* $\tilde{g} = \tilde{0}$,

2° $\mathscr{I}^{\sim}_{\psi'}(a\tilde{g}) = \mathscr{I}^{\sim}_{\psi'}(\tilde{g})$ *for any number* a *such that* $|a| = 1$,

3° $\mathscr{I}^{\sim}_{\psi'}(a\tilde{g}_1+\beta\tilde{g}_2)\leqslant a\mathscr{I}^{\sim}_{\psi'}(\tilde{g}_1)+\beta\mathscr{I}^{\sim}_{\psi'}(\tilde{g}_2)$ *for any numbers* $a, \beta > 0$ *such that* $a+\beta = 1$,

4° $\|\tilde{g}\|^{\sim}_{(\psi')} = \inf\{\varepsilon^{-1}(1+\mathscr{I}^{\sim}_{\psi'}(\varepsilon\tilde{g})): \varepsilon > 0\}$.

Proof. If $\tilde{g} = \tilde{0}$, then obviously $\mathscr{I}^{\sim}_{\psi'}(\tilde{g}) = 0$. Conversely, let $\mathscr{I}^{\sim}_{\psi'}(\tilde{g}) = 0$. Then for every natural number m there is a $f_m\in L^{*\psi'}_{+}$ such that $\mathscr{I}_{\psi'}(f_m+g)\leqslant m^{-2}$. Since ψ and ψ' satisfy Young's inequality $uv\leqslant \psi(u)+\psi'(v)$ for $u, v\geqslant 0$, it means that for $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots$ we get

$$\left|\int_0^{2\pi} g(t)e^{-int}dt\right| = m\left|\int_0^{2\pi}(f_m(t)+g(t))\frac{1}{m}e^{-int}dt\right|$$

$$\leqslant m\left(\mathscr{I}_{\psi}\left(\frac{1}{m}e^{-in\cdot}\right)+\mathscr{I}_{\psi'}(f_m+g)\right)\leqslant m\left(2\pi\psi\left(\frac{1}{m}\right)+\frac{1}{m^2}\right).$$

Recalling that ψ satisfies the condition (0_1), we see that the right-hand side of this above inequality tends to 0 as $m\to\infty$. Hence

$$\int_0^{2\pi} g(t)e^{-int}dt = 0 \quad \text{for } n = 0, 1, 2, \ldots,$$

we have $\tilde{g} = \tilde{0}$.

Let a be a number such that $|a| = 1$. By the linearity of $L^{*\psi'}_{+}$ we get

$$\mathscr{I}^{\sim}_{\psi'}(a\tilde{g}) = \inf\{\mathscr{I}_{\psi'}(a(f+g)): f\in L^{*\psi'}_{+}\} = \inf\{\mathscr{I}_{\psi'}(f+g): f\in L^{*\psi'}_{+}\} = \mathscr{I}^{\sim}_{\psi'}(\tilde{g}).$$

For arbitrary numbers $a, \beta > 0$ such that $a+\beta = 1$ we now obtain

$$\mathscr{I}^{\sim}_{\psi'}(a\tilde{g}_1+\beta\tilde{g}_2) = \inf\{\mathscr{I}_{\psi'}(a(f_1+g_1)+\beta(f_2+g_2)): f_1, f_2\in L^{*\psi'}_{+}\}$$
$$\leqslant \inf\{a\mathscr{I}_{\psi'}(f_1+g_1)+\beta\mathscr{I}_{\psi'}(f_2+g_2): f_1, f_2\in L^{*\psi'}_{+}\} = a\mathscr{I}^{\sim}_{\psi'}(\tilde{g}_1)+\beta\mathscr{I}^{\sim}_{\psi'}(\tilde{g}_2).$$

Finally, we verify that

$$\|\tilde{g}\|^{\sim}_{(\psi')} = \inf\{\varepsilon^{-1}(1+\mathscr{I}_{\psi'}(\varepsilon(f+g))): \varepsilon > 0, f\in L^{*\psi'}_{+}\} = \inf\{\varepsilon^{-1}(1+\mathscr{I}^{\sim}_{\psi'}(\varepsilon\tilde{g})): \varepsilon > 0\}.$$

This means that *the space* $\tilde{L}^{*\psi'}_{+}$ *is a modular space with respect to the modular* $\mathscr{I}^{\sim}_{\psi'}(\cdot)$.

2.2. For $G\in (H^{*\psi})'$ let us designate

$$\mu'_{\psi}(G) = \mathscr{I}^{\sim}_{\psi'}(\tilde{g}),$$

where G is the Cauchy integral of $g\in L^{*\psi'}$.

From 2.1 and 1.4 it follows immediately that the functional $\mu'_{\psi}(\cdot)$ has the following properties on the space $(H^{*\psi})'$:

1° $\mu'_{\psi}(G) = 0$ if and only if $G = 0$,

2° $\mu'_{\psi}(aG) = \mu'_{\psi}(G)$ for any number a such that $|a| = 1$.

3° $\mu'_{\psi}(aG_1+\beta G_2)\leqslant a\mu'_{\psi}(G_1)+\beta\mu'_{\psi}(G_2)$ for any numbers $a, \beta > 0$ such that $a+\beta = 1$,

4° $\|G\|'_{\psi} = \inf\{\varepsilon^{-1}(1+\mu'_{\psi}(\varepsilon G)): \varepsilon > 0\}$.

Thus the space $(H^{*\psi})'$ *is a modular space with respect to the functional* $\mu'_{\psi}(\cdot)$.

We easily deduce from 4° that a sequence $\{G_n\}\subset (H^{*\psi})'$ is norm convergent to a $G\in (H^{*\psi})'$ if and only if $\mu'_{\psi}(a(G_n-G))\to 0$ as $n\to\infty$ for every $a > 0$. As in Section I, we say that a sequence $\{G_n\}\subset (H^{*\psi})'$ is modular convergent to a $G\in (H^{*\psi})'$ if $\mu'_{\psi}(a(G_n-G))\to 0$ as $n\to\infty$ for some $a > 0$ (depending, in general, on $\{G_n-G\}$).

2.3. *For any functions $F\in H^{*\psi}$ and $G\in(H^{*\psi})'$ the following inequality is satisfied*

$$2\pi|(F*G)(z)|\leqslant \mu_\psi(F)+\mu'_\psi(G) \quad \textit{for } z\in D.$$

Proof. Let $G\in(H^{*\psi})'$ be the Cauchy integral of $g\in L^{*\psi'}$. By virtue of III. 1.6 we then have

$$2\pi(F*G)(r)=\int_0^{2\pi}F(re^{-it})g(t)\,dt \quad \text{for } 0\leqslant r<1.$$

The application of Young's inequality for $F\in H^{*\psi}$, $f\in L^{*\psi'}_{+}$ and $0\leqslant r<1$ yields

$$2\pi|(F*G)(r)|=\Big|\int_0^{2\pi}F(re^{-it})\big(f(t)+g(t)\big)\,dt\Big|$$
$$\leqslant \mathscr{I}_\psi\big(F(re^{-i\cdot})\big)+\mathscr{I}_{\psi'}(f+g)\leqslant \mu_\psi(F)+\mathscr{I}_{\psi'}(f+g).$$

Hence

$$2\pi|(F*G)(r)|\leqslant \mu_\psi(F)+\mathscr{I}^{\sim}_{\psi'}(g^{\sim})=\mu_\psi(F)+\mu'_\psi(G)$$

for $F\in H^{*\psi}$, $G\in(H^{*\psi})'$ and $0\leqslant r<1$. For $z\in D$, $z=re^{it}$ we now get

$$2\pi|(F*G)(z)|=2\pi|(S_tF*G)(r)|\leqslant \mu_\psi(S_tF)+\mu'_\psi(G)$$
$$=\mu_\psi(F)+\mu'_\psi(G).$$

2.4. *If a sequence $\{G_n\}\subset(H^{*\psi})'$ is modular convergent to a $G\in(H^{*\psi})'$ then this sequence converges very weakly to G.*

Proof. Let $\{G_n\}\subset(H^{*\psi})'$ and $G\in(H^{*\psi})'$ and let $\mu'_\psi\big(\alpha(G_n-G)\big)\to 0$ as $n\to\infty$ for $\alpha>0$. Then there is an n_0 such that $\mu'_\psi\big(\alpha(G_n-G)\big)\leqslant 1$ for $n\geqslant n_0$. Hence we get for $n\geqslant n_0$

$$\|G_n-G\|'_\psi\leqslant \alpha^{-1}\Big(1+\mu'_\psi\big(\alpha(G_n-G)\big)\Big)\leqslant 2\alpha^{-1}.$$

Therefore

$$\sup_n\|G_n-G\|'_\psi\leqslant \sup\{2\alpha^{-1}, \|G_1-G\|'_\psi, \ldots, \|G_{n_0}-G\|'_\psi\}<\infty.$$

On account of 2.3 we get for $k=0,1,2,\ldots$, $\beta>0$ and $0\leqslant r<1$

$$2\pi|\gamma_k(G_n-G)|r^n=\alpha^{-1}\beta^{-1}2\pi\big|\big(\beta U_k*\alpha(G_n-G)\big)(r)\big|$$
$$\leqslant \alpha^{-1}\beta^{-1}\Big(\mu_\psi(\beta U_k)+\mu'_\psi\big(\alpha(G_n-G)\big)\Big)$$
$$\leqslant \alpha^{-1}\beta^{-1}\Big(2\pi\psi(\beta)+\mu'_\psi\big(\alpha(G_n-G)\big)\Big),$$

and, further,

$$2\pi|\gamma_k(G_n-G)|\leqslant \alpha^{-1}\beta^{-1}\Big(2\pi\psi(\beta)+\mu'_\psi\big(\alpha(G_n-G)\big)\Big).$$

It follows now that

$$\limsup_{n\to\infty}|\gamma_k(G_n-G)|\leqslant \alpha^{-1}\beta^{-1}\psi(\beta).$$

Since ψ satisfies condition (0_1) we see that the right-hand side of the above inequality tends to 0 as $\beta\to 0$. Thus $\gamma_k(G_n-G)\to 0$ as $n\to\infty$ for $k=0,1,2,\ldots$ By III. 5.2 we conclude that $\{G_n\}$ converges very weakly to G.

2.5. For a function $g\in L^1$ and $0\leqslant r<1$ we define

$$(T_rg)(t)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(t-\tau)+r^2}\,g(\tau)\,d\tau.$$

We shall demonstrate that

If $\mathscr{I}_{\psi'}(g)<\infty$ then $\mathscr{I}_{\psi'}\big(\frac{1}{2}(T_rg-g)\big)\to 0$ as $r\to 1-$ (cf. [15], IV(6.15)).

Proof. Applying Jensen's integral inequality, we easily get $\mathscr{I}_{\psi'}(T_rg)\leqslant \mathscr{I}_{\psi'}(g)$ for $0\leqslant r<1$. On the other hand, from the Fatou theorem ([2], p. 34) it follows that $\{(T_rg)(t)\}$ is convergent to $g(t)$ as $r\to 1-$ for almost every $t\in[0,2\pi)$. This means that the sequence $\big\{\psi'\big(|(T_rg)(t)|\big)\big\}$ converges to $\psi'\big(|g(t)|\big)$ as $r\to 1-$ for almost every $t\in[0,2\pi)$ and by Fatou lemma

$$\lim_{r\to 1-}\int_0^{2\pi}\psi'\big(|(T_rg)(t)|\big)\,dt=\int_0^{2\pi}\psi'\big(|g(t)|\big)\,dt.$$

It is known ([5]) that then we also have for every measurable set $E\subset[0,2\pi)$

$$\lim_{r\to 1-}\int_E\psi'\big(|(T_rg)(t)|\big)\,dt=\int_E\psi'\big(|g(t)|\big)\,dt.$$

From this it follows that

$$\limsup_{r\to 1-}\int_E\psi'\left(\frac{1}{2}|(T_rg)(t)-g(t)|\right)dt$$
$$\leqslant \limsup_{r\to 1-}\left(\frac{1}{2}\int_E\psi'\big(|(T_rg)(t)|\big)\,dt+\frac{1}{2}\int_E\psi'\big(|g(t)|\big)\,dt\right)$$
$$=\int_E\psi'\big(|g(t)|\big)\,dt.$$

The function $\psi'(|g(\cdot)|)$ is integrable on $[0,2\pi)$ and so, for every $\varepsilon>0$, there is a $\delta>0$ such that for every set $E\subset[0,2\pi)$ whose measure is $\operatorname{mes}E\leqslant\delta$ we have

$$\int_E\psi'\big(|g(t)|\big)\,dt\leqslant\varepsilon.$$

By virtue of the Egoroff theorem there exists a set $E \subset [0, 2\pi)$ whose measure is $\operatorname{mes} E \leqslant \delta$ and is such that $\{T_r g\}$ converges uniformly to g as $r \to 1-$ on a set $[0, 2\pi)\setminus E$. Hence we get

$$\limsup_{r\to 1-} \int_0^{2\pi} \psi'\big(\tfrac{1}{2}|(T_r g)(t) - g(t)|\big)\,dt \leqslant \limsup_{r\to 1-} \int_{[0,2\pi)\setminus E} \psi\big(\tfrac{1}{2}|(T_r g)(t) - g(t)|\big)\,dt + \int_E \psi'(|g(t)|)\,dt \leqslant \varepsilon.$$

From this we conclude that

$$\lim_{r\to 1-} \mathscr{I}_{\psi'}\big(\tfrac{1}{2}(T_r g - g)\big) = \lim_{r\to 1-} \int_0^{2\pi} \psi'\big(\tfrac{1}{2}|(T_r g)(t) - g(t)|\big)\,dt = 0.$$

2.6. *If $\mu'_\psi(G) < \infty$ for $G \in (H^{*\psi})'$ then $\mu'_\psi\big(\tfrac{1}{2}(T_r G - G)\big) \to 0$ as $r \to 1-$.*

Proof. Let $\mu'_\psi(G) < \infty$ for $G \in (H^{*\psi})'$. Then, in view of 2.2 and 2.1 we observe that there is a function $g \in L^{*\psi'}$ such that $\mathscr{I}_{\psi'}(g) < \infty$ and G is the Cauchy integral of g. Hence $\tfrac{1}{2}(T_r G - G)$ is the Cauchy integral of $\tfrac{1}{2}(T_r g - g)$. By 2.2 and 2.1 we get

$$\mu'_\psi\big(\tfrac{1}{2}(T_r G - G)\big) \leqslant \mathscr{I}_{\psi'}\big(\tfrac{1}{2}(T_r g - g)\big).$$

By virtue of 2.5 the right-hand side of this ineguality tends to 0 as $r \to 1-$. Hence $\mu'_\psi\big(\tfrac{1}{2}(T_r G - G)\big) \to 0$ as $r \to 1-$.

2.7. $(H^{*\psi})' = (H^{*\psi}_m)'$.

Proof. It suffices to show that $(H^{*\psi})' \subset (H^{*\psi}_m)'$. Let $G \in (H^{*\psi})'$. From 2.2, 4° it follows that $\mu'_\psi(\alpha G) < \infty$ for some $\alpha > 0$. By 2.6 for every $\varepsilon > 0$ there is an $0 \leqslant r < 1$ such that $\mu'_\psi\big(\tfrac{1}{2}\alpha(T_r G - G)\big) \leqslant \tfrac{1}{2}\alpha\varepsilon$. Let $\xi^\circ \in (H^{\circ\psi})^\#$ be a functional corresponding to a function G and let $\{F_n\} \subset H^{\circ\psi}$ be an arbitrary sequence such that $\mu_\psi(F_n) \to 0$ as $n \to \infty$. By 2.3 we get

$$\begin{aligned}|\xi^\circ(F_n)| &\leqslant |\xi^\circ(F_n) - T_r^\# \xi^\circ(F_n)| + |T_r^\# \xi^\circ(F_n)| \\ &= 2\pi\big|\big(F_n * (G - T_r G)\big)(1)\big| + |T_r^\# \xi^\circ(F_n)| \\ &\leqslant 2\alpha^{-1}\Big(\mu_\psi(F_n) + \mu'_\psi\big(\tfrac{1}{2}\alpha(G - T_r G)\big)\Big) + |T_r^\# \xi^\circ(F_n)| \\ &\leqslant 2\alpha^{-1}\mu_\psi(F_n) + \varepsilon + |T_r^\# \xi^\circ(F_n)|.\end{aligned}$$

Since $T_r^\# \xi^\circ \in (H^{\circ\psi}_{vw})^\# \subset (H^{\circ\psi}_m)^\#$, implies that

$$\limsup_{n\to\infty} |\xi^\circ(F_n)| \leqslant \varepsilon,$$

and hence $\xi^\circ(F_n) \to 0$ as $n \to \infty$. This proves that $\xi^\circ \in (H^{\circ\psi}_m)^\#$. By II. 6.4 there is a unique functional $\xi \in (H^{*\psi}_m)^\#$ such that $\xi(F) = \xi^\circ(F)$ for $F \in H^{\circ\psi}$.

It follows in view of III. 3.4 that the function

$$\frac{1}{2\pi}\sum_{n=0}^{\infty} \xi(U_n) z^n = \frac{1}{2\pi}\sum_{n=0}^{\infty} \xi^\circ(U_n) z^n = \sum_{n=0}^{\infty} \gamma_n(G) z^n = G(z) \qquad (z \in D),$$

belongs to $(H^{*\psi}_m)'$.

2.8. *Every functional $\xi \in (H^{*\psi})^\#$ can be uniquely represented in the form*

$$\xi = \xi_1 + \xi_2, \qquad \text{where } \xi_1 \in (H^{*\psi}_m)^\# \text{ and } \xi_2 \in (\tilde{H}^{*\psi})^\#.$$

Proof. Let ξ° be restriction of a functional $\xi \in (H^{*\psi})'$ to the domain $H^{\circ\psi}$. In view of III. 3.2 we see that there exists a unique function G analytic in D such that $\xi^\circ(F) = 2\pi(F * G)(1)$ for $F \in H^{\circ\psi}$; this function belongs to $(H^{*\psi})'$. By 2.7 it is $G \in (H^{*\psi}_m)'$. From III.3.3 it follows that $\xi_1(F) = 2\pi(F * G)(1)$ for $F \in H^{*\psi}$ belongs to $(H^{*\psi}_m)^\#$. Obviously, $\xi_2 = \xi - \xi_1$ belongs to $(\tilde{H}^{*\psi})^\#$. Let us also observe that this decomposition is unique.

3.1. *For every function $g \in L^{\circ\psi'}$ its Cauchy integral G belongs to $(H^{*\psi}_{vw})'$.*

Proof. Let G be the Cauchy integral of $g \in L^{\circ\psi'}$. Since $\mathscr{I}_{\psi'}(\alpha g) < \infty$ for every $\alpha > 0$, by 2.5 we have $\mathscr{I}_{\psi'}\big(\tfrac{1}{2}\alpha(T_r g - g)\big) \to 0$ as $r \to 1-$. This implies that $\|T_r g - g\|^*_{(\psi')} \to 0$ as $r \to 1-$. The function $T_r G - G$ is the Cauchy integral of $T_r g - g$ and thus, by 1.3,

$$\|T_r G - G\|'_\psi \leqslant \|T_r g - g\|^*_{(\psi')}.$$

Hence $\|T_r G - G\|'_\psi \to 0$ as $r \to 1-$ and this leads us to the conclusion $G \in (H^{*\psi}_{vw})'$.

3.2. Let us designate

$$L^{\circ\psi'}_+ = L^{\circ\psi'} \cap L^{*\psi'}_+.$$

As in 1.4 we equip $L^{\circ\psi'}$ with an equivalence relation $\simeq$, writing $f_1 \simeq f_2$ for $f_1, f_2 \in L^{\circ\psi'}$, if $f_1 - f_2 \in L^{\circ\psi'}_+$; the quotient space $L^{\circ\psi'}/\simeq\, = L^{\circ\psi'}/L^{\circ\psi'}_+$ we shall designate by $\tilde{L}^{\circ\psi'}$ and by $g^\simeq$ the equivalence class determined by g. Since $[L^{\circ\psi'}, \|\cdot\|^*_{(\psi')}]$ is complete and $L^{\circ\psi'}_+$ is, as we easily notice, its closed linear subspace, the space $\tilde{L}^{\circ\psi'}$ is, as is well known, a complete normed linear space where addition and scalar multiplication are defined as

$$g_1^\simeq + g_2^\simeq = (g_1 + g_2)^\simeq, \qquad a g^\simeq = (a g)^\simeq,$$

and the norm is given by

$$\|g^\simeq\|^\simeq_{(\psi')} = \inf\{\|f + g\|^*_{(\psi')} : f \in L^{\circ\psi'}_+\}.$$

We shall demonstrate that

The space $[(\tilde{L}^{\circ\psi'})^\#, \|\cdot\|^\#_{(\psi')}]$ is isometric isomorphic to the space $[H^{*\psi}, \|\cdot\|_{1\psi}]$. More specifically, for every functional $\eta^\simeq \in (\tilde{L}^{\circ\psi'})^\#$ there is a unique function

$F \in H^{*\varphi}$ *such that*

$$\eta^{\simeq}(g^{\simeq}) = \int_0^{2\pi} F(e^{-it}) g(t)\,dt \quad \text{for } g \in L^{\circ\varphi'}$$

and, conversely, every functional $\eta^{\simeq}$ *represented by this formula with a function* $F \in H^{*\varphi}$ *is a number of* $(\tilde{L}^{\circ\varphi'})^{\#}$ *and then* $\|\eta^{\simeq}\|^{\#}_{(\varphi')} = \|F\|_{1\varphi}$.

Proof. Let $\eta^{\simeq} \in (\tilde{L}^{\circ\varphi'})^{\#}$. The functional

$$\eta(g) = \eta^{\simeq}(g^{\simeq}) \quad \text{for } g \in L^{\circ\varphi'}$$

is clearly a member of $(L^{\circ\varphi'})^{\#}$ and its norm is $\|\eta\|^{\#}_{(\varphi')} = \|\eta^{\simeq}\|^{\#}_{(\varphi')}$. It is known ([3], p. 128; see I. 2.5) that for η there is a unique (precisely to a set of measure zero) function $f \in L^{*\varphi}$ such that

$$\eta(g) = \int_0^{2\pi} f(t) g(t)\,dt \quad \text{for } g \in L^{\circ\varphi'},$$

and, moreover $\|\eta\|^{\#}_{(\varphi')} = \|f\|^{*}_{1\varphi}$. We have, for $k = 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$,

$$\int_0^{2\pi} e^{-ikt} e^{-int}\,dt = 0.$$

This indicates that the functions $e^{-ik\cdot}$, $k = 1, 2, \ldots$, belong to $L_{+}^{\circ\varphi'}$. Hence

$$0 = \eta(e^{-ik\cdot}) = \int_0^{2\pi} f(t) e^{-ikt}\,dt \quad \text{for } k = 1, 2, \ldots$$

By III. 1.7 we now infer that the Cauchy integral F of $f(2\pi - t)$ is also the Poisson integral of $f(2\pi - t)$. This implies, on account of III. 1.8, that $F(e^{it}) = f(2\pi - t)$ for almost every $t \in [0, 2\pi)$ and $F \in H^{*\varphi}$. Obviously, $\|F\|_{1\varphi} = \|f\|^{*}_{1\varphi}$. Thus for $\eta^{\simeq} \in (\tilde{L}^{\circ\varphi'})^{\#}$ there is a unique function $F \in H^{*\varphi}$ such that

$$\eta^{\simeq}(g^{\simeq}) = \int_0^{2\pi} F(e^{i(2\pi - t)}) g(t)\,dt = \int_0^{2\pi} F(e^{-it}) g(t)\,dt$$

for $g \in L^{\circ\varphi'}$, and $\|\eta^{\simeq}\|^{\#}_{(\varphi')} = \|F\|_{1\varphi}$.

Now let $F \in H^{*\varphi}$. Let us consider the functional

$$\eta(g) = \int_0^{2\pi} F(e^{-it}) g(t)\,dt \quad \text{for } g \in L^{\circ\varphi'}.$$

This functional belongs to $(L^{\circ\varphi'})^{\#}$ since $F(e^{-i\cdot}) \in L^{*\varphi}$. Let $g \in L_{+}^{\circ\varphi'}$. The functional represented by the integral $\int_0^{2\pi} f(t) g(t)\,dt$ for $f \in L^{*\varphi}$ is modular continuous on $L^{*\varphi}$. Hence

$$\eta(g) = \lim_{r \to 1-} \int_0^{2\pi} F(re^{-it}) g(t)\,dt$$
$$= \lim_{r \to 1-} \sum_{n=0}^{\infty} \gamma_n(F) r^n \int_0^{2\pi} e^{-int} g(t)\,dt = 0.$$

This means that $\eta^{\simeq}(g^{\simeq}) = \eta(g)$ is well defined for $g^{\simeq} \in \tilde{L}^{\circ\varphi'}$ and belongs to $(\tilde{L}^{\circ\varphi'})^{\#}$.

3.3. *The space* $[\tilde{L}^{\circ\varphi'}, \|\cdot\|^{\simeq}_{(\varphi')}]$ *is isometric isomorphic to the space* $[(H^{*\varphi}_{vw})', \|\cdot\|'_{\varphi}]$. *This isomorphism establishes the operation of the Cauchy integral.*

Proof. Let Q be a polynomial. We have $Q(e^{i\cdot}) \in L^{\circ\varphi'}$ and clearly Q is the Cauchy integral of $Q(e^{i\cdot})$. By virtue of 1.3 and 3.2 we see that $\|Q\|'_{\varphi} \leqslant \|Q(e^{i\cdot})^{\simeq}\|^{\simeq}_{(\varphi')}$. On the other hand, by the very well known theorem stating the conditions of attaining the norm by functionals there is a functional $\eta^{\simeq} \in (\tilde{L}^{\circ\varphi'})^{\#}$ such that $\|\eta^{\simeq}\|^{\#}_{(\varphi')} = 1$ and $\|Q(e^{i\cdot})^{\simeq}\|^{\simeq}_{(\varphi')} = \eta^{\simeq}(Q(e^{i\cdot}))$. Hence, by 3.2 there is a function $F \in H^{*\varphi}$ such that $\|F\|_{1\varphi} = 1$ and

$$\|Q(e^{i\cdot})^{\simeq}\|^{\simeq}_{(\varphi')} = \int_0^{2\pi} F(e^{-it}) Q(e^{it})\,dt = 2\pi (F * Q)(1).$$

This implies that $\|Q(e^{i\cdot})^{\simeq}\|^{\simeq}_{(\varphi')} \leqslant \|Q\|'_{\varphi}$. Thus we have demonstrated that $\|Q\|'_{\varphi} = \|Q(e^{i\cdot})^{\simeq}\|^{\simeq}_{(\varphi')}$ for every polynomial Q.

By III. 4.5 polynomials form a dense set in $[(H^{*\varphi}_{vw})', \|\cdot\|'_{\varphi}]$. The space $[\tilde{L}^{\circ\varphi'}, \|\cdot\|^{\simeq}_{(\varphi')}]$ is complete. Hence $[(H^{*\varphi}_{vw})', \|\cdot\|'_{\varphi}]$ is isometric isomorphic via the Cauchy integral operator to a closed linear subspace of $[\tilde{L}^{\circ\varphi'}, \|\cdot\|^{\simeq}_{(\varphi')}]$. Applying 3.1 and taking into account that the Cauchy integral G of $g \in L^{\circ\varphi'}$ is identically equal to 0 in D if and only if $g \in L_{+}^{\circ\varphi'}$, we get 3.3.

3.4. Certain corollaries may be deduced from 3.3. Thus we infer that: Every function $G \in (H^{*\varphi}_{vw})'$ is the Cauchy integral of some $g \in L^{\circ\varphi'}$.

Further, $\tilde{L}^{\circ\varphi'}$ is isomorphic to the space $\{\tilde{g} \in \tilde{L}^{*\varphi'} : g \in L^{\circ\varphi'}\}$; this isomorphism is clearly a mapping of classes $g^{\simeq} \in \tilde{L}^{\circ\varphi'}$ onto classes $\tilde{g} \in \tilde{L}^{*\varphi'}$. Besides, on account of 1.4 and 3.3 we have $\|\tilde{g}\|^{\sim}_{(\varphi')} = \|g^{\simeq}\|^{\simeq}_{(\varphi')}$ for $g \in L^{\circ\varphi'}$.

3.5. $H^{\circ\varphi'} \subset (H^{*\varphi}_{vw})'$; *moreover* $\|F\|'_{\varphi} \leqslant \|F\|_{(\varphi')}$ *for every* $F \in H^{\circ\varphi'}$.

Proof. In view of 1.5 we have $H^{\circ\varphi'} \subset H^{*\varphi'} \subset (H^{*\varphi})'$ and $\|F\|'_{\varphi} \leqslant \|F\|_{(\varphi')}$ for every $F \in H^{\circ\varphi'}$. Hence, for $F \in H^{\circ\varphi'}$ and $0 \leqslant r < 1$, we get

$$\|T_r F - F\|'_{\varphi} \leqslant \|T_r F - F\|_{(\varphi')}.$$

For $F \in H^{\circ\psi'}$ we have $\|T_rF-F\|_{(\psi')} \to 0$ as $r\to 1-$. Thus, for $F\in H^{\circ\psi'}$, we have $\|T_rF-F\|'_\psi \to 0$ as $r\to 1-$. This indicates that $H^{\circ\psi'} \subset (H^{*\psi}_{vw})'$.

4.1. *For every $F\in H^{*\psi}$ the functional defined as*

$$(+)\qquad \eta^\circ(G) = \lim_{r\to 1-} 2\pi (F*G)(r) = 2\pi(F*G)(1) \quad \textit{for } G\in (H^{*\psi}_{vw})'$$

belongs to $((H^{*\psi}_{vw})')^{\#}$; moreover $\|\eta^\circ\|^{\#}_\psi = \|F\|_{1\psi}$. On the other hand, for every functional $\eta^\circ \in ((H^{*\psi}_{vw})')^{\#}$ there exists a unique function F analytic in D such that $(+)$ holds; this function belongs to $H^{*\psi}$ and is represented by

$$(++)\qquad F(z) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \eta^\circ(U_n) z^n \quad (z\in D).$$

Proof. Let $F\in H^{*\psi}$ and let η° be the functional defined in $(+)$. We have

$$|\eta^\circ(G)| = 2\pi |(F*G)(1)| \leqslant \|F\|_{1\psi}\|G\|'_\psi \quad \text{for every } G\in (H^{*\psi}_{vw})'$$

and so $\eta^\circ \in ((H^{*\psi}_{vw})')^{\#}$ and $\|\eta^\circ\|^{\#}_\psi \leqslant \|F\|_{1\psi}$.

Conversely, let $\eta^\circ \in ((H^{*\psi}_{vw})')^{\#}$. Taking into account 3.3, we set $\eta^{\simeq}(g^{\simeq}) = \eta^\circ(G)$, where G is the Cauchy integral of $g\in L^{\circ\psi'}$. It follows that $\eta^{\simeq} \in (\tilde{L}^{\circ\psi'})^{\#}$ and $\|\eta^{\simeq}\|^{\#}_{(\psi')} = \|\eta^\circ\|^{\#}_\psi$. By 3.2 there is a function $F\in H^{*\psi}$ such that

$$\eta^{\simeq}(g^{\simeq}) = \int_0^{2\pi} F(e^{-it})g(t)\,dt \quad \text{for } g\in L^{\circ\psi'} \text{ and } \|\eta^{\simeq}\|^{\#}_{(\psi')} = \|F\|_{1\psi}.$$

Since for a fixed $g\in L^{\circ\psi'}$ the functional represented by the integral $\int_0^{2\pi} f(t)g(t)\,dt$ for $f\in L^{*\psi}$ is modular continuous on $L^{*\psi}$, we get

$$\eta^{\simeq}(g^{\simeq}) = \lim_{r\to 1-}\int_0^{2\pi} F(re^{-it})g(t)\,dt \quad \text{for } g\in L^{\circ\psi'}.$$

In view of III. 1.6 we then have

$$\eta^\circ(G) = \lim_{r\to 1-} 2\pi(F*G)(r) = 2\pi(F*G)(1) \quad \text{for } G\in (H^{*\psi}_{vw})'$$

and moreover $\|\eta^\circ\|^{\#}_\psi = \|F\|_{1\psi}$. This function F is uniquely determined, since its coefficients are uniquely determined:

$$\gamma_n(F) = \lim_{r\to 1-}(F*U_n)(r) = (F*U_n)(1) = \frac{1}{2\pi}\eta^\circ(U_n) \quad \text{for } n = 0,1,2,\ldots$$

This accomplishes the proof of $(++)$.

4.2. *If $\eta\in ((H^{*\psi})')^{\#}$ then $T_r^{\#}\eta \in ((H^{*\psi})'_{vw})^{\#}$ for $0\leqslant r<1$.*

Proof. Let $\eta\in((H^{*\psi})')^{\#}$ and let r be a number such that $0\leqslant r<1$. Further, let $\{G_n\}\subset (H^{*\psi})'$ be a sequence very weakly converging to 0.

Then $\{T_rG_n(e^{it})\}$ is uniformly convergent to 0 sequence of continuous functions. Hence $\|T_rG_n(e^{it})\|^*_{(\psi')} \to 0$ as $n\to\infty$. This, on account of 1.3 yields $\|T_rG_n\|'_\psi \to 0$ as $n\to\infty$, and so we get

$$T_r^{\#}\eta(G_n) = \eta(T_rG_n) \to 0 \quad \text{as } n\to\infty.$$

This proves that $T_r^{\#}\eta\in ((H^{*\psi})'_{vw})^{\#}$.

4.3. *If $\eta\in((H^{*\psi})'_m)^{\#}$ then*

$$\eta(G) = \lim_{r\to 1-} T_r^{\#}\eta(G) \quad \textit{for every } G\in (H^{*\psi})'.$$

Proof. Let $G\in (H^{*\psi})'$. We deduce from 2.2 that $\mu'_\psi(\alpha G)<\infty$ for some $\alpha>0$, and from 2.6 that $\mu'_\psi(\tfrac12\alpha(T_rG-G)) \to 0$ as $r\to 1-$. Hence $\{T_rG\}$ is modular convergent to G as $r\to 1-$. This yields 4.3.

4.4. *For every function $F\in H^{*\psi}$ the functional defined by*

$$(+)\qquad \eta(G) = \lim_{r\to 1-} 2\pi(F*G)(r) = 2\pi(F*G)(1) \quad \textit{for } G\in (H^{*\psi})'$$

belongs to $((H^{*\psi})'_m)^{\#}$; moreover $\|\eta\|^{\#}_\psi = \|F\|_{1\psi}$. On the other hand, for every functional $\eta\in((H^{*\psi})'_m)^{\#}$ there is a unique function F analytic in D and such that $(+)$ holds; this function belongs to $H^{*\psi}$ and is defined by $(++)$.

Proof. Let $F\in H^{*\psi}$ and let η be the functional defined in $(+)$. The existence of the limit in $(+)$ follows from 2.7 and III. 2.9. Since $\mu_\psi(\alpha F)<\infty$ for some $\alpha>0$, it follows from I. 3.6 that for every $\varepsilon>0$ there is an $0\leqslant r<1$ such that $\mu_\psi(\tfrac12\alpha(T_rF-F)) \leqslant \tfrac12\alpha\varepsilon$. Now let $\{G_n\}\subset (H^{*\psi})'$ be an arbitrary sequence such that $\mu'_\psi(G_n)\to 0$ as $n\to\infty$. By virtue of 2.3 we get

$$\begin{aligned}|\eta(G_n)| &\leqslant |\eta(G_n) - T_r^{\#}\eta(G_n)| + |T_r^{\#}\eta(G_n)| \\ &= 2\pi\left|((F-T_rF)*G_n)(1)\right| + |T_r^{\#}\eta(G_n)| \\ &\leqslant 2\alpha^{-1}\left(\mu_\psi(\tfrac12\alpha(F-T_rF)) + \mu'_\psi(G_n)\right) + |T_r^{\#}\eta(G_n)| \\ &\leqslant \varepsilon + 2\alpha^{-1}\mu'_\psi(G_n) + |T_r^{\#}\eta(G_n)|.\end{aligned}$$

Since

$$|\eta(G)| = 2\pi|(F*G)(1)| \leqslant \|F\|_{1\psi}\|G\|'_\psi \quad \text{for } G\in (H^{*\psi})'$$

we have $\eta\in((H^{*\psi})')^{\#}$ and $\|\eta\|^{\#}_\psi \leqslant \|F\|_{1\psi}$. Applying 4.2 and 2.4 we get $T_r^{\#}\eta \in ((H^{*\psi})'_{vw})^{\#} \subset ((H^{*\psi})'_m)^{\#}$. Hence

$$\limsup_{n\to\infty}|\eta(G_n)| \leqslant \varepsilon.$$

This indicates that $\eta(G_n) \to 0$ as $n \to \infty$. This yields $\eta \in \big((H^{*\varphi})'_m\big)^{\#}$.

Conversely, let $\eta \in \big((H^{*\varphi})'_m\big)^{\#}$. Then the functional η°, being a restriction of η to $(H^{*\varphi}_{vw})'$, belongs to $\big((H^{*\varphi}_{vw})'\big)^{\#}$. By 4.1 there is a unique function F analytic in D and such that $(+)$ holds for $G \in (H^{*\varphi}_{vw})'$; this function belongs to $H^{*\varphi}$ and is determined by $(++)$. From 4.3 we conclude that $(+)$ holds for η and F for every $G \in (H^{*\varphi})'$. Further we notice that in $(++)$ η° can be replaced by η and that

$$\|F\|_{1\varphi} = \|\eta^\circ\|^{\#}_\varphi = \sup\{|\eta^\circ(G)|:\ G \in (H^{*\varphi}_{vw})',\ \|G\|'_\varphi \leqslant 1\}$$
$$\leqslant \sup\{|\eta(G)|:\ G \in (H^{*\varphi})',\ \|G\|'_\varphi \leqslant 1\} = \|\eta\|^{\#}_\varphi.$$

This accomplishes the proof of 4.4.

4.5. *Every functional* $\eta \in \big((H^{*\varphi})'\big)^{\#}$ *is represented uniquely in the form*

$$\eta = \eta_1 + \eta_2, \quad \textit{where} \quad \eta_1 \in \big((H^{*\varphi})'_m\big)^{\#} \quad \textit{and} \quad \eta_2 \in \big((H^{*\varphi})'^{\sim}\big)^{\#}.$$

$\big((H^{*\varphi})'^{\sim}\big)^{\#}$ *designates the space of such functionals* $\eta \in \big((H^{*\varphi})'\big)^{\#}$ *that* $\eta(G) = 0$ *for every* $G \in (H^{*\varphi}_{vw})'$.

The proof of this theorem, being quite similar to that of 2.8, is omitted.

4.6. *For every function* $F \in H^{\circ\varphi}$ *the functional* η *defined by* $(+)$ *on* $(H^{*\varphi})'$ *belongs to* $\big((H^{*\varphi})'_{vw}\big)^{\#}$. *Conversely, for every functional* $\eta \in \big((H^{*\varphi})'_{vw}\big)^{\#}$ *the function* F *defined by* $(++)$ *belongs to* $H^{\circ\varphi}$.

Proof. Let $F \in H^{\circ\varphi}$ and let η be a functional defined by $(+)$ on $(H^{*\varphi})'$. Further, let $\{G_n\} \subset (H^{*\varphi})'$ be a sequence very weakly converging to 0, and such that $\sup\limits_n \|G_n\|'_\varphi < M < \infty$. Since $F \in H^{\circ\varphi}$, by the application of I.3.6 we see that for every $\varepsilon > 0$ there is an $0 \leqslant r < 1$ such that $\|T_r F - F\|_{1\varphi} \leqslant \varepsilon M^{-1}$. Hence we get

$$|\eta(G_n)| \leqslant |\eta(G_n) - T_r^{\#}\eta(G_n)| + |T_r^{\#}\eta(G_n)|$$
$$= 2\pi\big|\big((F - T_r F) * G_n\big)(1)\big| + |T_r^{\#}\eta(G_n)|$$
$$\leqslant \|F - T_r F\|_{1\varphi}\|G_n\|'_\varphi + |T_r^{\#}\eta(G_n)| \leqslant \varepsilon + |T_r^{\#}\eta(G_n)|.$$

By 4.2 we have $T_r^{\#}\eta \in \big((H^{*\varphi})'_{vw}\big)^{\#}$ and so

$$\limsup_{n\to\infty} |\eta(G_n)| \leqslant \varepsilon.$$

This means that $\eta(G_n) \to 0$ as $n \to \infty$, and thus $\eta \in \big((H^{*\varphi})'_{vw}\big)^{\#}$. Conversely, let $\eta \in \big((H^{*\varphi})'_{vw}\big)^{\#}$ and F let be a function defined by $(++)$. By 2.4 and 4.4 we get $F \in H^{*\varphi}$ and

$$\|T_r F - F\|_{1\varphi} = \|T_r^{\#}\eta - \eta\|^{\#}_\varphi \quad \text{for } 0 \leqslant r < 1.$$

In view of the fact that $T_r F \in H^{\circ\varphi}$ for $0 \leqslant r < 1$ and that $H^{\circ\varphi}$ is closed in $[H^{*\varphi}, \|\cdot\|_{1\varphi}]$ it suffices to show only that $\|T_r^{\#}\eta - \eta\|^{\#}_\varphi \to 0$ as $r \to 1-$. Let us suppose this is not so. Then there exist a number $\varepsilon > 0$, a sequence $\{r_n\}$ and $\{G_n\} \subset (H^{*\varphi})'$ such that $0 \leqslant r_n < 1$, $r_n \to 1$ $(n \to \infty)$, $\|G_n\|'_\varphi \leqslant 1$ and $|T_{r_n}^{\#}\eta(G_n) - \eta(G_n)| \geqslant \varepsilon$ for $n = 1, 2, \ldots$ By III. 5.3 the ball $\{G \in (H^{*\varphi})':\ \|G\|'_\varphi \leqslant 1\}$ is sequentially very weakly compact. Thus we can find a very weakly convergent subsequence $\{G_{n_k}\}$ of $\{G_n\}$. Applying a procedure similar to that used in the proof of II.4.3, we conclude that $\{T_{r_{n_k}} G_{n_k} - G_{n_k}\}$ converges very weakly to 0. Hence we get

$$\varepsilon \leqslant |T_{r_{n_k}}^{\#}\eta(G_{n_k}) - \eta(G_{n_k})| = |\eta(T_{r_{n_k}} G_{n_k} - G_{n_k})| \to 0 \quad \text{as } k \to \infty.$$

This is a contradiction. Thus we have proved that $\|T_r^{\#}\eta - \eta\|^{\#}_\varphi \to 0$ as $r \to 1-$. This means that $F \in H^{\circ\varphi}$.

4.7. By $(H^{*\varphi})''$ we shall designate the class of all functions F analytic in D for which

$$\|F\|''_\varphi = \sup\{2\pi|(F * G)(z)|:\ G \in (H^{*\varphi})',\ \|G\|'_\varphi \leqslant 1,\ z \in D\} < \infty.$$

The following equality is true:

$$[(H^{*\varphi})'',\ \|\cdot\|''_\varphi] = [H^{*\varphi},\ \|\cdot\|_{1\varphi}].$$

Proof. Let $F \in H^{*\varphi}$. Then for arbitrary $G \in (H^{*\varphi})'$ and $z \in D$ we have

$$2\pi|(F * G)(z)| \leqslant \|F\|_{1\varphi}\|G\|'_\varphi.$$

This implies that $\|F\|''_\varphi \leqslant \|F\|_{1\varphi}$ and further $F \in (H^{*\varphi})''$. Let now η° be a functional on $(H^{*\varphi}_{vw})'$ defined by $(+)$ for F. In view of 4.1 we get

$$\|F\|_{1\varphi} = \|\eta^\circ\|^{\#}_\varphi = \sup\{2\pi|(F * G)(1)|:\ G \in (H^{*\varphi}_{vw})',\ \|G\|'_\varphi \leqslant 1\}$$
$$\leqslant \sup\{2\pi|(F * G)(z)|:\ G \in (H^{*\varphi})',\ \|G\|'_\varphi \leqslant 1,\ z \in D\} = \|F\|''_\varphi.$$

Thus we have $H^{*\varphi} \subset (H^{*\varphi})''$ and $\|F\|''_\varphi = \|F\|_{1\varphi}$ for every $F \in H^{*\varphi}$. Let now $F \in (H^{*\varphi})''$. Then for every $0 \leqslant r < 1$ we have $T_r F \in H^{*\varphi}$. Hence we get for $0 \leqslant r < 1$

$$\|T_r F\|_{1\varphi} = \|T_r F\|''_\varphi = \sup\{2\pi|(F * G)(zr)|:\ G \in (H^{*\varphi})',\ \|G\|'_\varphi \leqslant 1,\ z \in D\}$$
$$\leqslant \sup\{2\pi|(F * G)(z)|:\ G \in (H^{*\varphi})',\ \|G\|'_\varphi \leqslant 1,\ z \in D\} = \|F\|''_\varphi.$$

From this we conclude that

$$\sup\{\|T_r F\|_{1\varphi}:\ 0 \leqslant r < 1\} \leqslant \|F\|''_\varphi.$$

This implies that (see [5]) $F \in H^{*\varphi}$. Thus $(H^{*\varphi})'' \subset H^{*\varphi}$.

4.8. As in Section III, we can consider subspaces $\big((H^{*\varphi})'_m\big)'$ and $\big((H^{*\varphi})'_{vw}\big)'$ of the space $(H^{*\varphi})''$ corresponding to the spaces of functionals $\big((H^{*\varphi})'_m\big)^{\#}$ and $\big((H^{*\varphi})'_{vw}\big)^{\#}$, respectively. On account of Theorems 4.4 and 4.6 we easily obtain the following identities

$$\big[\big((H^{*\varphi})'_m\big)',\ \|\cdot\|''_\varphi\big] = [(H^{*\varphi})'',\ \|\cdot\|''_\varphi] = [H^{*\varphi},\ \|\cdot\|_{1\varphi}]$$

and

$$[((H^{*\varphi})'_{vw})', \|\cdot\|''_{\varphi}] = [H^{\circ\varphi}, \|\cdot\|_{1\varphi}].$$

5.1. *The equality* $(H^{*\varphi})' = (H^{*\varphi}_{vw})'$ *occurs if and only if* φ *satisfies the condition* (∇_2).

Proof. Let φ satisfy (∇_2). Then (see [3]) its complementary function φ' satisfies condition (Δ_2). Then we have $L^{*\varphi'} = L^{\circ\varphi'}$. By 1.4 and 3.3 we get $(H^{*\varphi})' = (H^{*\varphi}_{vw})'$.

Conversely, suppose φ does not satisfy (∇_2). Then there exists a sequence of positive numbers $\{u_n\}$ such that

$$2^{n+1}\varphi(2^{-n}u_n) > \varphi(u_n) > 2^n \quad \text{for } n = 1, 2, \ldots$$

In the interval $(0, 2\pi)$ we find a sequence $\{E_n\}$ of pairwise disjoint sets such that their measures are $\operatorname{mes} E_n = \frac{1}{4}(\varphi(u_n))^{-1}$ and we define a sequence of real functions

$$f_n(t) = \begin{cases} u_n & \text{when } 2\pi - t \in E_n, \\ \varphi_{-1}\left(\dfrac{1}{2^{n+1}\pi}\right) & \text{for other } t \text{ from } [0, 2\pi). \end{cases}$$

Next we define a sequence of analytic functions

$$F_n(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log f_n(t)\,dt\right) \qquad (z \in D).$$

It is known ([15], Chap. VII (7.33)) that $F_n \in N'$ and that these functions are such that $|F_n(e^{it})| = f_n(t)$ for almost every t from $[0, 2\pi)$. In view of this fact and also of

$$\mathscr{I}_{\varphi}(f_n) \leqslant \varphi(u_n)\cdot\tfrac{1}{4}(\varphi(u_n))^{-1} + \frac{1}{2^{n+1}\pi}\,2\pi \leqslant 1$$

following from I. 3.3 we infer that $F_n \in H^{*\varphi}$ and $\|F_n\|_{1\varphi} \leqslant 1$ for $n = 1, 2, \ldots$ Let us notice that $|F_n(e^{it})| = f_n(t) \to 0$ as $n \to \infty$ for almost every $t \in [0, 2\pi)$. Thus, by Ostrowski's theorem ([8]) it follows that $\{F_n\}$ very weakly converges to 0. Now we define a function

$$f(t) = \begin{cases} 2^{-n}u_n & \text{for } t \in E_n,\ n = 1, 2, \ldots, \\ 0 & \text{for other } t \text{ from } [0, 2\pi). \end{cases}$$

For this function we have

$$\mathscr{I}_{\varphi}(2f) = \sum_{n=1}^{\infty} \varphi(2^{-n+1}u_n)\cdot\tfrac{1}{4}(\varphi(u_n))^{-1} \leqslant \sum_{i=1}^{\infty} 2^{-n+1}\cdot\tfrac{1}{4} = \tfrac{1}{2}.$$

Hence $\|f\|_{1\varphi} \leqslant \frac{1}{2}$ and $\|f\|_{(\varphi)} \leqslant 1$. It is known ([3] p. 73) that then $\dfrac{d\varphi}{du}(f(\cdot)) \in L^{*\varphi'}$, where $\dfrac{d\varphi}{du}$ denotes a right-hand side derivative of φ. We finally define

$$g(t) = \begin{cases} \dfrac{d\varphi}{du}(f(t))\operatorname{sgn} F_n(e^{-it}) & \text{for } t \in E_n,\ n = 1, 2, \ldots \\ 0 & \text{for other } t \text{ from } [0, 2\pi). \end{cases}$$

Clearly, g also belongs to $L^{*\varphi'}$. Thus, applying 1.3, we see that the Cauchy integral G of g belongs to $(H^{*\varphi})'$. Moreover, by III. 1.6 and on account of the fact that the functional represented by the integral $\int_0^{2\pi} f(t)g(t)\,dt$ is modular continuous on $L^{*\varphi}$, we get

$$2\pi(F_n * G)(1) = \lim_{r\to 1-} \int_0^{2\pi} F_n(re^{-it})g(t)\,dt = \int_0^{2\pi} F_n(e^{-it})g(t)\,dt$$

for $n = 1, 2, \ldots$ We verify that

$$\int_{E_n} F_n(e^{-it})g(t)\,dt = \int_{E_n} |F_n(e^{-it})|\frac{d\varphi}{du}(f(t))\,dt$$
$$= u_n\frac{d\varphi}{du}(2^{-n}u_n)\cdot\frac{1}{4}(\varphi(u_n))^{-1} \geqslant 2^n\varphi(2^{-n}u_n)\cdot\frac{1}{4}(\varphi(u_n))^{-1} > \frac{1}{8}$$

and

$$\left|\int_{[0,2\pi)\setminus E_n} F_n(e^{-it})g(t)\,dt\right| \leqslant \varphi_{-1}\left(\frac{1}{2^{n+1}\pi}\right)\int_0^{2\pi} |g(t)|\,dt.$$

The integral appearing in the above inequality is finite, since $L^{*\varphi'} \subset L^1$. Thus we get

$$\liminf_{n\to\infty} 2\pi|(F_n * G)(1)| \geqslant \tfrac{1}{8}.$$

This proves that $G \notin (H^{*\varphi}_{vw})'$.

5.2. *If* φ *does not satisfy the condition* (∇_2) *then* $((H^{*\varphi})'^{\sim})^{\#}$ *is a non-trivial space, i.e. there exist non-trivial functionals* $\eta \in ((H^{*\varphi})')^{\#}$ *such that* $\eta(G) = 0$ *for every* $G \in (H^{*\varphi}_{vw})'$. *For these functionals there are no functions* F *analytic in* D *and such that* (+) *is satisfied.*

Proof. Since $(H^{*\varphi}_{vw})'$ is a closed linear subspace of $[(H^{*\varphi})', \|\cdot\|'_{\varphi}]$, the functional

$$p(G) = \inf\{\|G - F\|'_{\varphi}\colon F \in (H^{*\varphi}_{vw})'\}, \qquad (G \in (H^{*\varphi})')$$

is a pseudonorm on $(H^{*\psi})'$ such that $p(G) = 0$ if and only if $G \in (H^{*\psi}_{vw})'$. This pseudonorm is non-trivial, since by 5.1 we have $(H^{*\psi})' \neq (H^{*\psi}_{vw})'$. We take an element $G_0 \in (H^{*\psi})' \setminus (H^{*\psi}_{vw})'$ and put

$$\eta(aG_0) = ap(G_0) \quad \text{for any number } a.$$

Further, by the Hahn-Banach theorem, we extend η to the whole space $(H^{*\psi})'$ so that $|\eta(G)| \leqslant p(G)$ for every $G \in (H^{*\psi})'$. Clearly, η is a non-trivial functional and belongs to $((H^{*\psi})'^{\sim})^{\#}$. Let us suppose that for $\eta \in ((H^{*\psi})'^{\sim})^{\#}$, $\eta \neq 0$, there is a function F analytic in D and satisfying $(+)$. Then its coefficients are

$$\gamma_n(F) = (F * U_n)(1) = \frac{1}{2\pi}\eta(U_n) \quad \text{for } n = 0, 1, 2, \ldots$$

Since $U_n \in (H^{*\psi}_{vw})'$ for $n = 0, 1, 2, \ldots$, it follows that $\gamma_n(F) = 0$ for $n = 0, 1, 2, \ldots$ This implies that $F = 0$ and, further, $\eta(G) = 2\pi(F*G)(1) = 0$ for every $G \in (H^{*\psi})'$. This contradicts our assumption that $\eta \neq 0$.

5.3. *In the space $(H^{*\psi})'$ modular convergence is equivalent to norm convergence if and only if ψ satisfies condition (∇_2).*

Proof. We need only to show that modular convergence in $(H^{*\psi})'$ implies norm convergence if and only if ψ satisfies (∇_2). Let then ψ satisfy (∇_2). Then its complementary function ψ' satisfies (Δ_2). It is known (see I.2.4) that then modular convergence in $L^{*\psi'}$ implies norm convergence. Let $\{G_n\} \subset (H^{*\psi})'$ be a sequence such that $\mu'_\psi(G_n) \to 0$ as $n \to \infty$. By 2.2 and 2.1 we know that there exists a sequence $\{g_n\} \subset L^{*\psi'}$ such that G_n are Cauchy integrals of corresponding g_n and $\mathscr{I}_{\psi'}(g_n) \leqslant \mu'_\psi(G_n) + \frac{1}{n}$ for $n = 1, 2, \ldots$ From this we get $\mathscr{I}_{\psi'}(g_n) \to 0$ as $n \to \infty$ and $\|g_n\|^*_{(\psi')} \to 0$ as $n \to \infty$. This in turn implies by 1.3 that $\|G_n\|'_\psi \to 0$ as $n \to \infty$.

Let us assume that ψ does not satisfy (∇_2). Then in view of 5.2 there exist non-trivial norm continuous functionals on $(H^{*\psi})'$ which are not representable by $(+)$, and, by 4.4, they are not modular continuous. This implies in consequence that norm convergence and modular convergence are not equivalent on $(H^{*\psi})'$.

5.4. *The equation $(H^{*\psi})' = H^{*\psi'}$ occurs if and only if ψ satisfies simultaneously conditions (Δ_2) and (∇_2).*

Proof. In view of 1.5 it suffices to show that $(H^{*\psi})' \subset H^{*\psi'}$ if and only if ψ satisfies simultaneously the conditions (Δ_2) and (∇_2). Let ψ satisfy both (Δ_2) and (∇_2). Let $G \in (H^{*\psi})'$. Applying 1.2, we see that G is the Cauchy integral of some function $g \in L^{*\psi'}$. Since the complementary function ψ' also satisfies (Δ_2) and (∇_2), it follows from Ryan's theorem (see I.2.6) that the conjugate function $\hat{g}$ also belongs to $L^{*\psi'}$. Thus we find that a function

$$h(t) = \frac{1}{2}\left(g(t) + i\hat{g}(t) + \frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx\right), \quad 0 \leqslant t < 2\pi,$$

belongs to $L^{*\psi'}$. We verify that its Poisson integral equals

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-\tau)+r^2} h(\tau)\,d\tau = \frac{1}{4\pi}\left(\int_0^{2\pi} \frac{1+re^{i(t-\tau)}}{1-re^{i(t-\tau)}} g(\tau)\,d\tau + \int_0^{2\pi} g(x)\,dx\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1-re^{i(t-\tau)}} g(\tau)\,d\tau = G(re^{it}), \quad 0 \leqslant r < 1.$$

Hence $G(e^{it}) = h(t)$ for almost every $t \in [0, 2\pi)$ and $G \in H^{*\psi'}$. This means that $(H^{*\psi})' \subset H^{*\psi'}$.

Conversely, let $(H^{*\psi})' \subset H^{*\psi'}$. We take an arbitrary function $g \in L^{*\psi'}$. By 1.3 its Cauchy integral G is an element of $(H^{*\psi})'$. From our assumption $G \in H^{*\psi'}$. It follows that $G(e^{i\cdot}) \in L^{*\psi'}$ and that

$$\hat{g}(t) = i\left(\frac{1}{2\pi}\int_0^{2\pi} g(x)\,dx + g(t) - 2G(e^{it})\right), \quad 0 \leqslant t < 2\pi,$$

belongs to $L^{*\psi'}$. This leads us to the conclusion that the mapping $g \to \hat{g}$ sends $L^{*\psi'}$ into itself. On account of I. 2.6, ψ' satisfies (Δ_2) and (∇_2). Hence ψ also satisfies (Δ_2) and (∇_2).

5.5. *The following conditions are equivalent:*

1° *ψ satisfies simultaneously (Δ_2) and (∇_2),*

2° *$H^{*\psi}$ is a reflexive space (in the norm sense),*

3° *$H^{\circ\psi}$ is a reflexive space (in the norm sense),*

4° *$H^{*\psi} = H^{\circ\psi}$ and $(H^{*\psi})' = (H^{*\psi}_{vw})'$,*

5° *$(H^{*\psi})^{\#} = (H^{*\psi}_{vw})^{\#}$,*

6° *$(H^{*\psi})' = H^{*\psi'}$.*

The equivalence of 1° and 6° follows from 5.4. The equivalence of 1° and 4° we get from I. 3.8 ([5]) and 5.1. The equivalence of any 2°, 3°, 5° with 1° we obtain by assembling the results of I. 3.8, III. 3.1, III. 3.2, III. 3.3, III. 3.4, III. 3.8, 2.7, 4.1, 4.4, 4.6, 5.1, 5.2 and 5.3. For instance we deduce 1° from 5° by III. 3.4, III. 3.8, 2.7 and 5.1.

5.6. The results given in 1.4 and 3.3 can be presented in another form if we consider the operator: $g(t) \to h(e^{it}) = e^{-it}g(2\pi - t)$ for g defined on $(0, 2\pi)$. This operator maps isometrically the space $L^{*\varphi'}$ onto itsey and, respectively for $L^{\circ\varphi'}$. For $g \in L^1$ and $n = 0, 1, 2, \ldots$ the equalitlf $\int_0^{2\pi} g(t)e^{-int}dt = 0$ holds if and only if $\int_0^{2\pi} g(2\pi - t)e^{-it}e^{i(n+1)t}dt = 0$. From this and III. 1.8 we deduce that this operator maps the space $L_+^{*\varphi'}$ onto $H^{*\varphi'}$ and, respectively $L_+^{\circ\varphi'}$ onto $H^{\circ\varphi'}$. Thus, we have

The space $(H^{*\varphi})'$ is isometric isomorphic to the quotient space $L^{*\varphi'}/H^{*\varphi'}$ (with the norm similar to that of 1.4) *and the space $(H^{*\varphi}_{vw})'$ is isometric isomorphic to the quotient space $L^{\circ\varphi'}/H^{\circ\varphi'}$ (with the norm similar to that of* 3.2). *This isomorphism establishes the operator*

$$G(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{h(e^{-it})e^{-it}}{1 - ze^{-it}}dt = \frac{1}{2\pi i}\int_C \frac{h(\zeta)}{1 - z\zeta}d\zeta \quad (z \in D),$$

where C is the boundary of D with the positive orientation. Moreover, then for $F \in H^{*\varphi}$ and $h(e^{i\cdot}) \in L^{*\varphi'}$ we have

$$\lim_{r\to 1-}(F * G)(r) = \lim_{r\to 1-}\frac{1}{2\pi}\int_0^{2\pi} F(re^{-it})h(e^{-it})e^{-it}dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} F(e^{-it})h(e^{-it})e^{-it}dt = \frac{1}{2\pi i}\int_C F(\zeta)h(\zeta)d\zeta.$$

The results of this section generalize the well known results for linear functionals in Hardy spaces H^p, $1 < p < \infty$ (see [12]).

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