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Some remarks on strong $(\mathfrak{M}, \varphi)$-summability*

by

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Abstract. In this paper it is proved that the method of strong (A, φ)-summability of sequences given in [8], and the method of strong φ-summability of functions given in [9], are special cases of a very general method of strong $(\mathfrak{M}, \varphi)$-summability which was introduced in [10].

In the course of my investigation of modular spaces connected with strong summability I introduced a new method of strong $(\mathfrak{M}, \varphi)$-summability.

1. Let E be an abstract set, and let $\mathscr{E}$ be a σ-algebra of subsets of the set E. $\mathscr{E}_0$ will denote a fixed σ-ring from $\mathscr{E}$.

Now, by $\mathfrak{T}$ we shall denote a locally compact Hausdorff topological space, and let $\tau' \notin \mathfrak{T}$. Let us write $\mathfrak{T}_0 = \mathfrak{T} \cup \{\tau'\}$. $\mathfrak{M} = \{\mu_\tau\}$, $\tau \in \mathfrak{T}$ will denote a family of non-negative measures, defined on the σ-algebra $\mathscr{E}$. Functions on E measurable with respect to $\mathscr{E}$ and sets belonging to $\mathscr{E}$, will be called $\mathfrak{M}$-measurable.

By $\mathfrak{X}$ we shall denote the space whose elements are classes of $\mathfrak{M}$-measurable real-valued functions, where two functions belong to the same class if and only if they differ on a set belonging to the σ-ring of $\mathfrak{M}$-zero sets $\mathfrak{R} = \bigcap_{\tau \in \mathfrak{T}} \mathfrak{R}_\tau$. Here $\mathfrak{R}_\tau$ denotes the σ-ring of sets in E of μ_τ measure zero.

Let a φ-function φ (see [1]) and a family $\mathfrak{M}$ of measures be given. We write for any $x \in \mathfrak{X}$

$$(*) \qquad \sigma_\varphi(\tau, x) = \int_E \varphi(|x(t)|)\, d\mu_\tau(t)$$

and we say that the integral transformation $\sigma_\varphi(\tau, x)$ tends to zero, as $\tau \to \tau'$, if for any number $\varepsilon > 0$ there exists a set Z compact in $\mathfrak{T}$ such that $\tau \notin Z$ implies $\sigma_\varphi(\tau, x) < \varepsilon$.

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Now we introduce

$$\mathfrak{X}_\varphi^* = \{x\in\mathfrak{X}\colon \sigma_\varphi(\tau, \lambda x) < \infty \text{ for } \tau\in\mathfrak{T} \text{ and } \sigma_\varphi(\tau, \lambda x)\to 0 \text{ as } \tau\to\tau'; \text{ for some } \lambda>0\}.$$

The relation $\leqslant$ defines a partial order in $\mathfrak{X}_\varphi^*$ and $\mathfrak{X}_\varphi^*$ is a linear lattice with respect to this relation, which is order complete. The elements of the space $\mathfrak{X}_\varphi^*$ are called *strongly* $(\mathfrak{M}, \varphi)$*-summable to zero*.

In $\mathfrak{X}_\varphi^*$ we define a modular $\varrho_\varphi(x)$, in the sense of the definition given by J. Musielak and W. Orlicz in [3], [4], by the following formula: $\varrho_\varphi(x) = \sup_{\tau\in\mathfrak{T}}\int_E \varphi(|x|)\,d\mu_\tau$ if $\int_E\varphi(|x|)\,d\mu_\tau<\infty$ for $\tau\in\mathfrak{T}$ and $\int_E\varphi(|x|)\,d\mu_\tau\to 0$ as $\tau\to\tau'$, and $\varrho_\varphi(x)=\infty$ elsewhere.

The theory of these modular spaces was developed in [10] under the following assumptions on the family of measures $\mathfrak{M}$:

1° *For every set* $K\in\mathscr{E}_0$, $\sigma_\varphi(\tau, x_{\chi_K})$ *is a continuous function of the variable* $\tau\in\mathfrak{T}$ *(where* χ_K *means the characteristic function of the set* K*).*

2° *For every* $\mathfrak{M}$*-measurable function* x *for which* $\sigma_\varphi(\tau, x)$ *is finite for all* $\tau\in\mathfrak{T}$*, and* $\sigma_\varphi(\tau, x)\to 0$ *as* $\tau\to\tau'$ *the integral remainders are uniformly small on set* Z *compact in* $\mathfrak{T}$.

3° *The family of measures* $\mathfrak{M}$ *is uniformly bounded.*

4° *For an arbitrary set* $K\in\mathscr{E}_0$ *and for any* $\varepsilon>0$ *there exist a set* Z *compact in* $\mathfrak{T}$ *such that* $\tau\notin Z$ *implies* $\mu_\tau K<\varepsilon$.

5° *Let us denote* $A(K)=\sup_{\tau\in\mathfrak{T}}\mu_\tau K$.

(a) *For an arbitrary set* $K\in\mathscr{E}_0$ *there exist a set* Z *compact in* $\mathfrak{T}$ *such that* $A(K)=\sup_{\tau\in Z}\mu_\tau K$.

(b) *There exist constants* $\delta>0$ *and* $0<c\leqslant 1$ *such that for every number* η *satisfying the inequalities* $0<\eta\leqslant\delta$ *there exist a set* $K\in\mathscr{E}_0$ *such that* $c\eta\leqslant A(K)\leqslant\eta$.

We show that the assumptions on the family of measures $\mathfrak{M}$ are sufficiently general in order to cover the special cases of purely atomic measures or atomless measures considered in [8], [9] and also in [2], [4], [6].

2. Let us first consider the case where $\mathfrak{M}$ *is a family of finite, atomless measures*. Then we take $\mathfrak{T}=[\tau^*, \infty[$, where τ^* is a positive number which may be chosen arbitrarily depending on the concrete family of measures, and $\tau'=\infty$. As $\mathscr{E}$ we take the σ-algebra of all Lebesgue measurable subsets of $[0, \infty[$. In place of $\mathfrak{X}$ we shall write X; thus the elements of X are classes of equivalence of real, measurable, finite functions, with respect to the relation of equality almost everywhere. We take measures μ_τ, $\tau\in[\tau^*, \infty[$, absolutely continuous with respect to the Lebesgue measure. Then, applying the Radon–Nikodym theorem, we may find a non-negative function $a(t, \tau)$ defined in the product $[0, \infty[\times[\tau^*, \infty[$ such that it is measurable with respect to the variable t for every $\tau\in[\tau^*, \infty[$ and satisfies the condition

$$(**)\qquad \mu_\tau A=\int_0^\infty a(t, \tau)\chi_A(t)\,dt$$

where $\chi_A(t)$ is the characteristic function of the set $A\in\mathscr{E}$. For such a family of measures, the integral transformation $(*)$ is of the form $\sigma_\varphi(\tau, x) = \int_0^\infty a(t, \tau)\varphi(|x(t)|)\,dt$.

We now show that if the function (kernel) $a(t, \tau)$ possesses properties (1), (2) and (I)–(V) given in [9], p. 116, then the family of measures $\mathfrak{M}=\{\mu_\tau\}$, $\tau\in\mathfrak{T}$, where μ_τ are defined by $(**)$, satisfies conditions 1°–5° given in Section 1 of this Note.

It is easily observed that conditions (1) and (2) in [9] imply that the measures $(**)$ are non-negative.

Let us suppose that the function $x\in X$ satisfies the conditions

$$(+)\qquad \sigma_\varphi(\tau, x)<\infty \quad \text{for } \tau\in[\tau^*, \infty[;\quad \sigma_\varphi(\tau, x)\to 0 \text{ as } \tau\to\infty.$$

Let us take any numbers $\bar{t}\in]0, \infty[$, $\bar{\tau}\in]\tau^*, \infty[$, and let $\tau, \tau_0\in]\tau^*, \bar{\tau}[$, $\tau_0<\tau$. Finally, let $K\subset[0, \bar{t}]$, $K\in\mathscr{E}_0$, be an arbitrary set. Then

$$\begin{aligned}|\sigma_\varphi(\tau, x\chi_K)-\sigma_\varphi(\tau_0, x\chi_K)| &= \Big|\int_E \varphi(|x(t)|\chi_K(t))\,d(\mu_\tau-\mu_{\tau_0})\Big| \leqslant \int_K |a(t, \tau)-a(t, \tau_0)|\varphi(|x(t)|)\,dt\\ &= \int_{K\setminus[\tau_0, \tau]}|a(t, \tau)-a(t, \tau_0)|\varphi(|x(t)|)\,dt+\int_{[\tau_0, \tau]}|a(t, \tau)-a(t, \tau_0)|\varphi(|x(t)|)\,dt.\end{aligned}$$

But, by property (II) a) in [9], there exists a constant $c>0$ such that $a(t, \tau)\leqslant c$ for $t\in[0, \bar{t}]$ and $\tau\in[\tau^*, \bar{\tau}]$. Moreover, by property (II) b) in [9], we have the inequality $|a(t, \tau)-a(t, \tau_0)|\leqslant L|\tau-\tau_0|^\alpha$ for $\tau, \tau_0\in[\tau^*, \bar{\tau}]$ and $\tau, \tau_0<t$ or $\tau, \tau_0\geqslant t$, where $0<\alpha\leqslant 1$ and the constant L does not depend on $t\in[0, \bar{t}[$. Thus we may write

$$|\sigma_\varphi(\tau, x)-\sigma_\varphi(\tau_0, x)|\leqslant 2c\int_{[\tau_0, \tau]}\varphi(|x(t)|)\,dt+L|\tau-\tau_0|^\alpha\int_{K\setminus[\tau_0, \tau]}\varphi(|x(t)|)\,dt.$$

From property (I) in [9] it follows that for an arbitrary set $K\subset[0, \infty[$ there exist constants $d>0$ and $\tau_0>\sup K$ such that $a(t, \tau_0)\geqslant d$ for every $t\in K$. Hence we obtain for a function $x\in X$ satisfying conditions $(+)$,

$$\int_K\varphi(|x(t)|)\,dt\leqslant\frac{1}{d}\int_K a(t, \tau_0)\varphi(|x(t)|)\,dt\leqslant\frac{1}{d}\int_E a(t, \tau_0)\varphi(|x(t)|)\,dt<\infty.$$

Finally, we have $|\sigma_\varphi(\tau, x) - \sigma_\varphi(\tau_0, x)| < \varepsilon$ for τ sufficiently near τ_0, and so the family of measures $\mathfrak{M}$ possesses property 1°.

Let $x \in X$ be an arbitrary function satisfying conditions (+). From condition (III) in [9] it follows that for every $\varepsilon > 0$ and every $\tau_1 \in]\tau^*, \infty[$ there exists a set $K \in \mathscr{E}_0$ such that $\int_{K'} a(t, \tau)\varphi(|x(t)|)\,dt = \int_{K'} \varphi(|x|)\,d\mu_\tau < \varepsilon$. This means that the integral remainders are uniformly small in the set $[\tau^*, \tau_1]$, i.e. the family of measures $\mathfrak{M}$ possesses the property 2°.

Properties 3° and 4° of the family $\mathfrak{M}$ follow from conditions (IV) and (V) in [9] immediately.

Let us also remark that in the case of a family $\mathfrak{M}$ of finite and atomless measures the function $A_y = \sup_{\tau \geqslant \tau^*} \mu_\tau([y, y+1])$ introduced in [9] is finite for every $y \geqslant 0$ and is a continuous function of the variable $y \geqslant 0$ if the function $a(t, \tau)$ satisfies conditions (1), (2), (I)–(V) given in [9]. If we write $k = \sup_{y \geqslant 0} A_y$, then the continuity of A_y implies that for $0 < \eta < k$ there exists a y_0 such that $\eta = A_{y_0}$. Hence it is easily seen that the family of measures $\mathfrak{M}$ possesses property 5°, and the inequality which appears in 5° (b) may be replaced by the equality $\eta = A([y_0, y_0+1])$.

Thus we have proved that the space of functions strongly φ-summable to zero, defined by means of a family of finite atomless measures, is a special case of the space $\mathfrak{X}_\varphi^*$, and that theorems given in [10] generalize the respective theorems from [9].

3. In the case where $\mathfrak{M}$ *is a family of finite and purely atomic measures*, as both the topological space $\mathfrak{T}$ and the abstract set E we take the set N of natural numbers. Measures μ_n, $n \in N$, are defined on the σ-algebra $\mathscr{E}$ of all subsets of the set N. Then there exists a non-negative matrix $A = (a_{n\nu})$, $n, \nu \in N$, such that

$$(***) \qquad \mu_n K = \sum_i a_{n\nu_i} \quad \text{for } K = \{\nu_i\} \in \mathscr{E} \text{ and } \mu_n \emptyset = 0.$$

In the case of this family of measures, the integral transformation (*) is of the form $\sigma_\varphi(n, x) = \sum_{\nu=1}^{\infty} a_{n\nu}\varphi(|t_\nu|)$. It is easily verified that if the matrix A is non-negative and possesses no column consisting of zero only, and if it possesses properties 3(a)–(d) in [8], p. 243, then the family of measures $\mathfrak{M} = \{\mu_n\}$, $n \in N$, where μ_n are defined by (***), satisfies conditions 1°–5° given in Section 1 of this Note.

In connection with property 5° (b) let us remark that in [8] it was shown that if the matrix A possesses properties 3(b) and (c) in [8], then there exist constants $c \in]0, 1[$, $\eta_0 \in]0, \infty[$ such that for every $\eta \in]0, \eta_0]$ there exists a ν for which the inequality $c\eta \leqslant A_\nu \leqslant \eta$ is satisfied, where $A_\nu = \sup_n a_{n\nu}$.

Condition 5° (b) plays an important role in the theory of strongly φ-summable sequences. It is connected with the condition of equisplittability of a family of measures formulated in [5]; moreover, if we reformulate it in terms of [7], it corresponds to condition (D) given in [7].

If $\mathfrak{M}$ is a family of finite, purely atomic measures, we get a special case of space $\mathfrak{X}_\varphi^*$; this is then the space of sequences strongly (A, φ)-summable to zero [8]. Other special cases of such spaces have been investigated in [2], [4], [6] and [8].

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