STUDIA MATHEMATICA, T. XLVI. (1973)

A general ratio ergodic theorem for Abel sums

by

RYOTARO SATO (Sakado)

Abstract. The main purpose of the present paper is to prowe the following result: Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space and T a linear contraction on $L^1(X)$. If $\{p_n; n \geqslant 0\}$ is a T-admissible sequence then for any $f \in L^1(X)$, the limit

$$\lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} r^n fT^n(x)}{\sum_{n=0}^{\infty} r^n p_n(x)}$$

exists and is finite almost everywhere on $\{x; \sum_{n=0}^{\infty} p_n(x) > 0\}$. This result is the analogue for Abel sums of Chacon's general ratio ergodic theorem for Cesàro sums.

1. Introduction. The main purpose of this paper is to obtain the analogue for Abel sums of Chacon's general ratio ergodic theorem [7], [8] for Cesàro sums. Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space with positive measure μ and let $L^1(X)$ be the Banach space of equivalent classes of integrable complex-valued functions on X. Let T be a linear contraction on $L^1(X)$. A sequence $\{p_n; n \geqslant 0\}$ of non-negative measurable functions on X is said to be T-admissible if $f \in L^1(X)$ and $|f| \leqslant p_n$ for some n imply $|fT| \leqslant p_{n+1}$. We shall prove below the following

THEOREM 1. *If T is a linear contraction on $L^1(X)$ and $\{p_n; n \geqslant 0\}$ is a T-admissible sequence then for any $f \in L^1(X)$, the limit*

$$\lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} r^n fT^n(x)}{\sum_{n=0}^{\infty} r^n p_n(x)}$$

exists and is finite almost everywhere on $\{x; \sum_{n=0}^{\infty} p_n(x) > 0\}$.

In Section 4 we shall see that ratio ergodic limits for Abel sums coincide with ratio ergodic limits for Cesàro sums almost everywhere, and in the last section we shall consider a general ratio ergodic theorem with weighted averages.

2. Preliminaries. It is known (cf. [10]) that for a linear contraction T on $L^1(X)$ there exists a unique positive linear contraction τ on $L^1(X)$, called the linear modulus of T, such that $|fT| \leqslant |f|\tau$ for all $f \in L^1(X)$ and $g\tau = \sup\{|fT|;\ f \in L^1(X)$ and $|f| \leqslant g\}$ for all $0 \leqslant g \in L^1(X)$. In what follows, we shall denote by T a linear contraction on $L^1(X)$ and by τ its linear modulus. It follows easily that a sequence of non-negative measurable functions on X is T-admissible if and only if it is τ-admissible. Hence such a sequence will be called, simply, an *admissible sequence*. Let $\{p_n;\ n \geqslant 0\}$ be an admissible sequence and $E \in \mathscr{B}$. Following Akcoglu [1], we define a possibly finite sequence $\{a_n\}$ of non-negative measurable functions on X as follows:

$$a_0 = 1_E p_0, \qquad a_n = 1_E\Big(p_n - \sum_{k=1}^{n} a_{n-k}\tau^k\Big),$$

where 1_E denotes the indicator function of E. Let

$$\Omega_E(\{p_n\}) = \sum_k \int a_k\, d\mu$$

where the summation is taken over the set of indices k for which a_k is defined. It follows directly that if $0 \leqslant g \in L^1(X)$ then $\{g\tau^n;\ n \geqslant 0\}$ is an admissible sequence and $\Omega_E(\{g\tau^n\}) \leqslant \|g\|$.

Let $\{f_n;\ n \geqslant 0\}$ be a sequence of measurable functions on X and define

$$S_r(\{f_n\})(x) = \sum_{k=0}^{\infty} r^k f_k(x)$$

for $r \in (0, 1)$ and $x \in X$ if it exists. It follows easily that if $f \in L^1(X)$ and $f_n = fT^n$ for all $n \geqslant 0$ then for almost all $x \in X$, the series

$$S_r(\{fT^n\})(x) = \sum_{k=0}^{\infty} r^k fT^k(x)$$

has at least unit radius of convergence as a power series in r.

The following lemma is the exact analogue for Abel sums of Akcoglu's maximal ergodic theorem [1] for Cesàro sums and an extension of the maximal ergodic theorem of Edwards [11].

LEMMA 1. *Let $\{p_n;\ n \geqslant 0\}$ and $\{q_n;\ n \geqslant 0\}$ be two admissible sequences and $E \in \mathscr{B}$. Then $\limsup_{r\uparrow 1}[S_r(\{p_n\})(x) - S_r(\{q_n\})(x)] > 0$ a.e. on E implies that*

$$\Omega_E(\{p_n\}) \geqslant \Omega_E(\{q_n\}).$$

Proof. If $\varepsilon > 0$ is given, choose a sequence $\{g_n;\ n \geqslant 0\}$ of strictly positive integrable functions such that

$$\sum_{n=0}^{\infty} \|g_n\| < \varepsilon.$$

Suppose that $x \in E$ and $\limsup_{r\uparrow 1}[S_r(\{p_n\})(x) - S_r(\{q_n\})(x)] > 0$. It follows that for some $r_0 \in (0, 1)$ and $n_0 \geqslant 0$,

$$\sum_{k=0}^{n_0} r_0^k\big(p_k(x) - q_k(x)\big) > 0.$$

Here we may assume that

$$\sum_{k=0}^{m} r_0^k\big(p_k(x) - q_k(x)\big) \leqslant 0 \quad \text{for } 0 \leqslant m < n_0.$$

Hence it follows from ([3], Lemma 2.2) that

$$\sum_{k=0}^{n_0}\big(p_k(x) - q_k(x)\big) > 0.$$

If $\sum_{k=0}^{m}\big(p_k(x) - q_k(x)\big) \geqslant 0$ for all $m > n_0$ then clearly $\limsup_{m\to\infty}\sum_{k=0}^{m}\big(p_k(x) - q_k(x)\big) \geqslant 0$. If $\sum_{k=0}^{m}\big(p_k(x) - q_k(x)\big) < 0$ for some $m > n_0$, let

$$m_0 = \min\Big\{m > n_0;\ \sum_{k=0}^{m}\big(p_k(x) - q_k(x)\big) < 0\Big\}.$$

We now choose $r_1 \in (0, 1)$ and $n_1 > m_0$ such that

$$\sum_{k=0}^{n_1} r_1^k\big(p_k(x) - q_k(x)\big) > 0,$$

$$\sum_{k=0}^{m} r_1^k\big(p_k(x) - q_k(x)\big) \leqslant 0 \quad \text{for } m_0 \leqslant m < n_1,$$

$$r_1^{-m_0}\sum_{k=0}^{m_0-1} r_1^k\big(p_k(x) - q_k(x)\big) < g_0(x) + \sum_{k=0}^{m_0-1}\big(p_k(x) - q_k\big).$$

Then the argument of ([3], p. 604) shows that

$$0 < r_1^{-m_0}\sum_{k=0}^{m_0-1} r_1^k\big(p_k(x) - q_k(x)\big) + \sum_{k=m_0}^{n_1}\big(p_k(x) - q_k(x)\big)$$

$$< g_0(x) + \sum_{k=0}^{n_1}\big(p_k(x) - q_k(x)\big) \leqslant \sum_{k=0}^{n_1}\big(g_0\tau^k(x) + p_k(x) - q_k(x)\big).$$

Since $\limsup_{r\uparrow 1}[S_r(\{g_0\tau^n + p_n\})(x) - S_r(\{q_n\})(x)] > 0$, an induction argument shows that if we let $g = \sum_{n=0}^{\infty} g_n$ then

$$\limsup_{m\to\infty}\sum_{k=0}^{m}\big(g\tau^k(x) + p_k(x) - q_k(x)\big) \geqslant 0.$$

Hence we may apply Akcoglu's maximal ergodic theorem [1] to infer that

$$\Omega_E(\{g\tau^n+p_n\}) \geqslant \Omega_E(\{q_n\}).$$

Since $\Omega_E(\{g\tau^n+p_n\}) = \Omega_E(\{g\tau^n\})+\Omega_E(\{p_n\}) \leqslant \varepsilon+\Omega_E(\{p_n\})$, this completes the proof of Lemma 1.

The following lemma, which is a special case of Theorem 1, is a direct consequence of Lemma 1. Since the argument is similar to that given in [1] for the proof of a ratio ergodic theorem for admissible sequences, we omit the details.

LEMMA 2. *Let $\{p_n;\ n\geqslant 0\}$ be an admissible sequence. Then for any $f\in L^1(X)$, the limit*

$$\lim_{r\uparrow 1}\frac{S_r(\{f\tau^n\})(x)}{S_r(\{p_n\})(x)}$$

exists and is finite a.e. on $\{x;\ \sum_{n=0}^{\infty}p_n(x)>0\}$.

Let $f\in L^1(X)$ and $0\leqslant g\in L^1(X)$. Then we shall denote

$$A_r(f,g)(x) = \frac{S_r(\{fT^n\})(x)}{S_r(\{g\tau^n\})(x)},$$

$$a_r(f,g)(x) = \frac{S_r(\{f\tau^n\})(x)}{S_r(\{g\tau^n\})(x)}.$$

As an immediate corollary of Lemma 2, we have the following:

$$a(f,g)(x) = \lim_{r\uparrow 1}a_r(f,g)(x)$$

exists and is finite a.e. on $\{x;\ \sum_{n=0}^{\infty}g\tau^n(x)>0\}$.

It is well-known [6] that τ decomposes X into two disjoint measurable sets C and D, called, respectively, the *conservative* and *dissipative parts of* X, such that if $0\leqslant g\in L^1(X)$ then $\sum_{n=0}^{\infty}g\tau^n(x)=0$ or ∞ a.e. on C, and $\sum_{n=0}^{\infty}g\tau^n(x)<\infty$ a.e. on D. A set $E\in\mathscr{B}$ is called *invariant* if $f\in L^1(X)$ is supported on E then so is $f\tau$. It is also known [6] that C is invariant and the invariant subsets of C form a σ-field $\mathscr{I}$ with respect to C.

For a measurable set E, T_E and τ_E will denote the linear operators on $L^1(X)$ such that $fT_E=(fT)1_E$ and $f\tau_E=(f\tau)1_E$ for $f\in L^1(X)$. We now define the linear operators P_T and P_τ on $L^1(X)$ as follows:

$$fP_T = \lim_{n\to\infty}\sum_{k=0}^{n}(f1_D)T_D^kT_C+f1_C,$$

$$fP_\tau = \lim_{n\to\infty}\sum_{k=0}^{n}(f1_D)\tau_D^k\tau_C+f1_C.$$

It may be readily seen that P_T and P_τ are contractions on $L^1(X)$.

LEMMA 3. *If $f\in L^1(X)$ and $0\leqslant g\in L^1(X)$ then*

$$\limsup_{r\uparrow 1}|A_r(fP_T-f,g)(x)| = 0$$

a.e. on $A = C\cap\{x;\ \sum_{n=0}^{\infty}g\tau^n(x)>0\}$.

Proof. It follows easily that $\limsup_{r\uparrow 1}|A_r(fT^n-f,g)(x)|=0$ a.e. on A for each $n\geqslant 1$. Thus

$$\limsup_{r\uparrow 1}\Big|A_r(f,g)(x)-A_r\Big(\sum_{k=0}^{n}(f1_D)T_D^kT_C+(f1_D)T_D^{n+1}+f1_C,\,g\Big)(x)\Big| = 0$$

a.e. on A for each $n\geqslant 0$. Hence it follows that

$$\limsup_{r\uparrow 1}|A_r(fP_T-f,g)(x)|$$
$$\leqslant \limsup_{r\uparrow 1}\Big|A_r\Big(\sum_{k=n+1}^{\infty}(f1_D)T_D^kT_C,\,g\Big)(x)\Big|+\limsup_{r\uparrow 1}\Big|A_r\big((f1_D)T_D^{n+1},g\big)(x)\Big|$$

a.e. on A. Put for each n,

$$f_n = \sum_{k=n+1}^{\infty}(f1_D)T_D^kT_C \quad\text{and}\quad h_n=(f1_D)T_D^{n+1}.$$

It follows that $\lim_{n\to\infty}\|f_n\|=0$. Let E be a measurable subset of A with $\mu(E)<\infty$. If $\varepsilon>0$ is given, let

$$E_n = E\cap\{x;\ \limsup_{r\uparrow 1}|A_r(f_n,g)(x)|>\varepsilon\}$$

and

$$F_n = E\cap\{x;\ \limsup_{r\uparrow 1}a_r(|f_n|,g)(x)>\varepsilon\}.$$

Then clearly $E_k\subset F_k$ for each k, and Lemma 1 implies that

$$\|f_k\|\geqslant \Omega_{F_k}(\{|f_k|\tau^n;\ n\geqslant 0\})$$
$$\geqslant \varepsilon\Omega_{F_k}(\{g\tau^n;\ n\geqslant 0\})\geqslant \varepsilon\int_{F_k}g\,d\mu.$$

Hence $\lim_{k\to\infty}\int_{F_k}g\,d\mu=0$. An induction argument now implies that $\lim_{k\to\infty}\int_{F_k}g\tau^n\,d\mu=0$ for each $n\geqslant 0$. Since $\sum_{n=0}^{\infty}g\tau^n(x)=\infty$ a.e. on A, this demonstrates that $\lim_{k\to\infty}\mu(F_k)=0$ and hence $\lim_{k\to\infty}\mu(E_k)=0$. An analogous argument as above also implies that

$$\lim_{k\to\infty}\mu\big(E\cap\{x;\ \limsup_{r\uparrow 1}|A_r(h_k,g)(x)|>\varepsilon\}\big)=0.$$

Thus the lemma is proved.

LEMMA 4. *Let* $X = C$ *and* $\mu(X) < \infty$, *and let* $f \in L^1(X)$ *and* $0 \leqslant g \in L^1(X)$. *Then*

(i) $\{x;\ \sum_{n=0}^{\infty} g\tau^n(x) > 0\} = \{x;\ E\{g|\mathscr{I}\}(x) > 0\}$,

(ii) $\alpha(f, g)(x) = \dfrac{E\{f|\mathscr{I}\}(x)}{E\{g|\mathscr{I}\}(x)}$ *a.e. on* $\{x;\ \sum_{n=0}^{\infty} g\tau^n(x) > 0\}$.

Proof. (i) is known (see, for example, ([12], pp. 28–29])). Hence, to prove (ii), we may assume without loss of generality that $X = \{x;\ \sum_{n=0}^{\infty} g\tau^n(x) > 0\}$ and $f \geqslant 0$. Then, by virtue of Lemma 1, a slightly modified argument of ([12], pp. 29–30) is sufficient for the proof, and we omit the details.

Combining Lemmas 3 and 4, we have the following

LEMMA 5. *Let* $f \in L^1(X)$ *and* $0 \leqslant g \in L^1(X)$. *Then*

(i) $\alpha(f, g)1_C$ *is* $\mathscr{I}$-*measurable*,

(ii) *for each* $A \in \mathscr{I}$, $\int_A \alpha(f, g)(gP_\tau)\,d\mu = \int_A fP_\tau\,d\mu$.

Proof. Let μ' be a finite measure equivalent to μ and let $\varrho = d\mu'/d\mu$. Then $L^1(X, \mathscr{B}, \mu') = \{f';\ f'\varrho \in L^1(X)\}$. Define a positive linear contraction τ' on $L^1(X, \mathscr{B}, \mu')$ by $f';\ \tau' = (f'\varrho)\tau/\varrho$ for $f' \in L^1(X, \mathscr{B}, \mu')$. Now, applying Lemmas 3 and 4 to the contraction τ', the lemma follows easily.

3. Proof of Theorem 1. By virtue of Lemma 2 it suffices to prove that for $f \in L^1(X)$ and $0 < g \in L^1(X)$, the limit

(1) $$A(f, g) = \lim_{r\uparrow 1} A_r(f, g)(x)$$

exists and is finite a.e. Clearly (1) exists and is finite a.e. on D. On the other hand, since $\limsup_{r\uparrow 1} |A_r(fP_T - f, g)(x)| = 0$ a.e. on C by Lemma 3, it may be assumed without loss of generality that f is supported on C. Here we utilize a theorem due to Akcoglu and Brunel [2], which states that T decomposes uniquely C into two disjoint invariant sets Γ and Δ such that

(i) there exists a measurable function s on Γ satisfying $|s| = 1$ a.e. on Γ and $fT = (sf)\tau\bar{s}$ for any $f \in L^1(\Gamma)$, where $\bar{s}$ is the complex conjugate function of s and $L^1(\Gamma)$ is the subspace of $L^1(X)$ consisting of all functions that vanish a.e. on $X - \Gamma$,

(ii) $\{fT - f;\ f \in L^1(\Delta)\}$ is dense in $L^1(\Delta)$ in the norm topology.

If $f \in L^1(\Gamma)$ then $fT^n = (sf)\tau^n\bar{s}$ for each $n \geqslant 0$, and hence it follows from Lemma 2 that the limit (1) exists and is finite a.e. on Γ and the limit satisfies

$$A(f, g)(x) = \overline{s(x)}\,\alpha(sf, g)(x) \quad \text{a.e.}$$

Next suppose that $f \in L^1(\Delta)$. It follows from (ii) that for each $k \geqslant 1$ there exists a function $h_k \in L^1(\Delta)$ such that $\|f - (h_kT - h_k)\| < 1/k$. Let $f_k = h_kT - h_k$. Since $\lim_{r\uparrow 1} A_r(f_k, g)(x) = 0$ a.e., it follows that

$$\limsup_{r\uparrow 1} |A_r(f, g)(x) - A_r(f - f_k, g)(x)| = 0 \quad \text{a.e.}$$

On the other hand, since $\lim_{k\to\infty} \|f - f_k\| = 0$, the same argument as in the proof of Lemma 3 implies that if E is a measurable subset of Δ with $\mu(E) < \infty$, then

$$\lim_{k\to\infty} \mu(E \cap \{x;\ \limsup_{r\uparrow 1} |A_r(f - f_k, g)(x)| > \varepsilon\} = 0$$

for each $\varepsilon > 0$. Therefore the limit (1) exists a.e. and the limit satisfies

$$A(f, g)(x) = 0 \quad \text{a.e.}$$

This completes the proof of the theorem.

4. Identification of the limit. Let $\{p_n;\ n \geqslant 0\}$ be an admissible sequence. Chacon's general ratio ergodic theorem [7], [8] for Cesàro sums states that for any $f \in L^1(X)$, the limit

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} fT^k(x)}{\sum_{k=0}^{n} p_k(x)}$$

exists and is finite a.e. on $\{x;\ \sum_{k=0}^{\infty} p_k(x) > 0\}$. In this section we shall prove the following

THEOREM 2. *If* $\{p_n;\ n \geqslant 0\}$ *is an admissible sequence then for any* $f \in L^1(X)$,

$$\lim_{r\uparrow 1} \frac{\sum_{k=0}^{\infty} r^k fT^k(x)}{\sum_{k=0}^{\infty} r^k p_k(x)} = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} fT^k(x)}{\sum_{k=0}^{n} p_k(x)}$$

a.e. *on* $\{x;\ \sum_{k=0}^{\infty} p_k(x) > 0\}$.

Proof. Let $f \in L^1(X)$ and $0 < g \in L^1(X)$, and define

$$R(f, g)(x) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} fT^k(x)}{\sum_{k=0}^{n} g\tau^k(x)}.$$

It is clear that $A(f,g)(x) = R(f,g)(x)$ a.e. on D. Lemma 3 implies that $A(f,g)(x) = A(fP_T, gP_\tau)(x)$ a.e. on C. On the other hand, Akcoglu's maximal ergodic theorem [1] for Cesàro sums together with an argument similar to that given in the proof of Lemma 3 implies that $R(f,g)(x) = R(fP_T, gP_\tau)(x)$ a.e. on C. Thus it follows from [2] that $R(f,g)(x) = 0$ a.e. on Δ, and hence $A(f,g)(x) = R(f,g)(x)$ a.e. on Δ. Since $fT = (sf)\tau\bar{s}$ for any $f \in L^1(\Gamma)$, it also follows from [6] and Lemma 5 that $A(f,g)(x) = R(f,g)(x)$ a.e. on Γ. Therefore in order to complete the proof it suffices to show that

$$\lim_{r\uparrow 1} \frac{\sum_{k=0}^{\infty} r^k g\tau^k(x)}{\sum_{k=0}^{\infty} r^k p_k(x)} = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} g\tau^k(x)}{\sum_{k=0}^{n} p_k(x)} \tag{2}$$

a.e. on $\{x; \sum_{k=0}^{\infty} p_k(x) > 0\}$. Clearly (2) holds good for almost all $x \in D \cap \{x; \sum_{k=0}^{\infty} p_k(x) > 0\}$. Let $a > 0$ be any positive number and set

$$B^+(a) = C \cap \left\{x; \lim_{r\uparrow 1} \frac{\sum_{k=0}^{\infty} r^k g\tau^k(x)}{\sum_{k=0}^{\infty} r^k p_k(x)} > a\right\}.$$

We now choose a sequence $\{f_i; i \geq 0\}$ of strictly positive functions in $L^1(X)$ such that $\lim_{i\to\infty} \|f_i\| = 0$. It follows easily from the same argument as in the proof of Lemma 1 that for each $i \geq 0$,

$$\limsup_{n\to\infty} \Big[\sum_{k=0}^{n} \big(g\tau^k(x) + f_i\tau^k(x)\big) - \sum_{k=0}^{n} a p_k(x)\Big] \geq 0$$

a.e. on $B^+(a)$. Thus

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} g\tau^k(x) + \sum_{k=0}^{n} f_i\tau^k(x)}{\sum_{k=0}^{n} a p_k(x)} \geq 1$$

a.e. on $B^+(a)$. Since $\lim_{i\to\infty} \|f_i\| = 0$, we may apply Akcoglu's maximal ergodic theorem [1] for Cesàro sums and the method used in the proof of Lemma 3 to infer that if E is a measurable subset of $B^+(a)$ with $\mu(E) < \infty$ and $\varepsilon > 0$, then

$$\lim_{i\to\infty} \mu\left(E \cap \left\{x; \lim_{n\to\infty} \frac{\sum_{k=0}^{n} f_i\tau^k(x)}{\sum_{k=0}^{n} a p_k(x)} > \varepsilon\right\}\right) = 0.$$

Hence it follows that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} g\tau^k(x)}{\sum_{k=0}^{n} p_k(x)} \geq a \quad \text{a.e. on } B^+(a).$$

An analogous argument is also applied to obtain that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} g\tau^k(x)}{\sum_{k=0}^{n} p_k(x)} \leq a$$

a.e. on

$$B^-(a) = C \cap \left\{x; \lim_{r\uparrow 1} \frac{\sum_{k=0}^{\infty} r^k g\tau^k(x)}{\sum_{k=0}^{\infty} r^k p_k(x)} < a\right\}.$$

This completes the proof of Theorem 2.

5. A general ratio ergodic theorem with weighted averages. In this section we shall extend Theorem 1 to a more general form. Let $\{w_n; n \geq 1\}$ be a sequence of non-negative numbers whose sum is one, and let $\{u_n; n \geq 0\}$ be the sequence defined by $u_n = w_n u_0 + \ldots + w_1 u_{n-1}$, $u_0 = 1$. Then we have the following

THEOREM 3. *If $\{p_n; n \geq 0\}$ is an admissible sequence then for any $f \in L^1(X)$, the limit*

$$\lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} u_n r^n f T^n(x)}{\sum_{n=0}^{\infty} u_n r^n p_n(x)}$$

exists and is finite a.e. on $\{x; \sum_{n=0}^{\infty} u_n p_n(x) > 0\}$.

Before the proof we note that if $w_1 = 1$, $w_n = 0$ for $n \geq 2$ then $u_n = 1$ for all $n \geq 0$ and hence the above theorem contains Theorem 1 as a special case, and that if T is positive and $p_n = gT^n$ for all $n \geq 0$ for some $0 \leq g \in L^1(X)$ then the above theorem reduces to the ergodic theorem of Báez-Duarte [3] (see also [11]).

The following proof is analogous to that given in [9] for the proof of Baxter's ergodic theorem [4], [5] in a stronger form.

Proof of Theorem 3. Let Y be the positive integers, $\mathscr{P}$ all possible subsets of Y, and λ the measure on $(Y, \mathscr{P})$ defined by $\lambda(\{1\}) = 1$ and $\lambda(\{i\}) = 1-w_1-\ldots-w_{i-1}$ for $i \geqslant 2$. Let $\{c_n; n \geqslant 1\}$ be the sequence defined by $c_n = w_n/(1-w_1-\ldots-w_{n-1})$, $c_1 = w_1$. Let S be the linear operator on $L^1(Y)$ satisfying

$$d_1 S = \sum_{n=1}^{\infty} c_n d_n \quad \text{and} \quad d_n S = (1-c_{n-1}) d_{n-1} \quad \text{for } n \geqslant 2$$

where d_n denotes the indicator function of the set $\{n\}$. Then it is known [9] that S is a contraction on $L^1(Y)$ and $d_1 S^n(1) = u_n$ for each $n \geqslant 0$. Taking $(X\times Y, \mathscr{B}\otimes\mathscr{P}, \mu\times\lambda)$ to be the direct product of $(X, \mathscr{B}, \mu)$ and $(Y, \mathscr{P}, \lambda)$ and $T\times S$ the direct product of T and S, it follows that $T\times S$ is a linear contraction on $L^1(X\times Y)$ and satisfies $fd_1(T\times S)^n(x, 1) = fT^n(x) d_1 S^n(1) = u_n fT^n(x)$ for all $n \geqslant 0$. Now define a sequence $\{\tilde{p}_n; n \geqslant 0\}$ of non-negative measurable functions on $X\times Y$ as follows:

$$\tilde{p}_n(x, i) = p_n(x) d_1 S^n(i) \quad \text{for } (x, i) \in X\times Y.$$

It is easily checked that $\{\tilde{p}_n; n \geqslant 0\}$ is a $T\times S$-admissible sequence. Hence Theorem 1 completes the proof of the present theorem.

Remark. The method above may be applied to obtain the exact analogue for Cesàro sums (see [13]).

References

[1] M. A. Akcoglu, *Pointwise ergodic theorems*, Trans. Amer. Math. Soc. 125 (1966), pp. 296–309.

[2] — and A. Brunel, *Contractions on L_1-spaces*, Trans. Amer. Math. Soc. 155 (1971), pp. 315–325.

[3] L. Báez-Duarte, *An ergodic theorem of Abelian type*, J. Math. Mech. 15 (1966), pp. 599–607.

[4] G. Baxter, *An ergodic theorem with weighted averages*, J. Math. Mech. 13 (1964), pp. 481–488.

[5] — *A general ergodic theorem with weighted averages*, J. Math. Mech. 14 (1965), pp. 277–288.

[6] R. V. Chacon, *Identification of the limit of operator averages*, J. Math. Mech. 11 (1961), pp. 961–968.

[7] — *Operator averages*, Bull. Amer. Math. Soc. 68 (1962), pp. 351–353.

[8] — *Convergence of operator averages*, Ergodic Theory (Proc. Internat. Sympos., Tulane Univ., New Orleans, La., 1961), New York 1963.

[9] — *Ordinary means imply recurrent means*, Bull. Amer. Math. Soc. 70 (1964), pp. 796–797.

[10] — and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc. 15 (1964), pp. 553–559.

[11] D. A. Edwards, *On potentials and general ergodic theorems for resolvents*, Z. Wahrscheinlichkeitstheorie verw. Geb. 20 (1971), pp. 1–8.

[12] S. R. Foguel, *The ergodic theory of Markov processes*, New York 1969.

[13] R. Sato, *On a general ratio ergodic theorem with weighted averages*, Proc. Amer. Math. Soc. 35 (1972), pp. 177–178.

DEPARTMENT OF MATHEMATICS, JOSAI UNIVERSITY
SAKADO, JAPAN

Received January 5, 1972 (459)