STUDIA MATHEMATICA, T. XLVI. (1973)

A finiteness result on the ring of analytic functions defined on a Banach space

by

M. J. FIELD (Coventry, England)

Abstract. It is shown that irreducibility of the germ of an analytic function defined on a Banach space E is equivalent to irreducibility when restricted to suitable finite dimensional subspaces of E. A number of applications of the result are given.

Recently the study of complex analysis on Banach spaces has been receiving increasing attention (for example [1], [2], [4], [6], and [7]). In this note our main result is to show that irreducibility of a germ in $\mathcal{O}_0(E)$ (E Banach) is equivalent to irreducibility when it is restricted to some suitable finite dimensional subspace of E(1). This result is a useful "theorem proving machine" in that it enables one to establish a number of theorems in complex analysis in Banach spaces using a combination of easy analytic methods and classical results rather than the algebraic methods used in [6]. I am particularly grateful to Professor J. Eells for introducing me to the field of complex analysis.

1. A lemma on functions satisfying analytic conditions. Let E be a complex Banach space and B denote a ball, centre zero, in E. For definitions and background see [1] and [2]. Our notation will follow these papers.

LEMMA 1. *If $f\colon B \to C$ is a continuous function satisfying the following condition:*

There exists a subset V of B such that if F is any two-dimensional complex subspace of E then $V \cap F$ is a neighbourhood of 0 in V and $f \mid V \cap F$ is analytic. Then:

1° *f is G^∞ at 0 (that is, for every $k = 1, 2, \ldots$ and for every $h \in E$ the map $E \ni h \to \delta_0^k f = \left(\dfrac{d}{dt}\right)^k f(th)|_{t=0}$ is well defined and $\delta_0^k f$ is a homogeneous polynomial of degree k).*

2° *$\delta_0^k f \in P^k(E, C)$ (that is, $\delta_0^k f$ is a continuous polynomial of degree k).*

(1) I have recently learnt from J. P. Ramis that P. Mazet (Orsay) has also proved a similar type of result to the one given in this paper.

3° *The series* $\sum_{0}^{\infty} \frac{1}{k!} \delta_0^k f$ *converges normally at* $0 \in E$. *The series therefore defines an analytic function in a neighbourhood of* 0.

Proof. 1° For every two-dimensional subspace F of E $\delta_0^k(f \mid V \cap F) = (\delta_0^k f) \mid F$ is a homogeneous polynomial of degree k. So by Corollary 3 in [2] $\delta_0^k f$ is a homogeneous polynomial of degree k.

2° This follows using a Taylor expansion for f and the Baire property of E (see proof of Theorem 5 [2]).

3° $f = \sum_{0}^{\infty} \frac{1}{k!} \delta_0^k f$ in an absorbing subset of E. Thus, by Proposition 5.2 of [1], it converges normally in a neighbourhood of 0.

2. The main theorem. For properties of rings of germs of analytic functions we refer the reader to [6]. In particular let $\mathcal{O}_0(E)$ denote the ring of germs of analytic functions at the origin of E. $\mathcal{O}_0(E)$ is an integral domain and we have the notion of *irreducibility* of germs. We may now state the main result.

THEOREM 1. $f \in \mathcal{O}_0(E)$ *is irreducible if and only if there exists a finite dimensional subspace* F *of* E *such that* $(f \mid F)_0 \in \mathcal{O}_0(F)$ *is irreducible. Further for all (closed) subspaces* H *of* E *containing* F *we have* $(f \mid H)_0 \in \mathcal{O}_0(H)$ *irreducible.*

Proof. 1. Suppose an F exists with $(f \mid F)_0 \in \mathcal{O}_0(F)$ irreducible then we may easily check that for $H \supset F$, $(f \mid H)_0$ is irreducible. In particular f is irreducible. We leave details to the reader.

2. We now construct F. We may suppose, without loss of generality, that f is a Weierstrass polynomial [6]:

$$f(Z', Z) = Z^p + \ldots + a_p(Z'); \qquad (Z', Z) \in E' \oplus \boldsymbol{C}a = E.$$

Consider $f \mid M$, where M is a finite dimensional subspace of E. Using classical theory (for example [5]) $f \in \mathcal{O}_0(M)$ factorizes as a product of $p(M)$ irreducible factors (counting multiplicities). Clearly $M_1 \supseteq M_2 \Rightarrow p(M_1) \leqslant p(M_2)$. Thus we may find a finite dimensional subspace L' of E such that, for all finite dimensional subspaces $M \supset L'$, we have $p(M) = p(L')$. Set $L = L' \oplus \boldsymbol{C}a$. Let $(f \mid L)_0 = f_1 \ldots f_k$, $k = p(L)$ and $f_j \in \mathcal{O}_0(L)$ irreducible. Using the Weierstrass division theorem we may suppose that each f_j is a Weierstrass polynomial: This uniquely defines the f_j, since f is a Weierstrass polynomial. Suppose $L_i \supset L$, $i = 1, 2$, are finite dimensional. Then we may write:

$$f \mid L_i = f_1^i \ldots f_k^i, f_j^i \in \mathcal{O}_0(L_i),$$

where the f_j^i are Weierstrass polynomials. By rearranging we may suppose $f_j^i \mid L = f_j$. Clearly then $f_j^1 \mid L_1 \cap L_2 = f_j^2 \mid L_1 \cap L_2$ — since the f_j^i are uniquely defined. Thus we may set $f_j^i = f_j$.

Suppose $p(L) > 1$, then the above shows that the f_i are *uniquely* defined on some subset V of E which has the property that for every complex finite dimensional subspace N of E, $N \cap V$ contains a neighbourhood of 0. We suppose that f is defined as an analytic function on some ball B, centre 0, in E.

We now prove that, on the assumption $p(L) > 1$, we obtain a contradiction and hence $p(L)$ must equal 1 and we may take $F = L$.

From the above remarks we have:

$$\mathrm{A}_1 \qquad f(Z', Z) = \prod_{i=1}^{p(L)} \big(Z^{p_i} + b_1^i(Z')Z^{p_i - 1} + \ldots + b_{p_i}^i(Z')\big) \quad \text{on } V,$$

where:

1. b_j^i are analytic on $N \cap V$ for all finite dimensional subspaces N of E'.

2. $f_i(Z', Z) = Z^{p_i} + \ldots + b_{p_i}^i(Z')$ is such that $(f_i \mid H)_0 \in \mathcal{O}_0(H)$ is irreducible for all finite dimensional subspaces $H \supset L$.

For brevity of exposition we will now assume the known result that $\mathcal{O}_0(E)$ is a unique factorization domain ([6]). Thus, since $\mathrm{D}f \neq 0$ ('Df' denotes the *discriminant* of f) we may factorize f as:

$$f(Z', Z) = \prod_{i=1}^{p} \big(Z - a_i(Z')\big),$$

where $a : B \cap E' \to \boldsymbol{C}$ and is continuous. We wish to prove that there exists a subset $J_k \subset \{1, \ldots, p\}$ for $k = 1, \ldots, p(L)$ such that:

$$\mathrm{A}_2 \qquad f_k(Z', Z) = \prod_{j \in J_k} \big(Z - a_j(Z')\big) \quad \text{on } V.$$

In fact we prove more: A_1 and A_2 hold in some neighbourhood of $0 \in E$. To prove A_2 we restrict attention to finite dimensional subspaces H of E containing L. A_2 then follows straightforwardly, using $\mathrm{D}f_k \not\equiv 0$, and in fact defines b_j^i on $B \cap E'$ as continuous functions. We omit details. Using Lemma 1 we see easily that the b_j^i are then analytic on some neighbourhood of 0 in E'. Contradiction, since we have now factored f as a product of analytic germs none of which are units.

We give three examples of the use of this theorem.

COROLLARY 1. *We could have avoided the assumption that* $\mathcal{O}_0(E)$ *was a unique factorization domain in the above proof. That* $\mathcal{O}_0(E)$ *is a unique factorization domain is then an immediate consequence of the theorem together with the classical result.*

Corollary 2. (Nullstellensatz for Principal ideals). *If* $g \in \mathcal{O}_0(E)$ *is irreducible and* $f \in \mathcal{O}_0(E)$ *is identically zero on* $V(g)$ *(the zero set of* g*), then there exists* $h \in \mathcal{O}_0(E)$ *such that* $f = g \cdot h$.

Proof. Just a question of obtaining a factorization of f and g on suitably large finite dimensional subspaces of E, applying the classical result and dividing to obtain $h \in \mathcal{O}_0(E)$.

Corollary 3. *If* X *is an analytic subset of a complex Banach manifold* U *then: If for all* $x \in X$ *the germ* X_x *does not contain a principal germ* ([6]), *the pair* $(U-X, U)$ *possesses the property of extension* ([6]).

Proof. From [6] all we must prove is the special case where U is an open ball in E, $X = V(f_1, f_2)$, where $f_1, f_2\colon U \to \boldsymbol{C}$ and $h\colon U-X \to \boldsymbol{C}$ is analytic. Using the theorem we can reproduce the situation on sufficiently large finite dimensional subspaces of E and apply the classical extension theorem to obtain a function $\bar{h}\colon U \to \boldsymbol{C}$ which is analytic on $U-X$ and also analytic on all finite dimensional (affine) subspaces of U. The result follows immediately from work in [1] and the fact that $U-X$ is open, connected and non-empty.

Problem. Localise Theorem 1.

References

[1] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1971), pp. 77–112.

[2] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings*, Studia Math. 39 (1971), pp. 59–76.

[3] E. Hille and E. G. Philips, *Functional analysis and semigroups*, Colloq. Amer. Math. Soc. 1957.

[4] S. J. Greenfield and N. R. Wallach, *The Hilbert ball and bi-ball are holomorphically inequivalent*, Bull. Amer. Math. Soc. 77, March 1971.

[5] Narasimhan, *Introduction to the theory of analytic spaces*, 1966.

[6] J. P. Ramis, *Sous-ensembles Analytiques d'une variété Banachique Complexe*, 1970.

[7] *Seminaire de Géométrie Analytique*. Publications Mathematique D'Orsay, 1968–69.

UNIVERSITY OF WARWICK
COVENTRY, GREAT BRITAIN

Received November 12, 1971 (434)

STUDIA MATHEMATICA, T. XLVI. (1973)

Formally real rings of distributions

by

MANGHO AHUJA* (Cape Girardeau, Mo.)

Let $\mathcal{D}$ denote the set of test functions, and its dual $\mathcal{D}'$ denote the set of Schwartz distributions [6]. Let $\mathcal{D}'_+$ denote the set of those elements of $\mathcal{D}'$, which have support in the positive cone $\boldsymbol{R}^n_+$, where

$$\boldsymbol{R}^n_+ = \{(t_1, t_2, \ldots, t_n)\colon t_i \in R, t_i \geqslant 0 \text{ for } i = 1, 2, \ldots, n\}.$$

It is well known that the set $\mathcal{D}'_+$ is a commutative ring under the operations addition, $+$, and convolution $*$. Moreover the ring $\mathcal{D}'_+$ has no zero divisors ([6], p. 173) and hence can be embedded into a quotient field M. In the one dimensional case, where $n = 1$, M is the quotient field of Mikusinski operators [3].

Let $(\mathcal{D}'_+)_r$ denote the set of all T in $\mathcal{D}'_+$, for which $T(\varphi)$ is a real number, whenever φ is a real valued test function. The aim of this paper is to show that, whereas $\mathcal{D}'_+$ and M cannot be (linearly) ordered, the ring $(\mathcal{D}'_+)_r$ and its quotient field M_r are both formally real and hence can be (linearly) ordered.

1. Let $\mathcal{D}_r$ denote the subset of $\mathcal{D}$ consisting of the real valued test functions, and let $\mathcal{D}'_r$ denote its real dual, i. e. the set of real valued continuous linear functionals on $\mathcal{D}_r$. Let $(\mathcal{D}'_r)_+ = \{T \in \mathcal{D}'_r\colon \text{support } T \subset \boldsymbol{R}^n_+\}$.

The relation between $(\mathcal{D}'_+)_r$ and $(\mathcal{D}'_r)_+$ is far from superficial.

Theorem I. $(\mathcal{D}'_+)_r$ *and* $(\mathcal{D}'_r)_+$ *are isomorphic as convolution algebras over the reals.*

Proof. Let $T \in \mathcal{D}'_+$ and $\varphi \in \mathcal{D}$. Let $T = T_1 + iT_2$, and $\Phi(x) = \alpha(x) + i\beta(x)$ be their decompositions into real and imaginary parts. Then $T(\Phi) = (T_1 + iT_2)(\alpha + i\beta)$. It follows that if $T \in (\mathcal{D}'_+)_r$, then

$$T(\Phi) = T_1(\alpha) + iT_1(\beta).$$

Let $\tilde{T}$ denote the restriction of T_1 to $\mathcal{D}_r$. Then $\theta\colon T \to \tilde{T}$ furnishes the desired isomorphism. ■

* These results are taken from the author's doctoral dissertation [7] at the University of Colorado, written under the direction of Prof. G. H. Meisters.