On the isomorphism of cartesian products of locally convex spaces

by

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Abstract. The following relation $\mathcal{R}$ between topological linear spaces is studied: $(X, Y) \in \mathcal{R}$ iff every continuous linear operator $T: X \to Y$ is compact. The results concerning the relation $\mathcal{R}$ are applied to give conditions which guarantee that the isomorphism of certain product spaces $\times X_i$ and $\times Y_i$ implies near isomorphisms $X_i \sim Y_i$ (i.e. the existence of Fredholm operators from $X_i$ onto $Y_i$) for $i = 1, 2, \ldots$, and to establish some criteria of quasi-equivalence of all bases in product spaces $X \times Y$.

§ 1. Let $X$ and $Y$ be locally convex spaces (I-c's) $(1)$. A linear operator $T: X \to Y$ will be called a near-isomorphism (наделипоморфизм) if the following conditions are satisfied:

a) $T(X)$ is closed in $Y$ and $T$ is an open map from $X$ onto $T(X)$,

b) $\sigma(T) = \dim \ker T < \infty$,

c) $\beta(T) = \operatorname{codim} T(X) = \dim Y / T(X) < \infty$ (cf. [24]) $(2)$. The I-c's $X$ and $Y$ are said to be nearly isomorphic (наделипоморфными) $(X \sim Y)$ $(2)$ if there exists a near-isomorphism $T$ from $X$ onto $Y$.

In this paper we give some general conditions under which from (near) isomorphism cartesian products of I-c's $X_i \times X_i$ and $Y_i \times Y_i$ there follows that the factors are (near) isomorphic (Section II). The binary relation $(X, Y) \in \mathcal{R}$ defined on the set of pairs of I-c's by the condition "every continuous linear operator from $X$ to $Y$ is compact" plays a very important role here. The greater part of this paper, Sections I, III is an examination of this relation. Our methods lead effectively to an answer to the question of the isomorphism of a wide class of spaces which are not distinguishable by their diametral dimension: $\Gamma(X_i \times X_i) = \Gamma(Y_i \times Y_i)$, cf. [2], [17], [21]. In particular, we give a complete isomorphic classification of spaces of the form $X_i \times X_i$, where $X_i$ are finite or infinite centers of Riesz scales which are Montel spaces (§ 13).

$(1)$ We consider only Hausdorff locally convex spaces.

$(2)$ In [24] $T$ is called an $\omega$-map; one says also that $T$ is a Fredholm operator or $\Phi$-operator.

$(3)$ If $X$ and $Y$ are isomorphic we shall write $X \cong Y$. 
In §§ 14–15 we consider finite and infinite products of lcs's in particular of $L_k(b, r)$ spaces, which were defined in [7]. Finally, as an application we quote some results (obtained jointly with Dragilev) about quasi-equivalence of bases in nuclear spaces belonging to some classes (cf. § 16).

This paper, besides some new results, contains the proofs of all results announced in [27].

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I. COMPACT OPERATORS IN LINEAR TOPOLOGICAL SPACES

§ 2. Let $X$ and $Y$ be lcs's. A linear operator $T: X \to Y$ is said to be compact if there exists a neighborhood $U$ in $X$ such that its image $T(U)$ is precompact in $Y$ if $Y$ is a Montel space it is sufficient to require the boundedness of $T(U)$ in $Y$.

Now we shall define the relation $\mathcal{R}$ being important in the sequel.

DEFINITION 1. We shall say that an ordered pair of lcs's $(X, Y)$ satisfies the condition $\mathcal{R}$ if $(X, Y, \mathcal{R})$ if every linear continuous operator $T: X \to Y$ is compact.

Further (Section III) we shall describe a wide class of pairs of Köthe spaces which satisfy condition $\mathcal{R}$. Before we are going to consider some examples.

§ 3. EXAMPLE 1. An lcs $X$ will be called pre-Montel if every bounded set $A \subset X$ is precompact. The space $X$ is a Montel space (cf. [5]) if it is barrelled and pre-Montel. The following proposition characterizes the class of pre-Montel spaces in terms of the relation $\mathcal{R}$.

PROPOSITION 1. A necessary and sufficient condition for an lcs $X$ to be pre-Montel is: $(X, Y, \mathcal{R})$ for every normed space $X$.

Proof. Sufficiency. Let $X$ be a pre-Montel space, $X$ a normed space and $T: X \to Y$ a continuous linear operator. Then $T$ is bounded. Thus $T$ maps the ball $U$ in $X$ into a bounded, and hence precompact, set $T(U)$ in $Y$. Therefore $T$ is compact operator.

Necessity. Suppose $(X, Y, \mathcal{R})$ for every normed space $X$. We shall show that every bounded set $A \subset X$ is precompact.

Let $(\|y\|, \lambda xL)$ be a system of pseudonorms defining the topology of $X$. By the definition of a bounded set, there exists a function $m = m(A) > 0$, $\lambda L$ such that:

$$p(y) = \sup \{m(A) \|y\|; \lambda L \leq 1\}, \quad y \in A.$$  

As $X$ we shall take the normed space of all $y \in X$ with $p(y) < \infty$. The set $A$ is contained in the unit ball of $X$. The operator $T$ equal to the identity imbedding of $X$ into $Y$ is continuous. Thus, since $(X, Y, \mathcal{R})$, it is precompact. Therefore $U = T(U)$ is precompact and therefore the set $A$ is precompact in $Y$. The proposition is proved.

EXAMPLE 2. We shall say, according to Grothendieck [12], that an lcs is of type $(S)$ if for every neighborhood $U$ of zero in $X$ there exists a neighborhood $V$ of zero in $X$ which is totally bounded with respect to $U$ (cf. [29], p. 339). The class of spaces of type $(S)$ can be characterized in terms of the relation $\mathcal{R}$ as follows:

PROPOSITION 2. A lcs $X$ is of type $(S)$ iff $(X, Y, \mathcal{R})$ for every Banach space $Y$.

Proof. Sufficiency. Let $(X, Y, \mathcal{R})$ for every Banach space $Y$. We shall associate with every absolutely convex neighborhood $U$ of zero in $X$ a seminormed space $X_U$, which is $X$ with the seminorm $p_U(z) = \inf \{z > 0; z/lxU\}, \quad x \in X$.

We shall denote by $X_U$ the completion of the factor-space $X_U/N_U$ where $N_U = \{z \in X; p(z) = 0\}$. The canonical map $\pi_U: X \to X_U$ is continuous. Since $(X, X_U, \mathcal{R}), \pi_U$ is compact. Thus, there exists a neighborhood $V$ of zero in $X$ for which $\pi_U(V)$ is precompact in $X_U$. Therefore $V$ is totally bounded with respect to $U$, i.e. $X$ is a space of type $(S)$.

Necessity. Suppose $X$ be a space of type $(S)$, $Y$ an arbitrary Banach space and $T: X \to Y$ an arbitrary linear operator. Then there exists a neighborhood $U$ of zero in $X$ such that $T(U) \subset K$, where $K$ is the unit ball in $Y$. By the assumption, there exists a neighborhood $V = V(U)$ of zero in $X$ which is totally bounded with respect to $U$, so $T(V)$ is totally bounded with respect to $T(U)$, and hence with respect to $X$. Thus the set $T(V)$ is totally bounded in $Y$. By the completeness of $Y$, $T(V)$ is precompact in $Y$, i.e., the operator $T: X \to Y$ is compact. So it is proved that $(X, Y, \mathcal{R})$ for every Banach space $Y$.

COROLLARY 1. The relation $\mathcal{R}$ is not a partial order in the class of all lcs's.

Indeed, every complete space $X$ of type $(S)$ is pre-Montel space, and therefore for every infinite dimensional Banach space $X, (X, Y, \mathcal{R})$ and $(Y, X, \mathcal{R})$ hold simultaneously whereas $X \neq Y$ (see also Corollary 2).

EXAMPLE 3. Let $A_1$ be the space of all holomorphic functions in the unit disc and $A_\infty$ the space of all entire functions of one variable. Then $(A_1, A_\infty, \mathcal{R})$. This fact is a particular case of Corollary 5 (cf. § 9). But $(A_\infty, A_1, \mathcal{R})$ (cf. Corollary 9).

(*) Every complete space of type $(S)$ is a pre-Montel space; the contrary is not true.
EXAMPLE 4. \((\phi, \psi) \in R\) [6]. This fact and Dowady’s lemma (see §6) were used in [6] to prove the unconnectedness of the group of automorphisms of the space \(A_0 \times A'\).

§ 4. We are going to demonstrate some simple but important properties of the relation \(R\).

LEMMA 1. Let \((X, Y) \in R\). Then \((X_1, Y_1) \in R\) for every subspace \((1)\) \(X_1\) which is topologically complemented in \(X\) and any subspace \(Y_1\) of \(Y\).

Indeed, let \(I_1: X_1 \rightarrow Y_1\) be an arbitrary linear, continuous operator. By the assumption, there exists a subspace \(X_1\) in \(X\) such that \(X = \bigcap_{i=1}^{n} X_i \cap X_1\). Let \(T: X \rightarrow Y\) be the linear, continuous operator

\[
T(z) = \begin{cases} T_0 z & \text{for } x \in X_0, \\ 0 & \text{for } x \in X \setminus X_1. \end{cases}
\]

Since \((X, Y) \in R\), this operator is compact and therefore \(T_0\) is compact too. Hence \((X_1, Y_1) \in R\).

LEMMA 2. If \((X, Y) \in R\) and \(X \preceq Y\), then \(X\) is a finite-dimensional space.

Indeed, since there exists an isomorphism \(T: X \rightarrow Y\) and \((X, Y) \in R\), it follows that \(T\) is compact; hence there exists a neighborhood \(U\) of zero in \(X\) such that \(T(U)\) is precompact in \(Y\). On the other hand \(T(U)\) is a neighborhood in \(Y\) (because \(T\) is an isomorphism). Hence \(Y\) is a locally compact space, thus it is finite dimensional (cf. [5], p. 29). Because \(X \preceq Y\) the space \(X\) is finite dimensional too. The lemma is proved.

Remark. The requirement in Lemma 1 that \(X_1\) is topologically complemented in \(X\) is not dispensable.

Indeed, let \((z)\) be an arbitrary analytic map of the unit disc \(\{z: |z| < 1\}\) onto the complex plane. Then the operator \(T_0: z(\bar{z}) \rightarrow z(t(\bar{z}))\) maps isomorphically the space \(A_0\) into \(A_1\) [22]. By Lemma 2 \((T_0(A_0), A_0) \in R\) (by Lemma 1 it follows moreover that the subspace \(T_0(A_0)\) is not complemented in \(A_1\)).

COROLLARY 2. The relation \((A_0, A_1) \in R\) is not valid.

In the opposite case by Lemma 1 there would be \((A_0, A_0) \in R\) where \(X_0 = T_0(A_0)\), and this contradicts Lemma 2.

COROLLARY 3. If \((X, Y) \in R\) then no infinite dimensional, complemented subspace \(X_1 \subset X\) is isomorphic to any subspace \(Y_1\) of \(Y\). In particular, the space \(X\) is not isomorphic to any subspace \(Y_1\) of \(Y\).

On the other hand as in Remark after Lemma 2 the space \(Y\) may be isomorphic to a subspace of \(X\).

In particular, we have a new proof of the following

COROLLARY 4. ([19], [24]) There is no complemented subspace of \(A_1\) isomorphic to any subspace of the space \(A_0\).

Let \((X_1, \lambda \cdot L)\) be a collection of lca’s \(X_1\), where \(L\) is a linearly ordered set of indices. The product and the sum of the family \(X_1\) (cf. [20], p. 130, 135) will be written \(X_1 \times \sum L X_1\), respectively. The following lemmas will be useful in §5.

LEMMA 3. Let \(X = \bigcap_{i=1}^{n} X_i\). Then \(i\) from \((X_i, Y_i) \in R, j = 1, 2, \ldots, n\) it follows that \((X, Y) \in R, ii\) from \((Y, X_j) \in R, j = 1, 2, \ldots, n\) it follows that \((Y, X) \in R\).

LEMMA 3a. Let \(X\) be an lca in which there exists at least one continuous norm (not a seminorm) and \(Y = \bigoplus_{i=1}^{n} Y_i\). Then from \((Y_i, X) \in R\) for all \(i\) \(X\) there follows \((X, Y) \in R\).

LEMMA 3b. Let \(X\) be an lca in which there exists at least one bounded absolutely convex absorbing set \(Y = \bigoplus_{i=1}^{n} Y_i\). Then from \((X, Y_i) \in R\) for all \(i\) \(X\) there follows \((X, Y) \in R\).

LEMMA 3a and 3b follow from the statements: a) Under the assumptions of Lemma 3a, for every continuous operator \(T: X \rightarrow Y\), there exists a finite set \(A = A(T) \subset L\) such that from \(y = (y_i) \in Y\) and \(y_i = 0\) for \(i \notin L = A\) it follows that \(T(y) = 0\); b) under the assumptions of Lemma 4b for every linear continuous operator \(T: X \rightarrow Y\) there exists a finite set \(A = A(T) \subset L\) such that from \(y = (y_i) \in Y\) it follows that \(y_i = 0\) for \(i \notin A\).

II. NEAR-ISOMORPHISMS AND ISOMORPHISMS OF THE CARTESIAN PRODUCTS OF LC'S

§ 5. Let \(X, Y\) be lca’s. Recall that by the index of a near-isomorphism \(T: X \rightarrow Y\) one understands the number \(\text{ind } T = \alpha(T) - \beta(T)\), where \(\alpha(T) = \dim \ker T, \beta(T) = \dim \text{codim } T(X, Y)\).

LEMMA 4. (cf. [16], [23], [24]). Let \(T: X \rightarrow Y\) be a near-isomorphism and let \(S: X \rightarrow Y\) be linear compact operator. Then \(T + S\) is a near-isomorphism and \(\text{ind}(T + S) = \text{ind} T\).

LEMMA 5. (cf. [11], [24], [26]). An operator \(T: X \rightarrow Y\) is a near-isomorphism iff there exists an operator \(y: Y \rightarrow X\) satisfying conditions

a) \(X = I_T + B\) where \(B: X \rightarrow X\) is a compact (finite-dimensional) operator,

b) \(T y = I y + C\) where \(C: Y \rightarrow Y\) is a compact (finite-dimensional) operator.
§ 6. The following lemma will play an important role.

**Douday's Lemma**. Let \( X = X_1 \times X_2 \) and \( Y = Y_1 \times Y_2 \) be locally compact spaces, \( (X_1, Y_1, \mathbb{R}) \in \mathcal{R} \), and let \( T : X \to Y \) be a near-isomorphism given by the matrix \( \left[ \begin{array}{cc} T_{11} & T_{12} \\ 0 & T_{22} \end{array} \right] \) where \( T_{ij} : X_j \to Y_i \). Then the operator \( T : X \to Y \) given by the matrix

\[
\begin{bmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{bmatrix}
\]

is a near-isomorphism too. If moreover \( (Y_1, X_1, \mathbb{R}) \in \mathcal{R} \), then the operators \( T_{11} : X_1 \to Y_1 \) and \( T_{22} : X_2 \to Y_2 \) are near-isomorphisms and

\[
\text{ind} T = \text{ind} T_{11} + \text{ind} T_{22}.
\]

**Proof.** From \( (X_1, Y_1, \mathbb{R}) \in \mathcal{R} \) it follows that the operator \( T_{11} : X_1 \to Y_1 \) is compact so that the operator \( S : X \to Y \) given by the matrix

\[
\begin{bmatrix}
0 & 0 \\
T_{11} & 0
\end{bmatrix}
\]

is also compact. By Lemma 4, the operator \( \tilde{T} = T - S : X \to Y \) is a near-isomorphism and

\[
\text{ind} T = \text{ind} \tilde{T}.
\]

By Lemma 5 there exists an operator \( \psi : Y \to X \) such that \( \psi \tilde{T} = I_X + B \), \( \tilde{T} \psi = I_Y + C \), where \( B \) and \( C \) are compact in \( X \) and \( Y \) respectively. Let \( \psi \) be given by the matrix \( [\psi_{ij}] \). By the second assumption \( (Y_1, X_2, \mathbb{R}) \in \mathcal{R} \) so, the operator \( L : Y \to X \) defined by the matrix

\[
\begin{bmatrix}
0 & 0 \\
\psi_{11} & \psi_{12}
\end{bmatrix}
\]

is compact. Therefore the operator \( \tilde{\psi} = \psi - L : Y \to X \) is a near-isomorphism and

\[
\tilde{\psi} \tilde{T} \tilde{\psi} = \tilde{T}(\psi - L) = I_Y + \tilde{C},
\]

where the operators \( \tilde{B} = B - L\tilde{T}, \tilde{C} = C - L\tilde{L} \) are compact in \( X \) and \( Y \), respectively. Taking into account that the operators \( \tilde{T} \) and \( \tilde{\psi} \) are given by the upper triangular matrices from (3) we have

\[
T_{i1} \psi_{i1} = I_{X_1} + \tilde{C}_{i1}, \quad \psi_{i1} T_{i1} = I_{Y_1} + \tilde{B}_{i1}, \quad i = 1, 2,
\]

where \( \tilde{B}_{i1}, \tilde{C}_{i1} \) are compact in \( X_i \) and \( Y_i \), respectively. Hence, by Lemma 5, \( T_{11} \) and \( T_{22} \) are near-isomorphisms.

We shall check now the relation (1). By (4), it follows that

\[
\begin{bmatrix}
\psi_{11} & \psi_{12} \\
0 & \psi_{22}
\end{bmatrix}
\begin{bmatrix}
T_{11} & 0 \\
0 & T_{22}
\end{bmatrix}
\begin{bmatrix}
I_{X_1} & \psi_{12} T_{12} \\
0 & I_{Y_1}
\end{bmatrix}
= \begin{bmatrix}
I_{X_1} & \psi_{12} T_{12} \\
0 & I_{Y_1}
\end{bmatrix}
+ \begin{bmatrix}
\tilde{B}_{11} & 0 \\
0 & \tilde{B}_{22}
\end{bmatrix}
\]

The first summand on the right-hand side of (5) is a near-isomorphism of \( X \) into itself with the index equal to zero. Since the second summand is a compact operator in \( X \) the right-hand side of a near-isomorphism with the index equal to zero. And, because \( \text{ind} AB = \text{ind} A + \text{ind} B \), we have now: \( \text{ind} T_{11} + \text{ind} T_{22} = - \text{ind} \tilde{\psi} = - \text{ind} \tilde{T} = \text{ind} T \). So the lemma is proved.

The last proposition can be extended, by induction, as follows:

**Proposition 3.** Let \( X = \bigtimes_{i=1}^n X_i, \quad Y = \bigtimes_{j=1}^m Y_j, \quad (X_i, Y_j, \mathbb{R}) \in \mathcal{R} \). Let \( T : X \to Y \) be a near-isomorphism defined by the matrix \( [T_{ij}] \) where \( T_{ij} : X_i \to Y_j \), \( i, j = 1, \ldots, n \). Then the operators \( T_{ii} : X_i \to Y_i \) are near-isomorphisms for \( i = 1, \ldots, n \) and

\[
\text{ind} T = \sum_{i=1}^n \text{ind} T_{ii}.
\]

§ 7. As a consequence of Douday's lemma we mention the following criteria for the (near) isomorphism of cartesian products in terms of the (near) isomorphism of the factors. From Douday's lemma there follows immediately

**Proposition 4.** Let \( X = X_1 \times X_2, \quad Y = Y_1 \times Y_2 \) be locally compact spaces, \( (X_1, Y_1, \mathbb{R}) \in \mathcal{R} \). Then \( X \approx Y \) iff \( X_1 \approx Y_1 \) and \( X_2 \approx Y_2 \).

Let \( X^{(0)} \) denote an arbitrary subspace of the lex \( X \) of codimension \( i \) (all such spaces are isomorphic) when \( i > 0 \) and when \( i < 0 \) an arbitrary space \( X \times X \) where \( \text{dim} Z = -i \).

In general, under the assumptions of Proposition 4, the isomorphism of \( X \) and \( Y \) does not imply the isomorphism of the factors. But the following is true:

**Theorem 1.** Under the conditions of Proposition 4, \( X \approx Y \) iff there exists an \( s \) such that \( Y_1 \approx X_1^{(s)}, \quad X_1 \approx X_1^{(s)} \).

**Proof.** Sufficiency. \( Y_1 \approx X_1^{(s)}, \quad X_1 \approx X_1^{(s)} \). By Douday's lemma, \( T_{11} : X_1 \to X_1, \quad T_{22} : X_2 \to Y_2 \) are near-isomorphisms. Hence \( Y_1 \approx X_1^{(s)} \) where \( s_i = \text{ind} T_{1i} \) and \( X_1 \approx X_1^{(s)} \) where \( s_i = \text{ind} T_{1i} \). But \( \text{ind} T = \text{ind} T_{11} + \text{ind} T_{22} = 0 \). So it is sufficient to take \( s = s_i = 0 \).
§ 8. We shall denote by \( \Phi_\omega \) the class of all 1ca’s \( X \) such that the space \( X^{(s)} \) is isomorphic to \( X \) only for \( i = 0 \). If \( X \in \Phi_\omega \), we put

\[ m(X) = \inf \{ i \geq 1 : X^{(s)} \cong X \}, \]

and, if \( X \in \Phi_\omega \), we put \( m(X) = \infty \).

Every \( 1 \text{ca} \) \( X \) belongs to one of the classes \( \Phi_i = \{ X : m(X) = s \} \), \( s = 1, 2, \ldots \). The classes \( \Phi_1 \) and \( \Phi_2 \) are non-empty; \( \Phi_1 \) contains, for example, infinite-dimensional Hilbert spaces, \( \Phi_2 \) all finite-dimensional \( 1 \text{c}a \)'s. Examples of infinite-dimensional \( 1 \text{c}a \)'s were given first in [23], [17] (see also [7]). It is unknown whether the classes \( \Phi_s \) for \( 1 < s < \infty \) are non-empty.

The next lemma gives a necessary condition for a space to belong to \( \Phi_\omega \).

**Lemma 6.** If \( \Gamma(X) \neq \Gamma(X^{(0)}) \), then \( X \in \Phi_\omega \). \(^{(1)}\)

Indeed, by Proposition 7 of [18] the inclusions \( \Gamma(X^{(0)}) \supseteq \Gamma(X) \supseteq \Gamma(X) \), \( s \geq 1 \) hold \(^{(1)}\). Hence \( \Gamma(X) \neq \Gamma(X^{(0)}) \), so \( X^{(s)} \not\cong X \) for \( s \geq 1 \), which implies \( X \in \Phi_\omega \).

**Theorem 2.** If \( X \rightarrow X_1 \rightarrow \Phi_i \), then under the assumptions of Theorem 1, \( X \simeq Y \) iff \( X \simeq X_1 \) and \( X_1 \simeq Y_1 \).

### III. SUFFICIENT CONDITIONS FOR \((X, Y) \in \mathcal{R}

The importance of the theorems of the former paragraph essentially depends upon how large is the class of spaces satisfying the assumptions. In this paragraph we shall take this problem into consideration.

§ 9. Definition 2 (cf. [7], [4]). Let \( X \) be a countably-normed space.

We shall say that \( X \in \mathcal{C}_i, i = 1, 2 \), if there exists in \( X \) an absolute basis \( \{ x_i \} \), and a system of norms \( \{ \| x_i \|_p \}, p = 1, 2, \ldots \), defining the topology of \( X \) such that

\[ \exists \forall \exists \forall: \| x_i \|_p \leq \| x_j \|_p, \quad \text{if} \quad k \geq k_0 = k_0(p, q), \quad \text{for} \quad i = 1, \]

\[ \forall \exists \forall: \| x_i \|_p \geq \| x_j \|_p, \quad \text{if} \quad k \geq k_0 = k_0(p, q), \quad \text{for} \quad i = 2. \]

This definition is somewhat different from that given in [7] and coincides with it if in the above conditions one may take a regular basis \( \{ x_i \} \) (cf. [7], p. 153).

**Example 5.** Following [18], we denote

\[ E_\alpha(a_0) = \lim \inf \{ \exp \lambda a_0 \}, \quad -\infty < a < +\infty. \]

\(^{(1)}\) For the definition of diametral dimension \( F_\alpha \) see § 112.

The space \( E_\alpha(a_\alpha) \) is called the center of the Riesz scale \( \{ \exp \lambda a_\alpha \} \). If \( a < +\infty \) then \( E_\alpha(a_\alpha) \) will be called finite, and if \( a = +\infty \) — infinite center of the scale.

If \( a_\alpha \neq \infty \), then \( E_\alpha(a_\alpha) \) is a Montel space.

**Proposition 5.** (cf. [7], p. 154.) If \( a < \infty \), then \( E_\alpha(a_\alpha) \in \mathcal{C}_1 \); if \( a = \infty \), then \( E_\alpha(a_\alpha) \in \mathcal{C}_1 \).

Some other examples of spaces of type \( \alpha \) will be considered in § 10.

The spaces belonging to different classes \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) have very different properties. The following theorem confirms it.

**Theorem 3.** Let \( X \in \mathcal{C}_1 \), \( Y \in \mathcal{C}_2 \), and \( Y \) be a Montel space. Then \((X, Y), \mathcal{R}\).

**Proof.** Let \( T : X \rightarrow Y \) be an arbitrary linear, continuous operator.

We have to show that \( T \) is compact.

According to Definition 2 we choose the bases \( \{ x_i \} \) and \( \{ y_i \} \) in \( X \) and \( Y \) respectively. Since they are absolute, we may assume that the topologies in \( X \) and \( Y \) are defined by the systems of norms:

\[ \| x_i \|_p = \sum_j \| x_j \|_p^p, \quad x = \sum_j \| x_j \|_p^p x_j \]

\[ \| y_i \|_p = \sum_j \| y_j \|_p^p, \quad y = \sum_j \| y_j \|_p^p y_j \]

Let \( \lambda_\alpha = T x_\alpha = \sum \lambda_\alpha x_\alpha, \quad x = \sum \lambda_\alpha x_\alpha, \quad y = T x = \sum \lambda_\alpha y_\alpha. \)

Then \( y = \sum \lambda_\alpha y_\alpha \). The continuity of the linear operator \( T \) means that there exists a function \( q = q(p) \) for which

\[ \sum \lambda_\alpha \| y_\alpha \|_p \leq C(p) < \infty, \quad p = 1, 2, \ldots \]

Since \( Y \) is a Montel space, to prove that \( T \) is compact it is sufficient to show that some neighborhood \( U_\alpha = \{ x \in X : \| x \|_p < 1 \} \) is mapped into a bounded set, that is,

\[ \sum \| y_\alpha \|_p \leq M(p) < \infty, \quad p = 1, 2, \ldots \]

if \( y = \sum \| y_\alpha \|_p \leq \sum \| y_\alpha \|_p \leq M(p) < \infty. \)

Certainly it will hold if there exists \( q = q_\alpha \) such that

\[ \sum \lambda_\alpha \| y_\alpha \|_p \leq M(p) < \infty. \]

Since \( Y \in \mathcal{C}_1 \), by Definition 2

\[ \exists \forall \exists \forall: \sum \| y_\alpha \|_p \leq \sum \| y_\alpha \|_p \leq M(p) < \infty \]

for \( i = i_\alpha(p) \).
and since $X cd_1$ for $q = q(p)_p$, there exists $q = q_1$ such that for every $g_1$ (we choose $g_1 = q(p)_p$):

$$|g_1|_{q_1}^2 \geq |g_2|_{q_2}^2, \quad h \geq h_2(p).$$

By (8), (9), (6), and the Cauchy inequality, we obtain:

$$\sum_{i=1}^n |a_i| \leq L(p) \sum_{i=1}^n \frac{|g_2|_{q_2}}{|g_2|_{p_2}} \left( \int_{a_i}^b |g_2|_{q_2}^2 \right)^{1/2} \leq L(p) \sum_{i=1}^n \frac{|g_2|_{q_2}}{|g_2|_{p_2}} \left( \sum_{i=1}^n \frac{|g_2|_{q_2}^2}{|g_2|_{p_2}^2} \right)^{1/2} \leq L(p) [C(p_2)]^{1/2} [C(p_2)]^{1/2} = M(p) < \infty, \quad p = 1, 2, \ldots,$$

which means that inequality (7) holds, and so the operator $T$ is compact.

The theorem is proved.

Applying Proposition 5 we have

**Corollary 5.** If $a < b$, then $(E(a), E(b)) \in \mathbb{R}$, independently of $a$ and $b$, provided $b, r, \infty$.

**§ 10.** A more exact description of the relation $(X, X)_r \in \mathbb{R}$ can be obtained for a special class of Köthe spaces which were considered in [7] by M. M. Dragilev.

**Definition 2.** (cf. [7], p. 169). Let $f$ be an increasing odd function on $(-\infty, +\infty)$ which is logarithmically convex on $[0, +\infty)$ (i.e., $\varphi(\mathfrak{e}) = \ln(f(e^{\mathfrak{e}}))$ is convex on $[0, +\infty)$); $r = (r_0, \infty)$. Denote by $L_r(b, r)$ the Köthe space generated by the matrix $[\exp(f(r_0 b))]$, i.e.,

$$L_r(b, r) = \lim_{n \to \infty} \sup_{r_0} \left| \sum_{i=1}^n |a_i| \exp(f_i(r_0 b_i)) \right| \leq C(r_0 < \infty, \quad 0 < \sigma < \delta, \quad 0 < \varphi(\sigma) < r).$$

Without loss of generality, we may assume that $f(u) \geq 0$ for $u \geq 0$.

**Definition 3.** (cf. [7], p. 170). We shall say that an increasing function $f$ defined on $[0, +\infty)$ increases rapidly if, for every $a < b_0$, $\lim_{t \to a^+} f(t) = \infty$, and it increases slowly if, for every $a < b_0$, $\lim_{t \to b^+} f(t) = \tau(a) < \infty$.

**Lemma 7.** (cf. [7]). Let $f$ be an increasing logarithmically convex function on $[0, +\infty)$. Then, for every $a > b$,

$$\lim_{t \to a^+} f(t) = \tau(a).$$

Moreover, either $a) \tau(a) = \infty$, or $b) \tau(a) < \infty$ for $1 < a < \infty$. For $a, r \in \mathbb{R}$.

So, under the conditions of Definition 3, the function $f$ is always slowly or rapidly increasing.

**Proposition 6.** (cf. [7], p. 170). Let $X = L_r(b, r)$ be a space satisfying conditions of Definition 3. Under the assumption that $f$ is slowly increasing, $X$ is isomorphic to a finite center of Bessel scale provided $r < \infty$, and $X$ is isomorphic to an infinite center provided $r = \infty$; in the first case $X cd_1$, in the second $X cd_1$.

Assuming that $f$ is rapidly increasing we have $X cd_1$ if $0 < r \leq \infty$ and $X cd_1$ if $0 < r < \infty$.

**Theorem 4.** Let $f_1, f_2$ satisfy conditions of Definition 3 and let $\varphi = f_1^{-1} f_2$ be rapidly increasing. Then

$$\{L_r(b, b), L_r(a, r)\} \in \mathbb{R}$$

if $0 < r < \infty$, $0 < s < \infty$ and

$$\{L_r(b, a), L_r(a, r)\} \in \mathbb{R}$$

if $-\infty < r < -1$, $0 < s < \infty$ independently of the choice of the sequences $a = (a_i), b = (b_i)$.

**Proof.** First, we shall consider only the case $0 < r < \infty$, $0 < s < \infty$. Let $T: L_r(b, a) \to L_s(b, s)$ be an arbitrary linear continuous operator. We represent $T$ as a matrix in bases of unit vectors in $X$ and $Y$. Arguing as in Theorem 2 we may assert the existence of a function $\varphi = \varphi(\sigma)$ such that

$$\left( \sum_{r_0}^\infty |\sigma| \exp(f_2(r_0 b)) \right) \leq C(r_0 < \infty), \quad 0 < \sigma < \delta, \quad 0 < \varphi(\sigma) < r.$$
We take $c_\varepsilon$ such that $0 < c_\varepsilon < s$ and $\varphi_0 = \varphi(c_\varepsilon) < \varphi$ because of (12) it is possible. Then, using (10) we obtain
\[ S_\varepsilon(\sigma) \leq C(\sigma_0) \sup \{ \exp \tilde{L}_\varepsilon(\sigma) : \varepsilon \in \mathcal{N}_\varepsilon \}, \]
where
\[ \tilde{L}_\varepsilon(\sigma) = f_1(\varphi_0) - f_1(\varphi(a_\varepsilon)) + f_1(\varphi(b_\varepsilon)). \]
By (13), for $\varepsilon \in \mathcal{N}_\varepsilon$ the inequality (14) holds
\[ (14) \quad \tilde{L}_\varepsilon(\sigma) \leq \sup \left\{ \frac{a}{\sigma} + f_1(\varphi_0) - f_1(\varphi(a_\varepsilon)) : 0 < a < \infty \right\}. \]
By Lemma 7 the expression under the sup divided by $f_1(\varphi(a_\varepsilon))$, for $a \to \infty$, tends to
\[ (15) \quad \frac{1}{\tau(C(\varphi_0))} + \frac{1}{\tau(C(\varphi))} - 1. \]
Choosing the constant $C$ sufficiently large it is possible to make the number (15) negative, because $\varphi_0 < \varphi$. But this means that for $a > a_\varepsilon = a(\varepsilon)$ the expression under the sup in (14) is negative. Hence
\[ S_\varepsilon(\sigma) \leq C(\sigma_0) \sup \{ f_1(\varphi_0) - f_1(\varphi(a_\varepsilon)) : 0 < a < a(\varepsilon) \} = M_1(\sigma). \]
Now we estimate the second summand:
\[ R_\varepsilon(\sigma) = \sum_{\varepsilon \in \mathcal{N}_\varepsilon} \sup_{\varphi, \psi} \{ \exp \tilde{L}_\varepsilon(\sigma) : \varepsilon \in \mathcal{N}_\varepsilon \}. \]
We take $c_\varepsilon = c_\varepsilon(\sigma)$, $\rho < c_\varepsilon = c_\varepsilon(\sigma)$, and $\rho = \varphi(c_\varepsilon)$. Using (10) we obtain
\[ (16) \quad R_\varepsilon(\sigma) \leq C(\sigma_0) \sup \{ \exp \tilde{L}_\varepsilon(\sigma) : \varepsilon \in \mathcal{N}_\varepsilon \}, \]
where
\[ \tilde{L}_\varepsilon(\sigma) = f_1(\varphi_0) - f_1(\varphi(a_\varepsilon)) + f_1(\varphi(b_\varepsilon)). \]
By (13) for $\varepsilon \in \mathcal{N}_\varepsilon$ we have the inequality
\[ (17) \quad \tilde{L}_\varepsilon(\sigma) = \sup \left\{ -f_1(\varphi(\sigma_\varepsilon b)) + f_1(\varphi(\sigma_\varepsilon)) - f_1(\varphi(b_\varepsilon)) : 0 < b < \infty \right\}. \]
Since the function $\varphi$ increases rapidly
\[ \forall \delta > 0 \exists \delta_\varepsilon = \delta_\varepsilon(\sigma) \forall \delta_\varepsilon > \delta_\varepsilon : \varphi(\sigma_\varepsilon b) > \varphi(\sigma_\varepsilon). \]
So for $b \gg b_\varepsilon$
\[ (18) \quad -f_1(\varphi(\sigma_\varepsilon b)) + f_1(\varphi(\sigma_\varepsilon)) + f_1(C(\varphi_0) \varphi(\sigma_\varepsilon)) \leq -f_1(\varphi(\varphi(\sigma_\varepsilon b))) + f_1(\varphi(\sigma_\varepsilon)) + f_1(C(\varphi_0) \varphi(\sigma_\varepsilon)). \]
By Lemma 7, the right-hand side of this inequality divided by $f_1(\varphi(\sigma))$ tends to
\[ (19) \quad -1 + \frac{1}{\tau(\varphi)} + \frac{1}{\tau(C(\varphi_0))} \]
for $\sigma \to \infty$. Choosing $\delta = \delta(\sigma)$ sufficiently large the right hand side of the inequality (18) is negative for $b \gg b_\varepsilon$. So, by (16), (17) and (18), we obtain
\[ R_\varepsilon(\sigma) \leq C(\sigma_0) \sup \left\{ -f_1(\varphi(\sigma_\varepsilon b)) + f_1(\varphi(\sigma_\varepsilon)) + f_1(C(\varphi_0) \varphi(\sigma_\varepsilon)) : 0 < b \leq b_\varepsilon \right\} = M_2(\sigma) < \infty. \]
So, inequality (11) holds for $M(\sigma) = M_1(\sigma) + M_2(\sigma)$. If $\sigma_0$ is chosen as in (12).

The proof of the second part of the theorem is analogous.

**Theorem 5.** (i) $L_\varepsilon(a, \sigma, r), L_\varepsilon(b, \infty, r) \in \mathfrak{C}$ if $0 < r < \infty$.
(ii) $L_\varepsilon(a, 0, b, \sigma) \in \mathfrak{C}$ if $-\infty < r < 0$ and the function $f$ increases rapidly.

This theorem may be proved by methods given in the proof of Theorem 3. If $f(\varphi) = \varphi$ the statement (i) coincides with Corollary 3.

**IV. LINEAR TOPOLOGICAL INVARIANTS OF CARTESIAN PRODUCTS**

§ 11. Let $\varepsilon$ be the class of los's and let $K$ be a certain set. One says that the map $\tau : \varepsilon \to K$ generates a linear topological invariant $\tau(X)$, $X \times \varepsilon$ if the fact that $X, Y \in \varepsilon$ and $X, Y$ are isomorphic implies that $\tau(X) = \tau(Y)$.

One of the easy to compute linear topological invariants is the diametral dimension ([17], [18]):
\[ \Gamma(X) = \gamma = (\gamma_0) : \forall U \exists V \gamma_0 d_\varepsilon(V, U) \to 0, n \to \infty \]
where $U$ and $V$ are neighborhoods of zero in the space $X$ and $d_\varepsilon(V, U)$, is the n-dimensional Kolmogorov diameter of $V$ with respect to $U$ ([2], [13], [14], [23]).

The consideration of invariants like $\Gamma(X)$ leads to the solution of isomorphism problems for many los's (see e. g. [7], [14], [15], [18]). But there exists some simple examples ([2], [7], [17], [21]) of spaces of the form $X \times Y$, $X \times Y_1$ for which $\Gamma(X \times Y_1) = \Gamma(X \times Y) = \Gamma(X)$, though $X \neq X \times Y_1$, $i = 1, 2$. The dimensions $\Gamma(Y_i)$ are "absorbed" by the dimension of the factor.
In these cases the comparison of the diametral dimensions does not give the means to distinguish non-isomorphic spaces $X$ and $X \times Y_i$, and the question of isomorphism of the spaces $X$ and $Y_i$ is unsolved if $\Gamma(Y_i) \neq \Gamma(Y_i)$. Theorem 1 allows us to define a new linear topological invariant stronger than the diametral dimension but defined not on the whole class of lea's. It is defined only for some special (but considerably large) classes of lea's which contain in particular all the spaces mentioned in the above examples.

Let $\sigma_i$ and $\sigma_i$ be two classes of lea's such that $(X_1, X_2) \in \mathcal{H}$ whenever $X_i \in \sigma_i$ and $X_j \in \sigma_j$.

We shall write $\sigma_i \times \sigma_j = (X = X_1 \times X_2, X_i \in \sigma_i, X_j \in \sigma_j)$ and with every $X \in \sigma_i \times \sigma_j$ we shall associate the set $\tilde{\Gamma}(X)$ of all different pairs $(\Gamma(X)^{(a)}, \Gamma(X)^{(b)})$, where $a = 0, \pm 1, \ldots$ From Theorem 1 we obtain immediately

**Theorem 6.** $\tilde{\Gamma}(X)$ is a linear topological invariant defined on $\sigma = \sigma_i \times \sigma_j$.

**Remark 1.** One may obtain in the same way other linear topological invariants $\check{\Gamma}(X)$ defined on $\sigma = \sigma_i \times \sigma_j$ instead of $\Gamma(X)$ considering another topological invariant $\tau(X)$ (for example $\Phi(X)$, the approximative dimension (cf. [14], [15])).

**Remark 2.** From Theorem 2, it follows that Theorem 5 is applicable to the class $\sigma = \sigma_i \times (M \cap \sigma_i)$ where $M$ is the class of all Montel lea's. Let $\delta_i$ denote the class of all spaces from $\sigma_i$ for which in Definition 2 it is possible to take as $(a_i) \delta_i$ a regular basis.

Recall that a basis $(a_i) \delta_i$ in $X$ is regular (cf. [7], p. 153) if there exists a system of norms $\|x\|_X$ defining the topology of $\tilde{X}$ and such that all sequences $\|x\|_X, \|y\|_X$ are monotone. Then we have

**Proposition 7.** (cf. [7]) Two spaces $X$ and $Y$ in $\delta_i \cup \delta_i$ are isomorphic iff they are both in the same class $\delta_i$ and $\Gamma(X) = \Gamma(Y)$.

This proposition differs from the Theorem 7 of [7] by dropping the requirement of the nuclearity of the spaces (which in fact is not used in the proof).

Using Theorems 5, 6 and Proposition 7 we obtain following

**Theorem 7.** If $X$ and $X$ are in $\delta_i \times (M \cap \delta_i)$, then $X = Y$ iff $\check{\Gamma}(X) = \check{\Gamma}(Y)$.

**§ 12.** It is quite simple to compute the invariant $\check{\Gamma}(X)$ if $X_1 \times \Phi_1, \ldots, X_2 \times \Phi_2$.

The product $X = X_1 \times X_2$ of two centers of the same type of Riesz scales is isomorphic to the center of a Riesz scale too. Therefore Theorem 7 combined with Proposition 18 of [18] gives us a complete isomorphic classification of all spaces of the form $X_1 \times X_2$, where $X_1, X_2$ are finite or infinite centers of compact Riesz scales.

The following statement on spaces of analytic functions of several complex variables is an immediate consequence of Theorem 7.
Theorem 9. Let \( A(E^n) \) be the space of all analytic functions in the unit polyhedron \( E^n \) in \( C^n \), and let \( A(C^n) \) be the space of all entire functions of \( n \) variables. Then
\[
A(E^n) \times A(C^n) \cong A(E^n) \times A(C^n) \quad \text{iff} \quad (a, b) = (a_1, b_1).
\]

§ 14. Assume that two binary relations are defined on a class \( \sigma \) of \( \alpha \)'s: the partial-order relation \( \alpha \) and the equivalence relation \( \beta \). Furthermore the relations \( \alpha, \beta \) are compatible, i.e., \( X_\alpha X, X_\beta Y \) and \( X \alpha X', Y \alpha Y' \) imply \( X_\beta Y \). Two spaces \( X, Y \) \( \sigma \) are comparable if either \( X \alpha Y \) or \( Y \alpha X \) or \( X \beta Y \). Assume that in every space \( X \) \( \sigma \) there is a continuous norm (or, equivalently, there is an absolutely convex neighborhood of zero containing no line). Finally, assume that \( X \alpha Y \) implies \( (X, Y) \in \sigma \).

Under these assumptions we have

Theorem 10. Let \( X = X_1, Y = X_1, Y_1 \), where \( X_1, Y_1 \) are \( \alpha \)-infinite-dimensional \( \lambda \)'s, \( L, M \) are linearly ordered sets and \( X_1, Y, \lambda, \sigma \) for \( \lambda \in L \) and \( \mu \in M \). Furthermore, \( X_\alpha X_1, Y_\alpha Y_1 \) for \( \lambda < \lambda_1 \) and \( \mu < \mu_1 \) and all the pairs \( X_\lambda Y_\mu \) are comparable. Then:

(a) If \( \lambda \) is infinite set, then \( X \cong Y \) iff there is an order-isomorphism \( \tau : L \to M \) with \( X_\lambda \cong Y_{\tau(\lambda)} \).

(b) If the set \( \lambda \) is finite, then \( X \cong Y \) iff the conditions (a) and (b) above hold.

In the case when \( L \) is finite this theorem can easily be reduced to Proposition 3 (§ 6). So we shall consider only infinite sets \( L \).

Proof. Necessity. Let \( \iota \) denote the identity injection of \( X_1 \) into \( X \) and \( \pi \) – the natural projection of \( X \) onto \( X_1 \). For the space \( Y \) we use respective notations \( q, p \). Let \( T : X \to Y \) be an isomorphism and \( S = T^{-1} : Y \to X \). Write \( T_\mu = q, T^{-1}_\mu = p, S_\mu = p, S^{-1}_\mu = q \). The statement (a) after Lemma 3b gives: for every \( \mu \) there exists a finite set \( Q(\mu) \subset L \) such that \( T^{-1}_\mu = 0 \) for \( \lambda < \mu \) \( Q(\mu) \) and for every \( \lambda \in L \) there exists a finite set \( R = R(\lambda) \subset M \) such that \( S_\lambda = 0 \) for \( \mu \in \lambda \). Moreover
\[
\sum_{\mu \in Q(\mu)} T_{\mu, \lambda} = \delta_n \lambda,
\]
\[
\sum_{\lambda \in R(\lambda)} S_{\lambda, \mu} = \delta_n \mu.
\]

We shall show that for every \( \lambda \in L \) there exists exactly one \( \mu = \tau(\lambda) \in M \) such that \( X_\lambda \beta Y_\mu \). Uniqueness of \( \mu \) follows from compatibility of \( \alpha \) and \( \beta \) and monotony of the families \( \{X_\lambda \} \) and \( \{Y_\mu \} \). Suppose that for some \( \lambda \in L \) there is no \( \mu \in M \) with \( \mu \in \beta \). Then, for every \( \mu \in M \) either \( X_\lambda \alpha Y_\mu \) or \( Y_\mu \alpha X_\lambda \).

Therefore either \( (X_\lambda, Y_\mu) \in \sigma \) or \( (Y_\mu, X_\lambda) \in \sigma \). Hence \( S_{n, \mu} T_{n, \lambda} \) \( X_\lambda \to Y_\mu \) is compact for \( \mu \in M \). The last statement contradicts (23) since \( X_\lambda \) is infinite-dimensional.

Thus, the transformation \( \tau : L \to M \) satisfies the condition \( X_\lambda \beta Y_{\tau(\lambda)} \).

This transformation is bijective (applying the previous arguments to \( M \) and \( L \) taken in the reverse order we obtain a transformation \( \alpha : M \to L \) which is the inverse of \( \tau \)). Moreover \( \tau \) is an order-isomorphism. Indeed, let \( \lambda < \lambda_1 \), then since \( X_\lambda \beta Y_{\tau(\lambda)} \) and \( Y_{\tau(\gamma)} \alpha X_\gamma \), the relation \( Y_{\tau(\mu)} \alpha X_{\tau(\mu)} \) holds and therefore \( \tau(\lambda) \prec \tau(\lambda_1) \).

We shall show that \( X_\lambda \cong Y_{\tau(\lambda)} \). To that end rewrite (22) and (23) in the form
\[
S_{n, \mu} T_{n, \lambda} = I_{n, \mu} - \sum_{\mu \in Q(\mu)} S_{n, \mu} T_{n, \mu},
\]
\[
T_{n, \mu} S_{n, \mu} = I_{n, \mu} - \sum_{\lambda \in R(\lambda)} T_{n, \mu} S_{n, \lambda}.
\]

The sums on the right-hand side define compact operators in \( X_1 \) and \( X_1 \) respectively. Hence (see § 8) the operator \( T_{n, \mu} S_{n, \mu} \) is a near-isomorphism. This finishes the proof of necessity.

The sufficiency is a consequence of the following lemma.

Lemma 9. Let \( X = X_1, Y = X_1, X_1 \cong Y_1, \lambda \in L \), \( L \) an infinite set. Then \( X \cong Y \).

Proof. Let us consider first the case when \( L \) is a countable set. Without loss of generality we may assume that \( L = \{1, 2, ..., n \} \) and \( Y_1 \cong X_1 \), \( Y_2 \cong X_2 \).

Let us consider the decompositions: \( X_1 = X_1 \oplus Z_1 \), where \( \dim Z_1 = \alpha \), \( X_2 = X_2 \oplus W_2 \), where \( \dim W_2 = \alpha + 1 \), \( \alpha = 1, 2, ... \)

For every \( i \) take any isomorphism \( T_i : X_1 \to Y_i, S_i : Y_i \to W_2 \).

Then the required isomorphism \( T : X \to Y \) can be defined by \( y_i = T_i x_i \).

In general case we represent \( L \) as a union of countable pair-wise disjoint sets: \( L = \bigcup_{n=1} \bigcup_{p \neq q} \).

Then \( X \cong X_1 \oplus X_2 \oplus X_3 \). Lemma is proved.

It is easy to state an analogous theorem for infinite sums \( X = \sum_{n} X_n, Y = \sum_{n} Y_n \). In the proof of such a theorem one needs to use Lemma 3b instead of Lemma 3a.

§ 15. We shall consider the set \( \mathcal{G} \) of all pairs \( [f, r] \), where \( f \) is a function satisfying the conditions of Definition 3 and \( \mathcal{G} \) is an order \( \mathcal{G} \). We are defining an order on \( \mathcal{G} \).
\[ a) \ r_1 > 0, \ r_2 \leq 0 \] and \( f_1 \) is a rapidly increasing function,
\[ b) \ r_1 < 0, \ r_2 \leq 0 \] and \( f_2^{-1} \circ f_1 \) is a rapidly increasing function,
\[ c) \ r_1 > 0, \ r_2 > 0 \] and \( f_2^{-1} \circ f_1 \) is a slowly increasing function,
\[ d) \ r_1 < 0, \ r_2 = 0 \] and \( f_2^{-1} \circ f_1, f_1^{-1} \circ f_1 \) are slowly increasing functions and \( f_1 \) is a rapidly increasing function,
\[ e) \ r_2 = \infty, \ -\infty < r_1 < \infty \] and \( f_1^{-1} \circ f_1, f_2^{-1} \circ f_1 \) are slowly increasing functions.

We shall say that \( [f_1, r_1] \sim [f_2, r_2] \) if both \( f_1^{-1} \circ f_2 \) and \( f_2^{-1} \circ f_1 \) are slowly increasing functions and one of the following conditions is satisfied:
\[ a) \ -\infty < r_1, r_2 < 0 \] \( \beta \), \( 0 \leq r_1, r_2 < \infty \) \( \gamma \), \( r_1 = r_2 = 0 \), \( \delta \) \( r_1 = r_2 = \infty \) \( \varepsilon \) \( f_1 \) increases slowly, \( -\infty < r_1, r_2 < \infty \).

We shall call two pairs comparable if either \( [f_1, r_1] \preceq [f_2, r_2] \) or \( [f_1, r_1] \succ [f_2, r_2] \) or \( [f_1, r_1] \sim [f_2, r_2] \).

Remark that if \( [f_1, r_1] \sim [f_2, r_2] \) and \( [f_1, r_1] \sim [g, s] \) imply \( [f_1, r_1] \sim [g, s] \).

The results of Theorems 3, 4 together with Proposition 6 and Theorem 2 can now be joined in the following statement:

**Theorem 11.** If \( [f, r] \preceq [g, s] \) then \( L_q(a, r) \lessdot L_q(b, s) \leftrightarrow \exists \alpha \) independently of the choice of the sequences \( a, b \).

From the above observations it follows that the class \( \sigma \) of all spaces \( L_q(a, r) \) satisfies conditions of § 14, if we set
\[ L_q(a, r) \lessdot L_q(b, s) \leftrightarrow [f, r] \sim [g, s], \]
\[ L_q(a, r) \lessdot L_q(b, s) \leftrightarrow [f, r] \sim [g, s]. \]

In connection with Theorem 9 for this special case it is worth noticing the following:

**Proposition 8.** (Cf. [7], pp. 170-171.) Let \( q = q^{-f} \). Then \( L_q(a, r) \) is near-isomorphic to \( L_q(b, s) \) if and only if \( [f, r] \sim [g, s] \) and there exists an integer \( r \) such that either
\[ \lim_{k\to\infty} b_q/(a_{q+r}) \leq \lim_{k\to\infty} b_q/(a_{q+r}) < \infty, \]
when \( r = \infty \) or \( r = 0 \), or \( f \) is slowly increasing and
\[ \lim_{k\to\infty} b_q/(a_{q+r}) = 1 \]
for the other cases.

\( (*) \) In the definition of the space \( L_q(a, r) \) (11) it was assumed that \( b_{q+r} \to \infty \). The theorem is valid without this assumption for the first space.

**§ 16.** Finally we shall give some applications of the previous results to the problem of quasi-equivalence of bases in nuclear spaces. The results of this section were obtained jointly with M. M. Dragilev.

Let \( X \) be a nuclear countably-normed space with a basis \( \{ a_n \} \). Let \( \{ \xi_k(p), p = 1, 2 \ldots \} \) be a system of norms defining the topology of \( X \).

According to [9], we denote by \( K(X) \) the class of all different Köthe spaces \( L(\lambda_k, \xi_k(p)) \), where \( \{ \lambda_k \} \) is an arbitrary sequence of natural numbers tending to infinity and \( \lambda_k \) is an arbitrary sequence of positive numbers (two Köthe spaces are different, if they are set-theoretically different).

**Proposition 9.** (M. M. Dragilev [9]) \( (**) \) The class \( K(X) \) does not depend on the choice of the basis \( \{ a_n \} \) and the system of norms defining the topology of \( X \), and it is a linear topological invariant.

The following statement is a modification of the Theorem 4 of [8].

**Proposition 10.** Let \( X = X_1 \times \ldots \times X_n \) be a nuclear space, \( L \) a finite or countable set. Assume that \( X \approx X_1 \), where \( Y \times K(X_1) \) iff \( Y \approx X_1 \) for all \( \lambda \in L \). Moreover, in each space \( X_1 \), all bases are quasi-equivalent. Then all bases in \( X \) are quasi-equivalent.

Proof. Assume that for each \( \lambda \in L \), \( \{ a_n \}_{n \in \Lambda} \) is a basis of \( X_1 \), the sets \( N_1 \) are pair-wise disjoint and \( \bigcup N_1 = N = \{ 1, 2, \ldots \} \). Let \( \{ a_n \}_{n \in \Lambda} \) be the basis of \( X \) obtained by joining together the bases \( \{ a_n \}_{n \in \Lambda} \). Let \( \{ y_n \}_{n \in \Lambda} \) be another basis of \( X \). According to Proposition 9 (see also [4], lemma 2.0), there exist a sequence of natural numbers \( \{ n_k \} \) tending to infinity and numbers \( \lambda_k > 0 \) such that the basis \( \{ y_k \} \) is equivalent to the unit-vector-basis of the Köthe space \( Z \approx L(\lambda_k, \xi_k(p)) \).

Denote \( M_1 = \{ \lambda_k, \xi_k(p) \} \). Let \( Z_1 \) be the Köthe space generated by the matrix \( \{ a_n \}_{n \in \Lambda} \). By the assumptions \( X \approx Z \) implies \( X_1 \approx Z_1 \) for each \( \lambda_k, \) i.e. there exist integers \( n_k \) such that \( X_1 \approx Z \approx Z_1 \). If \( X \) is finite, we can choose \( n_k \) such that \( \Sigma n_k = 0 \). Without loss of generality we shall assume that \( n_k < 0 \) for \( \lambda \in L \).

Let \( Y_1 = \text{span}(Y_k; k \in M_1) \). Then \( X \approx Y_1 \) and therefore \( Y_1 \approx X_1 \).

Since all bases in \( X_1 \) are quasi-equivalent there exist isomorphisms \( T_k : X_1 \to Y_1 \), bijective transformations \( \sigma_k : M_1 \to N_1 \) and numbers \( \lambda_k > 0 \) such that \( y_k = \lambda_k T_\sigma_k a_k \) for \( k \in M_1 \). Consequently, there are defined: an isomorphism \( T : X = Y \times K(X_1) = (T, a_k) \) where \( x = (a_k) \), a bijective transformation \( \sigma : N = N \times \sigma(a_k) = \sigma(a_k) \) for \( k \in M_1 \) and numbers \( \lambda_k > 0 \) such that \( y_k = \lambda_k \sigma a_k \). This completes the proof of the proposition.

\( (***) \) This result was announced by M. M. Dragilev at the seminar on the theory of functions and functional analysis at the Rostov University in 1966.
Теорема 12. Пусть $X = X_1 \times X_2$ — это ядерное пространство, $X_1 \times d_1$, $X_2 \times d_1$. Каждое пространство базо имеет ядерное пространство. Для всех пространств $X_2$ являются разборка-эквивалентными.

Доказательство. В каждом пространстве $X_2$, где $i = 1, 2$, все базы являются разборка-эквивалентными (см. [9], Теорема 6). Следовательно, $X \times d_1$ эквивалентно $X_1 \times d_1$ (см. определение $d_1$ в § 9), поэтому, в соответствии с Теоремой 9, все пространства $X$ являются разборка-эквивалентными.

Теорема 13. Пусть $Y = Y_1 \times Y_2$, $L$-счетное, линейно упорядоченное множество. Предположим, что $Y_1$ является ядерным пространством и $Y_2$ является ядерным пространством. Тогда пространство $Y$ является разборка-эквивалентным.

Доказательство. В каждом пространстве $Y$ все базы являются разборка-эквивалентными (см. [7], Теорема 6). Для пространств $Y = Y_1 \times Y_2$ пространство $Y$ состоит из множеств, являющихся пространствами того же типа, что и пространственные $Y_1 \times d_1$, где $(y_1, d_1)$ (правда, диаметр $d_1$) приводит к диаметру $d_1$). Поэтому, чтобы доказать, что $Y$ является разборка-эквивалентным, достаточно рассмотреть теорему 9, которая указывает, что для каждой пары пространств $X$ и $Y$ такое множество $X$ является разборка-эквивалентным. 

References