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On the isomorphism of cartesian products of locally convex spaces

by

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Abstract. The following relation $\mathfrak{R}$ between topological linear spaces is studied: $(X, Y)\in\mathfrak{R}$ iff every continuous linear operator $T\colon X\to Y$ is compact. The results concerning the relation $\mathfrak{R}$ are applied to give conditions which guarantee that the isomorphism of certain product spaces $\mathop{\times}\limits_{i=1}^{\infty} X_i$ and $\mathop{\times}\limits_{i=1}^{\infty} Y_i$ implies near isomorphisms $X_i\approx Y_i$ (i.e. the existence of Fredholm operators from X_i onto Y_i) for $i=1,2,\ldots$, and to establish some criteria of quasi-equivalence of all bases is product spaces $X\times Y$.

§ 1. Let X and Y be locally convex spaces (lcs's) (¹). A linear operator $T\colon X\to Y$ will be called a *near-isomorphism* (почти изоморфизм) if the following conditions are satisfied:

a) $T(X)$ is closed in Y and T is an open map from X onto $T(X)$,

b) $\alpha(T)=\dim\operatorname{Ker}T<\infty$,

c) $\beta(T)=\operatorname{codim}T(X)=\dim Y/T(X)<\infty$ (cf. [24])(²). The lcs's X and Y are said to be *nearly isomorphic* (почти изоморфнымн) $(X\approx Y)$(³) if there exists a near-isomorphism T from X onto Y.

In this paper we give some general conditions under which from (near) isomorphism cartesian products of lcs's $X_1\times X_2$ and $Y_1\times Y_2$ there follows that the factors are (near) isomorphic (Section II). The binary relation $(X, Y)\in\mathfrak{R}$ defined on the set of pairs of lcs's by the condition "every continuous linear operator from X to Y is compact" plays a very important role here. The greater part of this paper, Sections I, III is an examination of this relation. Our methods lead effectively to an answer to the question of the isomorphism of a wide class of spaces which are not distinguishable by their diametral dimension: $\Gamma(X_1\times X_2)=\Gamma(Y_1\times Y_2)=\Gamma(X_1)$, cf. [2], [17], [21]. In particular, we give a complete isomorphic classification of spaces of the form $X_1\times X_2$, where X_i are finite or infinite centers of Riesz scales which are Montel spaces (§ 13).

(¹) We consider only Hausdorff locally convex spaces.

(²) In [24] T is called an σ-map; one says also that T is a Fredholm operator or Φ-operator.

(³) If X and Y are isomorphic we shall write $X\simeq Y$.

In §§ 14–15 we consider finite and infinite products of lcs's in particular of $L_f(b, r)$ spaces, which were defined in [7]. Finally, as an application we quote some results (obtained jointly with Dragilev) about quasi-equivalence of bases in nuclear spaces belonging to some classes (cf. § 16).

This paper, besides some new results, contains the proofs of all results announced in [27].

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I. COMPACT OPERATORS IN LINEAR TOPOLOGICAL SPACES

§ 2. Let X and Y be lcs's. A linear operator $T\colon X \to Y$ is said to be *compact* if there exists a neighborhood U in X such that its image $T(U)$ is precompact in Y (if Y is a Montel space it is sufficient to require the boundedness of $T(U)$ in Y).

Now we shall define the relation $\mathfrak{R}$ being important in the sequel.

DEFINITION 1. We shall say that an ordered pair of lcs's (X, Y) satisfies the condition $\mathfrak{R}$ $\big((X, Y)\in\mathfrak{R}\big)$ if every linear continuous operator $T\colon X \to Y$ is compact.

Further (Section III) we shall describe a wide class of pairs of Köthe spaces which satisfy condition $\mathfrak{R}$. Before we are going to consider some examples.

§ 3. EXAMPLE 1. An lcs X will be called *pre-Montel* if every bounded set $A \subset X$ is precompact. The space X is a *Montel space* (cf. [5]) if it is barreled and pre-Montel. The following proposition characterizes the class of pre-Montel spaces in terms of the relation $\mathfrak{R}$.

PROPOSITION 1. *A necessary and sufficient condition for an lcs Y to be pre-Montel is*: $(X, Y)\in\mathfrak{R}$ *for every normed space* X.

Proof. Sufficiency. Let Y be a pre-Montel space, X a normed space and $T\colon X \to Y$ a continuous linear operator. Then T is bounded. Thus T maps the ball U in X into a bounded, and hence precompact, set $T(U)$ in Y. Therefore T is compact operator.

Necessity. Suppose $(X, Y)\in\mathfrak{R}$ for every normed space X. We shall show that every bounded set A in Y is precompact.

Let $\{\|y\|_\lambda,\ \lambda\in L\}$ be a system of pseudonorms defining the topology of Y. By the definition of a bounded set, there exists a function $m = m(\lambda) > 0$, $\lambda\in L$ such that:

$$p(y) = \sup\{m(\lambda)\|y\|_\lambda\colon\ \lambda\in L\} \leqslant 1, \qquad y\in A.$$

As X we shall take the normed space of all $y\in Y$ with $p(y) < \infty$. The set A is contained in the unit ball U of X. The operator T equal to the identity imbedding of X into Y is continuous. Thus, since $(X, Y)\in\mathfrak{R}$, it is precompact. Therefore $U = T(U)$ is precompact and therefore the set A is precompact in Y. The proposition is proved.

EXAMPLE 2. We shall say, according to Grothendieck [12], that an lcs is of *type* (S) if for every neighborhood U of zero in X there exists a neighborhood V of zero in X which is totally bounded with respect to U (cf. [20], p. 239). The class of spaces of type (S) can be characterized in terms of the relation $\mathfrak{R}$ as follows:

PROPOSITION 2. *A lcs X is of type* (S) (4) *iff* $(X, Y)\in\mathfrak{R}$ *for every Banach space* Y.

Proof. Sufficiency. Let $(X, Y)\in\mathfrak{R}$ for every Banach space Y. We shall associate with every absolutely convex neighborhood U of zero in X a seminormed space X_U which is X with the seminorm

$$p_U(x) = \inf\{\lambda > 0\colon\ x/\lambda\in U\}, \qquad x\in X.$$

We shall denote by $\tilde{X}_U$ the completion of the factor-space X_U/N_U where $N_U = \{x\in X\colon\ p(x) = 0\}$. The canonical map $\pi_U\colon X \to \tilde{X}_U$ is continuous. Since $(X, \tilde{X}_U)\in\mathfrak{R}$, π_U is compact. Thus, there exists a neighborhood V of zero in X for which $\pi_U(V)$ is precompact in $\tilde{X}_U$. Therefore V is totally bounded with respect to U, i. e. X is a space of type (S).

Necessity. Let X be a space of type (S), Y an arbitrary Banach space and $T\colon X \to Y$ an arbitrary linear operator. Then there exists a neighborhood U of zero in X such that $T(U) \subset K$, where K is the unit ball in Y. By the assumption, there exists a neighborhood $V = V(U)$ of zero in X which is totally bounded with respect to U, so $T(V)$ is totally bounded with respect to $T(U)$, and hence with respect to K. Thus the set $T(V)$ is totally bounded in Y. By the completeness of Y, $T(V)$ is precompact in Y, i. e. the operator $T\colon X \to Y$ is compact. So it is proved that $(X, Y)\in\mathfrak{R}$ for every Banach space Y.

COROLLARY 1. *The relation $\mathfrak{R}$ is not a partial order in the class of all lcs's.*

Indeed, every complete space X of type (S) is pre-Montel space, and therefore for every infinite dimensional Banach space Y, $(X, Y)\in\mathfrak{R}$ and $(Y, X)\in\mathfrak{R}$ hold simultaneously whereas $X \neq Y$ (see also Corollary 2).

EXAMPLE 3. Let A_1 be the space of all holomorphic functions in the unit disc and A_∞ the space of all entire functions of one variable. Then $(A_1, A_\infty)\in\mathfrak{R}$. This fact is a particular case of Corollary 5 (cf. § 9). But $(A_\infty, A_1)\notin\mathfrak{R}$ (cf. Corollary 3).

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(4) Every complete space of type (S) is a pre-Montel space; the contrary is not true.

EXAMPLE 4. $(c_0, l^2) \in \mathfrak{R}$ [6]. This fact and Douady's lemma (see § 6) were used in [6] to prove the unconnectedness of the group of automorphisms of the space $c_0 \times l^2$.

§ 4. We are going to demonstrate some simple but important properties of the relation $\mathfrak{R}$.

LEMMA 1. *Let $(X, Y) \in \mathfrak{R}$. Then $(X_0, Y_0) \in \mathfrak{R}$ for every subspace*(5) *X_0 which is topologically complemented in X and any subspace Y_0 of Y.*

Indeed, let T_0: $X_0 \to Y_0$ be an arbitrary linear, continuous operator. By the assumption, there exists a subspace X_1 in X such that $X = X_0 \oplus X_1$. Let T: $X \to Y$ be the linear, continuous operator

$$T(x) = \begin{cases} T_0 x & \text{for } x \in X_0, \\ 0 & \text{for } x \in X_1. \end{cases}$$

Since $(X, Y) \in \mathfrak{R}$, this operator is compact and therefore T_0 is compact too. Hence $(X_0, Y_0) \in \mathfrak{R}$.

LEMMA 2. *If $(X, Y) \in \mathfrak{R}$ and $X \simeq Y$, then X is a finite-dimensional space.*

Indeed, since there exists an isomorphism T: $X \to Y$ and $(X, Y) \in \mathfrak{R}$, it follows that T is compact; hence there exists a neighborhood U of zero in X such that $T(U)$ is precompact in Y. On the other hand $T(U)$ is a neighborhood in Y (because T is an isomorphism). Hence Y is a locally compact lcs, thus it is finite dimensional (cf. [5], p. 29). Because $X \simeq Y$ the space X is finite dimensional too. The lemma is proved.

Remark. The requirement in Lemma 1 that X_0 is topologically complemented in X is not dispensable.

Indeed, let $t(z)$ be an arbitrary analytic map of the unit disc $\{z: |z| < 1\}$ onto the complex plane. Then the operator T_0: $x(z) \to x(t(z))$ maps isomorphically the space A_∞ into A_1 [22]. By Lemma 2 $(T_0(A_\infty), A_\infty) \notin \mathfrak{R}$ (by lemma 1 it follows moreover that the subspace $T_0(A_\infty)$ is not complemented in A_1).

COROLLARY 2. *The relation $(A_\infty, A_1) \in \mathfrak{R}$ is not valid.*

In the opposite case by Lemma 1 there would be $(A_\infty, X_0) \in \mathfrak{R}$ where $X_0 = T_0(A_\infty)$, and this contradicts Lemma 2.

COROLLARY 3. *If $(X, Y) \in \mathfrak{R}$ then no infinite dimensional, complemented subspace $X_0 \subset X$ is isomorphic to any subspace Y_0 of Y. In particular, the space X is not isomorphic to any subspace Y_0 of Y.*

On the other hand as in Remark after Lemma 2 the space Y may be isomorphic to a subspace of X.

(5) Here and in the sequel by a subspace of an lcs we mean a closed linear subspace.

In particular, we have a new proof of the following

COROLLARY 4. ([19], [4]) *There is no complemented subspace of A_1 isomorphic to any subspace of the space A_∞.*

Let $\{X_\lambda, \lambda \in L\}$ be a collection of lcs's X_λ, where L is a linearly ordered set of indices. The product and the sum of the family X_λ (cf. [20], p.p. 130, 133) will be written $\bigtimes_{\lambda \in L} X_\lambda$ and $\sum_{\lambda \in L} X_\lambda$, respectively. The following lemmas will be useful in §. 14.

LEMMA 3. *Let $X = \bigtimes_{j=1}^{n} X_j$. Then* i) *from $(X_j, Y) \in \mathfrak{R}$, $j = 1, 2, \ldots, n$ it follows that $(X, Y) \in \mathfrak{R}$,* ii) *from $(Y, X_j) \in \mathfrak{R}$, $j = 1, 2, \ldots, n$ it follows that $(Y, X) \in \mathfrak{R}$.*

LEMMA 3a. *Let X be an lcs in which there exists at least one continuous norm (not a seminorm!) and $Y = \bigtimes_{\lambda \in L} Y_\lambda$. Then from $(Y_\lambda, X) \in \mathfrak{R}$ for all $\lambda \in L$ there follows $(Y, X) \in \mathfrak{R}$.*

LEMMA 3b. *Let X be an lcs in which there exists at least one bounded absolutely convex absorbing set and $Y = \sum_{\lambda \in L} Y_\lambda$. Then from $(X, Y_\lambda) \in \mathfrak{R}$ for all $\lambda \in L$ there follows $(X, Y) \in \mathfrak{R}$.*

Lemmas 3a and 3b follow from the statements: a) Under the assumptions of Lemma 3a, for every continuous operator T: $Y \to X$, there exists a finite set $\Lambda = \Lambda(T) \subset L$ such that from $y = (y_\lambda) \in Y$ and $y_\lambda = 0$ for $\lambda \in L - \Lambda$ it follows that $T(y) = 0$; b) under the assumptions of Lemma 4b for every linear continuous operator T: $X \to Y$ there exists a finite set $\Lambda = \Lambda(T) \subset L$ such that from $y = (y_\lambda) \in T(X)$ it follows that $y_\lambda = 0$ for $\lambda \in \Lambda_0$.

II. NEAR-ISOMORPHISMS AND ISOMORPHISMS OF THE CARTESIAN PRODUCTS OF LCS'S

§ 5. Let X, Y be lcs's. Recall that by the *index* of a near-isomorphism T: $X \to Y$ one understands the number ind $T = \alpha(T) - \beta(T)$, *where* $\alpha(T) = \dim\ker T$, $\beta(T) = \operatorname{codim} T(X) = \dim Y/T(X)$.

LEMMA 4 (cf. [16], [23], [24]). *Let T: $X \to Y$ be a near-isomorphism and let S: $X \to Y$ be linear compact operator. Then $T + S$ is a near-isomorphism and* $\operatorname{ind}(T+S) = \operatorname{ind} T$.

LEMMA 5. (Cf. [11], [24], [26]). *An operator T: $X \to Y$ is a near-isomorphism iff there exists an operator ψ: $Y \to X$ satisfying conditions*

a) *$\psi T = I_X + B$ where B: $X \to X$ is a compact (finite-dimensional) operator,*

b) *$T\psi = I_Y + C$ where C: $Y \to Y$ is a compact (finite-dimensional) operator.*

§ 6. The following lemma will play an important role

Douady's Lemma[6]. *Let $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ be lcs's, $(X_1, Y_2) \in \mathfrak{R}$ and let $T\colon X \to Y$ be a near-isomorphism given by the matrix $[T_{ij}]$ where $T_{ij}\colon X_j \to Y_i$. Then the operator $T\colon X \to Y$ given by the matrix*

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

is a near-isomorphism too. If moreover $(Y_1, X_2) \in \mathfrak{R}$, then the operators $T_{11}\colon X_1 \to Y_1$ and $T_{22}\colon X_2 \to Y_2$ are near-isomorphisms and

$$\operatorname{ind} T = \operatorname{ind} T_{11} + \operatorname{ind} T_{22}. \tag{1}$$

Proof. From $(X_1, Y_2) \in \mathfrak{R}$ it follows that the operator $T_{21}\colon X_1 \to Y_2$ is compact so that the operator $S\colon X \to Y$ given by the matrix

$$\begin{bmatrix} 0 & 0 \\ T_{21} & 0 \end{bmatrix}$$

is also compact. By Lemma 4, the operator $\tilde{T} = T - S\colon X \to Y$ is a near-isomorphism and

$$\operatorname{ind} T = \operatorname{ind} \tilde{T}. \tag{2}$$

By Lemma 5 there exists an operator $\psi\colon Y \to X$ such that $\psi\tilde{T} = I_X + B$, $\tilde{T}\psi = I_Y + C$, where B and C are compact in X and Y respectively. Let ψ be given by the matrix $[\psi_{ij}]$. By the second assumption $(Y_1, X_2) \in \mathfrak{R}$ so, the operator $L\colon Y \to X$ defined by the matrix

$$\begin{bmatrix} 0 & 0 \\ \psi_{21} & 0 \end{bmatrix}$$

is compact. Therefore the operator $\tilde{\psi} = \psi - L\colon Y \to X$ is a near-isomorphism and

$$\tilde{T}\tilde{\psi} = \tilde{T}(\psi - L) = I_Y + \tilde{C}, \qquad \tilde{\psi}\tilde{T} = (\psi - L)\tilde{T} = I_X + \tilde{B}, \tag{3}$$

where the operators $\tilde{B} = B - L\tilde{T}$, $\tilde{C} = C - \tilde{T}L$ are compact in X and Y, respectively. Taking into account that the operators $\tilde{T}$ and $\tilde{\psi}$ are given by the upper triangular matrices from (3) we have

$$T_{ii}\psi_{ii} = I_{Y_i} + \tilde{C}_{ii}, \qquad \psi_{ii}T_{ii} = I_{X_i} + \tilde{B}_{ii}, \qquad i = 1, 2, \tag{4}$$

[6] The case when $X_1 = Y_1$, $X_2 = Y_2$ are Banach spaces is considered by Douady in [6]; our lemma is an immediate generalization of the result of Douady.

where $\tilde{B}_{ii}$, $\tilde{C}_{ii}$ are compact in X_i and Y_i respectively. Hence, by Lemma 5, T_{11} and T_{22} are near-isomorphisms.

We shall check now the relation (1). By (4), it follows that

$$\begin{bmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{bmatrix}\begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} I_{X_1} & \psi_{12}T_{22} \\ 0 & I_{X_2} \end{bmatrix} + \begin{bmatrix} \tilde{B}_{11} & 0 \\ 0 & \tilde{B}_{22} \end{bmatrix}. \tag{5}$$

The first summand on the right-hand side of (5) is a near-isomorphism of X into itself with the index equal to zero. Since the second summand is a compact operator in X the right-hand side is a near-isomorphism with the index equal to zero. And. because $\operatorname{ind} AB = \operatorname{ind} A + \operatorname{ind} B$, we have now: $\operatorname{ind} T_{11} + \operatorname{ind} T_{22} = -\operatorname{ind}\tilde{\psi} = \operatorname{ind}\tilde{T} = \operatorname{ind} T$. So the lemma is proved.

The last proposition can be extended, by induction, as follows:

Proposition 3. *Let $X = \bigtimes_{i=s+1}^{n} X_i$, $Y = \bigtimes_{j=1}^{n} Y_j$, $(\bigtimes_{i=1}^{s} X_i, \bigtimes_{j=s+1}^{n} Y_j) \in \mathfrak{R}$ $(\bigtimes_{j=1}^{s} Y_j, \bigtimes_{i=s+1}^{n} X_i) \in \mathfrak{R}$, $s = 1, \ldots, n-1$; let $T\colon X \to Y$ be a near-isomorphism defined by the matrix $[T_{ij}]$ where $T_{ij}\colon X_j \to Y_i$, $i, j = 1, \ldots, n$. Then the operators $T_{ii}\colon X_i \to Y_i$ are near-isomorphisms for $i = 1, \ldots, n$ and*

$$\operatorname{ind} T = \sum_{i=1}^{n} \operatorname{ind} T_{ii}.$$

§ 7. As a consequence of Douady's lemma we mention the following criteria for the (near) isomorphism of cartesian products in terms of the (near) isomorphism of the factors. From Douady's lemma there follows immediately

Proposition 4. *Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ be lcs's, and $(X_1, Y_2) \in \mathfrak{R}$, $(Y_1, X_2) \in \mathfrak{R}$. Then $X \approx Y$ iff $X_1 \approx Y_1$ and $X_2 \approx Y_2$.*

Let $X^{(i)}$ denote an arbitrary subspace of the lcs X of codimension i (all such spaces are isomorphic) when $i \geqslant 0$ and when $i < 0$ an arbitrary space $X \times Z$ where $\dim Z = -i$.

In general, under the assumptions of Proposition 4, the isomorphism of X and Y does not imply the isomorphism of the factors. But the following is true:

Theorem 1. *Under the conditions of Proposition 4, $X \simeq Y$ iff there exists an s such that $Y_1 \simeq X_1^{(s)}$, $Y_2 \simeq X_2^{(-s)}$.*

Proof. Sufficiency. $Y_1 \times Y_2 \simeq X_1^{(s)} \times X_2^{(-s)} \simeq X_1 \times X_2$.

Necessity. Let $T\colon X \to Y$ be an isomorphism. By Douady's lemma, $T_{11}\colon X_1 \to Y_1$, $T_{22}\colon X_2 \to Y_2$ are near-isomorphisms. Hence $Y_1 \simeq X_1^{(s_1)}$ where $s_1 = \operatorname{ind} T$, and $Y_2 \simeq X_2^{(s_2)}$ where $s_2 = \operatorname{ind} T_{22}$. But $\operatorname{ind} T = \operatorname{ind} T_{11} + \operatorname{ind} T_{22} = 0$. So it is sufficient to take $s = s_1 = -s_2$.

§ 8. We shall denote by Φ_∞ the class of all lcs's X such that the space $X^{(i)}$ is isomorphic to X only for $i=0$. If $X\notin\Phi_\infty$, we put

$$m(X)=\inf\{i\geqslant 1:\ X^{(i)}\simeq X\},$$

and, if $X\in\Phi_\infty$ we put $m(X)=\infty$.

Every lcs X belongs to one of the classes $\Phi_s=\{X:\ m(X)=s\}$, $s=1,2,\ldots$ The classes Φ_∞ and Φ_1 are non-empty: Φ_1 contains, for example, infinite-dimensional Hilbert spaces, Φ_∞-all finite-dimensional lcs's. Examples of infinite-dimensional spaces of the class Φ_∞ were given first in [2], [17] (see also [7]). It is unknown whether the classes Φ_s for $1<s<\infty$ are non-empty.

The next lemma gives a necessary condition for a space to belong to Φ_∞.

LEMMA 6. *If $\Gamma(X)\neq\Gamma(X^{(1)})$, then $X\in\Phi_\infty$.*(7)

Indeed, by Proposition 7 of [18] the inclusions $\Gamma(X^{(s)})\supset\Gamma(X^{(1)})\supset\Gamma(X)$, $s\geqslant 1$ hold(7). Hence $\Gamma(X)\neq\Gamma(X^{(s)})$, so $X^{(s)}\not\simeq X$ for $s\geqslant 1$, which implies $X\in\Phi_\infty$.

THEOREM 2. *If $X_1, X_2\in\Phi_1$, then under the assumptions of Theorem 1 $X\simeq Y$ iff $X_1\simeq Y_1$ and $X_2\simeq Y_2$.*

III. SUFFICIENT CONDITIONS FOR $(X, Y)\in\mathfrak{R}$

The importance of the theorems of the former paragraph essentialy depends upon how large is the class of spaces satisfying the assumpions. In this paragraph we shall take this problem into consideration

§ 9. Definition 2 (cf. [7], [4]). Let X be a countably-normed space. We shall say that $X\in d_i$, $i=1,2$, if there exists in X an absolute basis $\{x_k\}_{k=1}^\infty$ and a system of norms $\{|x|_p, p=1,2,\ldots\}$, defining the topology of X such that

$$\exists p\,\forall q\,\exists r:\ |x_k|_q^2\leqslant |x_k|_p|x_k|_r\quad \text{if } k\geqslant k_0=k_0(p,q),\quad \text{for } i=1,$$
$$\forall p\,\exists q\,\forall r:\ |x_k|_q^2\geqslant |x_k|_p|x_k|_r\quad \text{if } k\geqslant k_0=k_0\ (p,r),\quad \text{for } i=2.$$

This definition is somewhat different from that given in [7] and coincides with it if in the above conditions one may take a regular basis $\{x_k\}$ (cf. [7], p. 153).

EXAMPLE 5. Following [18], we denote

$$E_a(a_k)=\lim_{\lambda<a}\operatorname{proj} 1(\exp\lambda a_k),\qquad -\infty<a\leqslant+\infty.$$

(7) For the definition of diametral dimension $\Gamma(\cdot)$ see §12.

The space $E_a(a_k)$ is called the *center of the Riesz scale* $1(\exp\lambda a_k)$. If $a<+\infty$ then $E_a(a_k)$ will be called *finite*, and if $a=\infty$ — *infinite center* of the scale. If $a_k\nearrow\infty$, then $E_a(a_k)$ is a Montel space.

PROPOSITION 5. (cf. [7], p. 154.) *If $a<\infty$, then $E_a(a_k)\in d_2$; if $a=\infty$, then $E_a(a_k)\in d_1$.*

Other examples of the spaces of type d_i will be considered in § 10.

The spaces belonging to different classes d_1 and d_2 have very different properties. The following theorem confirms it.

THEOREM 3. *Let $X\in d_2$, $Y\in d_1$ and Y be a Montel space. Then $(X, Y)\in\mathfrak{R}$.*

Proof. Let T: $X\to Y$ be an arbitrary linear, continuous operator. We have to show that T is compact.

According to Definition 2 we choose the bases $\{x_k\}$ and $\{y_k\}$ in X and Y respectively. Since they are absolute, we may assume that the topologies in X and Y are defined by the systems of norms:

$$|x|_q=\sum_{k=1}^\infty|\xi_k|\,|x_k|_q,\qquad x=\sum_{k=1}^\infty\xi_kx_k\in X,$$
$$\|y\|_p=\sum_{i=1}^\infty|\eta_i|\,\|y_i\|_p,\qquad y=\sum_{i=1}^\infty\eta_iy_i\in Y.$$

Let $h_k=Tx_k=\sum_{i=1}^\infty t_{ik}y_i$, $x=\sum_{k=1}^\infty\xi_kx_k$, $y=Tx=\sum_{=1}^\infty\eta_iy_i$. Then $\eta_i=\sum_{k=1}^\infty t_{ik}\xi_k$. The continuity of the linear operator T means that there exists a function $q=q(p)$ for which

$$\sup_k\sum_{i=1}^\infty|t_{ik}|\,\|y_i\|_p/|x_k|_q\leqslant C(p)<\infty,\qquad p=1,2,\ldots\tag{6}$$

Since Y is a Montel space, to prove that T is compact it is sufficient to show that some neighborhood $U_{q_0}=\{x\in X:\ |x|_{q_0}\leqslant 1\}$ is mapped into a bounded set, that is,

$$\sum_{i=1}^\infty|\eta_i|\,\|y_i\|_p\leqslant M(p)<\infty,\qquad p=1,2,\ldots$$

if $y=\sum_{i=1}^\infty\eta_iy_i=Tx$, $x\in U_{q_0}$. Certainly it will hold if there exists $q=q_0$ such that

$$\sup_k\sum_{i=1}^\infty|t_{ik}|\,\|y_i\|_p/|x_k|_{q_0}\leqslant M(p)<\infty.\tag{7}$$

Since $Y\in d_1$, by Definition 2

$$\exists p_1\forall p\exists p_2=p_2(p):\ \|y_i\|_p^2\leqslant\|y_i\|_{p_1}\|y_i\|_{p_2}\qquad\text{for } i\geqslant i_0(p)\tag{8}$$

and since $X \epsilon d_2$ for $q = q(p_1)$, there exists $q = q_0$ such that for every q_2 (we choose $q_2 = q(p_2)$):

$$|x_k|_{q_0}^2 \geqslant |x_k|_{q_1} |x_k|_{q_2}, \qquad k \geqslant k_0(p). \tag{9}$$

By (8), (9), (6), and the Cauchy inequality we obtain:

$$\sum_{l=1}^{\infty} |t_{ik}| \frac{\|y_i\|_p}{|x_k|_{q_0}} \leqslant L(p) \sum_{l=1}^{\infty} \left(|t_{ik}| \frac{\|y_i\|_{p_1}}{|x_k|_{q_1}}\right)^{1/2} \left(|t_{ik}| \frac{\|y_i\|_{p_2}}{|x_k|_{q_2}}\right)^{1/2}$$
$$\leqslant L(p) \left(\sum_{i=1}^{\infty} |t_{ik}| \frac{\|y_i\|_{p_1}}{|x_k|_{q_1}}\right)^{1/2} \left(\sum_{i=1}^{\infty} |t_{ik}| \frac{\|y_i\|_{p_2}}{|x_k|_{q_2}}\right)^{1/2}$$
$$\leqslant L(p) \big(C(p_1)\big)^{1/2} \big(C(p_2)\big)^{1/2} = M(p) < \infty, \qquad p = 1, 2, \ldots,$$

which means that inequality (7) holds, and so the operator T is compact. The theorem is proved.

Applying Proposition 5 we have

COROLLARY 5. *If* $a < \infty$, *then* $\big(E_a(a_k), E_\infty(b_k)\big) \epsilon \mathfrak{R}$, *independently of* (a_k) *and* (b_k), *provided* $b_k \nearrow \infty$.

§ 10. A more exact description of the relation $(X, Y) \epsilon \mathfrak{R}$ can be obtained for a special class of Köthe spaces which were considered in [7] by M. M. Dragilev.

DEFINITION 2. (Cf. [7], p. 169). Let f be an increasing odd function on $(-\infty, +\infty)$ which is logarithmically convex on $[0, +\infty)$ (i. e. $\varphi(w) = \ln(f(\exp w))$ is convex on $(-\infty, +\infty)$); $r_p \nearrow r$, $b = (b_n)$, $b_n \nearrow \infty$. Denote by $L_f(b, r)$ the Köthe space generated by the matrix $[\exp f(r_p b_n)]$, i. e.

$$L_f(b, r) = \lim_{p\to\infty} \operatorname{proj} 1 [\exp f(r_p b_n)].$$

Without loss of generality we may assume that $f(u) \geqslant 0$ for $u \geqslant 0$.

DEFINITION 3. (Cf. [7], p. 170). We shall say that an increasing function φ defined on $[0, \infty)$ *increases rapidly* if, for every $a < 1$, $\varliminf_{t\to\infty} \varphi(at)/\varphi(t) = \infty$, and it *increases slowly* if, for every $a > 1$, $\varlimsup_{t\to\infty} \varphi(at)/\varphi(t) = \tau(a) < \infty$.

LEMMA 7. (Cf. [7]). *Let f be an increasing logarithmically convex function on* $[0, \infty)$. *Then, for every* $a > 1$,

$$\lim_{u\to\infty} f(au)/f(u) = \tau(a).$$

Moreover, either a) $\tau(a) \equiv \infty$, *or* b) $\tau(a) < \infty$ *for* $1 < a < \infty$ *and* $\tau(a) \nearrow \infty$ *for* $a \nearrow \infty$.

So, under the conditions of Definition 3, the function f is always slowly or rapidly increasing.

PROPOSITION 6. (Cf. [7], p. 170). *Let $X = L_f(b, r)$ be a space satisfying conditions of Definition 3. Under the assumption that f is slowly increasing, X is isomorphic to a finite center of Riesz scale provided $r < \infty$, and X is isomorphic to an infinite center* provided $r = \infty$; in the first case $X \epsilon d_1$, in the second $X \epsilon d_2$.

Assuming that f is rapidly increasing we have $X \epsilon d_1$ if $0 < r \leqslant \infty$ and $X \epsilon d_2$ if $-\infty < r < 0$.

THEOREM 4. *Let f_1, f_2 satisfy conditions of Definition 3 and let $\varphi = f_1^{-1} \circ f_2$ be rapidly increasing. Then*

$$\big(L_{f_1}(a, r), L_{f_2}(b, s)\big) \epsilon \mathfrak{R}$$

if $0 < r \leqslant \infty$, $0 < s \leqslant \infty$ and

$$\big(L_{f_2}(b, s), L_{f_1}(a, r)\big) \epsilon \mathfrak{R}$$

if $-\infty < r \leqslant 0$, $-\infty < s \leqslant \infty$ *independently of the choice of the sequences* $a = (a_k)$, $b = (b_k)$.

Proof. First, we shall consider only the case $0 < r \leqslant \infty$, $0 < s \leqslant \infty$. Let $T\colon L_{f_1}(a, r) \to L_{f_2}(b, s)$ be an arbitrary linear continuous operator. We represent T as a matrix in bases of unit vectors in X and Y. Arguing as in Theorem 2 we may assert the existence of a function $\varrho = \varrho(\sigma)$ such that

$$\sup_k \sum_{i=1}^{\infty} |t_{ik}| \exp\{f_2(\sigma b_i) - f_1(\varrho a_k)\} \leqslant C(\sigma) < \infty, \tag{10}$$
$$\text{for } 0 < \sigma < s, \ 0 < \varrho(\sigma) < r.$$

We must show that T is compact. Since the space $L_{f_2}(b, s)$ is Montel, it is sufficient, as in Theorem 2, to show the existence of $\varrho = \varrho_0 < r$ for which

$$\sup_k \sum_{i=1}^{\infty} |t_{ik}| \exp\{f_2(\sigma b_i) - f_1(\varrho_0 a_k)\} \leqslant M(\sigma) < \infty, \qquad 0 < \sigma < s. \tag{11}$$

We shall show that (11) holds if we take ϱ_0 so that

$$\inf\{\varrho(\sigma)\colon 0 < \sigma < s\} < \varrho_0 < r. \tag{12}$$

We fix an arbitrary σ: $0 < \sigma < s$ and C with $0 < C < \infty$. We designate

$$N_k = N_k(C, \sigma) = \{i\colon a_k \leqslant C\varphi(\sigma b_i)\}. \tag{13}$$

We break the sum in (11) into two summands which we shall estimate separately.

First we shall estimate:

$$S_k(\sigma) = \sum_{i \notin N_k} |t_{ik}| \exp\{f_2(\sigma b_i) - f_1(\varrho_0 a_k)\}.$$

We take σ_2 such that $0 < \sigma_2 < s$ and $\varrho_2 = \varrho(\sigma_2) < \varrho_0$ because of (12) it is possible. Then, using (10) we obtain

$$S_k(\sigma) \leqslant C(\sigma_2)\sup\{\exp L_{ik}(\sigma):\ i \notin N_k\},$$

where

$$L_{ik}(\sigma) = f_2(\sigma b_i) - f_1(\varrho_0 a_k) - f_2(\sigma_2 b_i) + f_1(\varrho_2 a_k).$$

By (13), for $i \notin N_k$ the inequality (14) holds

$$(14)\qquad L_{ik}(\sigma) \leqslant \sup\left\{f_1\left(\frac{a}{C}\right) + f_1(\varrho_2 a) - f_1(\varrho_0 a):\ 0 < a < \infty\right\}.$$

By Lemma 7 the expression under the sup divided by $f_1(\varrho_0 a)$, for $a \to \infty$, tends to

$$(15)\qquad \frac{1}{\tau(C\varrho_0)} + \frac{1}{\tau\left(\dfrac{\varrho_0}{\varrho_2}\right)} - 1.$$

Choosing the constant C sufficiently large it is possible to make the number (15) negative, because $\varrho_2 < \varrho_0$. But this means that for $a \geqslant a_0 = a_0(\sigma)$ the expression under the sup in (14) is negative. Hence

$$S_k(\sigma) \leqslant C(\sigma_2)\exp\sup\left\{f_1\left(\frac{a}{C}\right) + f_1(\varrho_2 a) - f_1(\varrho_0 a):\ 0 < a \leqslant a_0(\sigma)\right\} = M_2(\sigma).$$

Now we estimate the second summand:

$$R_k(\sigma) = \sum_{i \in N_k} |t_{ik}| \exp\{f_2(\sigma b_i) - f_1(\varrho_0 a_k)\}.$$

We take $\sigma_1 = \sigma_1(\sigma)$, $\sigma < \sigma_1 < s$ and $\varrho_1 = \varrho(\sigma_1)$. Using (10) we obtain

$$(16)\qquad R_k(\sigma) \leqslant C(\sigma_1)\sup\{\exp \tilde{L}_{ik}(\sigma):\ i \in N_k\},$$

where

$$\tilde{L}_{ik}(\sigma) = f_2(\sigma b_i) - f_1(\varrho_0 a_k) - f_2(\sigma_1 b_i) + f_1(\varrho_1 a_k).$$

By (13) for $i \in N_k$ we have the inequality

$$(17)\qquad \begin{aligned}\tilde{L}_{ik}(\sigma) &= -f_1\big(\varphi(\sigma_1 b_i)\big) + f_1\big(\varphi(\sigma b_i)\big) + f_1(\varrho_1 a_k) - f_1(\varrho_0 a_k)\\ &\leqslant \sup\{-f_1\big(\varphi(\sigma_1 b)\big) + f_1\big(\varphi(\sigma b)\big) + f_1\big(C\varrho_1\varphi(\sigma b)\big):\ 0 < b < \infty\}.\end{aligned}$$

Since the function φ increases rapidly

$$\forall \Delta > 0\ \exists b_0 = b_0(\Delta, \sigma)\ \forall b \geqslant b_0:\ \varphi(\sigma_1 b) \geqslant \Delta\varphi(\sigma b).$$

So for $b \geqslant b_0$

$$(18)\qquad \begin{aligned}&-f_1\big(\varphi(\sigma_1 b)\big) + f_1\big(\varphi(\sigma b)\big) + f_1\big(C\varrho_1\varphi(\sigma b)\big)\\ &\qquad\leqslant -f_1\big(\Delta\varphi(\sigma b)\big) + f_1\big(\varphi(\sigma b)\big) + f_1\big(C\varrho_1\varphi(\sigma b)\big).\end{aligned}$$

By Lemma 7, the right-hand side of this inequality divided by $f_1\big(\Delta\varphi(\sigma b)\big)$ tends to

$$-1 + \frac{1}{\tau(\Delta)} + \frac{1}{\tau\left(\dfrac{\Delta}{C\varrho_1}\right)}$$

for $b \to \infty$. Choosing $\Delta = \Delta(\sigma)$ sufficiently large the right hand side of the inequality (18) is negative for $b \geqslant b_1(\sigma)$. So, by (16), (17) and (18), we obtain

$$\begin{aligned}&R_k(\sigma)\\ &\leqslant C_1(\sigma)\exp\sup\{-f_1\big(\varphi(\sigma_1 b)\big) + f_1\big(\varphi(\sigma b)\big) + f_1\big(C\varrho_1\varphi(\sigma b)\big):\ 0 < b \leqslant b_1(\sigma)\}\\ &= M_1(\sigma) < \infty.\end{aligned}$$

So, inequality (11) holds for $M(\sigma) = M_1(\sigma) + M_2(\sigma)$ if ϱ_0 is chosen as in (12).

The proof of the second part of the theorem is analogous.

THEOREM 5. (i) $\big(L_f(a, r),\ L_f(b, \infty)\big) \in \mathfrak{R}$ *if* $0 < r < \infty$.

(ii) $\big(L_f(a, 0),\ L_f(b, r)\big) \in \mathfrak{R}$ *if* $-\infty < r < 0$ *and the function* f *increases rapidly.*

This theorem may be proved by methods given in the proof of Theorem 3. If $f(u) \equiv u$ the statement (i) coincides with Corollary 3.

IV. LINEAR TOPOLOGICAL INVARIANTS OF CARTESIAN PRODUCTS

§ 11. Let $\mathscr{E}$ be the class of lcs's and let K be a certain set. One says that the map $\tau:\ \mathscr{E} \to K$ generates a linear topological invariant $\tau(X)$, $X \in \mathscr{E}$ if the fact that $X, Y \in \mathscr{E}$ and X, Y are isomorphic implies that $\tau(X) = \tau(Y)$.

One of the easy to compute linear topological invariants is the *diametral dimension* ([17], [18]):

$$\Gamma(X) = \{\gamma = (\gamma_n):\ \forall\, U\, \exists\, V\ \gamma_n d_n(V, U) \to 0,\ n \to \infty\}$$

where U and V are neighborhoods of zero in the space X and $d_n(V, U)$, is the n-dimensional Kolmogorov diameter of V with respect to U ([2], [13], [14], [25]).

The consideration of invariants like $\Gamma(X)$ leads to the solution of isomorphism problems for many lcs's (see e. g. [7], [14], [15], [18]). But there exists some simple examples ([2], [7], [17], [21]) of spaces of the form $X \times Y_1$, $X \times Y_2$ for which $\Gamma(X \times Y_1) = \Gamma(X \times Y_2) = \Gamma(X)$, though $X \not\simeq X \times Y_i$, $i = 1, 2$. The dimensions $\Gamma(Y_i)$ are "absorbed" by the dimension of the factor.

In these cases the comparison of the diametral dimensions does not give the means to distinguish non-isomorphic spaces X and $X\times Y_i$ and the question of isomorphism of the spaces $X\times Y_1$ and $X\times Y_2$ is unsolved if $\Gamma(Y_1)\neq\Gamma(Y_2)$.

Theorem 1 allows us to define a new linear topological invariant stronger than the diametral dimension but defined not on the whole class of lcs's. It is defined only for some special (but considerably large) classes of lcs's which contain in particular all the spaces mentioned in the above examples.

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two classes of lcs's such that $(X_1, X_2)\in\mathfrak{R}$ whenever $X_1\in\mathscr{E}_1$ and $X_2\in\mathscr{E}_2$.

We shall write

$$\mathscr{E}_1\times\mathscr{E}_2 = \{X = X_1\times X_2\colon X_1\in\mathscr{E}_1, X_2\in\mathscr{E}_2\}$$

and with every $X\in\mathscr{E}_1\times\mathscr{E}_2$ we shall associate the set $\tilde{\Gamma}(X)$ of all different pairs $\big(\Gamma(X_1^{(s)}), \Gamma(X_2^{(-s)})\big)$, where $s = 0, \pm1, \ldots$ From Theorem 1 we obtain immediately

THEOREM 6. *$\tilde{\Gamma}(X)$ is a linear topological invariant defined on $\mathscr{E} = \mathscr{E}_1\times\mathscr{E}_2$.*

Remark 1. One may obtain in the same way other linear topological invariants $\tilde{\tau}(X)$ defined on $\mathscr{E} = \mathscr{E}_1\times\mathscr{E}_2$ instead of $\Gamma(X)$ considering another topological invariant $\tau(X)$ (for example $\Phi(X)$, the approximative dimension (cf. [14], [15])).

Remark 2. From Theorem 2, it follows that Theorem 5 is applicable to the class $\mathscr{E} = d_2\times(M\cap d_1)$ where M is the class of all Montel lcs's. Let $\tilde{d}_i$ denote the class of all spaces from d_i for which in Definition 2 it is possible to take as $\{x_k\}$ a regular basis.

Recall that a basis $\{x_k\}$ in X is *regular* (cf. [7], p. 153) if there exists a system of norm $\{\|x\|_p\}$ defining the topology of X and such that all sequences $\|x_k\|_p/\|x_k\|_q$ are monotone. Then we have

PROPOSITION 7. (cf. [7]) *Two spaces X and Y in $\tilde{d}_1\cup\tilde{d}_2$ are isomorphic iff they are both in the same class d_i and $\Gamma(X) = \Gamma(Y)$.*

This proposition differs from the Theorem 7 of [7] by dropping the requirement of the nuclearity of the spaces (which in fact is not used in the proof).

Using Theorems 5, 6 and Proposition 7 we obtain following

THEOREM 7. *If X and Y are in $\mathfrak{D} = \tilde{d}_2\times(M\cap\tilde{d}_1)$, then $X\simeq Y$ iff $\tilde{\Gamma}(X) = \tilde{\Gamma}(Y)$.*

§ 12. It is quite simple to compute the invariant $\tilde{\Gamma}(X)$ if $X_i\in\Phi_1$, $i = 1, 2$. In this case, the set $\tilde{\Gamma}(X)$ contains only one element: the pair $\big(\Gamma(X_1), \Gamma(X_2)\big)$. If $X_i\in\Phi_{m_i}$, where $m_i<\infty$, $i = 1, 2$, then $\tilde{\Gamma}(X)$ contains a finite number (equal to $\sup\{m_1, m_2\}$) of elements: however if one of m_i is infinity, then the set $\tilde{\Gamma}(X)$ is countable.

§ 13. Let us recall the notation

$$E_a(a_k) = \lim_{\lambda<a}\operatorname{proj} l(\exp\lambda a_k).$$

THEOREM 8. *Let $X = E_a(a_k^{(1)})\times E_\infty(a_k^{(2)})$, $Y = E_\beta(b_k^{(1)})\times E_\infty(b_k^{(2)})$, where $a, \beta<\infty$, $a_k^{(i)}, b_k^{(i)}\nearrow\infty$. Then X and Y are isomorphic iff there exists an integer s such that*

$$0<\varliminf_{k\to\infty} a_{k+s}^{(1)}/b_k^{(1)}\leqslant\varlimsup_{k\to\infty} a_{k+s}^{(1)}/b_k^{(1)}<\infty, \tag{19}$$

$$0<\varliminf_{k\to\infty} a_{k-s}^{(2)}/b_k^{(2)}\leqslant\varlimsup_{k\to\infty} a_{k-s}^{(2)}/b_k^{(2)}<\infty. \tag{20}$$

This theorem follows from Theorem 6 and the isomorphism criterion of centers of scales (Proposition 18 in [18]).

Remark. If

$$\varliminf_{k\to\infty} a_k^{(1)}/a_{k+1}^{(1)}>0, \quad \varliminf_{k\to\infty} a_k^{(2)}/a_{k+1}^{(2)}>0, \tag{21}$$

then, using the notation of Theorem 7, a necessary and sufficient condition for $X\simeq Y$ is that (19) and (20) be satisfied for $s = 0$. If one of the inequalities (21) is satisfied, then in Theorem 7 one may take $s = 0$ in the corresponding inequality (19) or (20).

This statement may be obtained from Corollary 5 and the following lemma.

LEMMA 8. *Let $X = E_a(a_k)$, $a\leqslant\infty$, $a_k\nearrow\infty$. Then*

(i) $\varliminf_{k\to\infty}\dfrac{a_k}{a_{k+1}}>0$ *iff* $X\in\Phi_1$,

(ii) $\varliminf_{k\to\infty}\dfrac{a_k}{a_{k+1}} = 0$ *iff* $X\in\Phi_\infty$.

Using the fact that from Proposition 18 of [10], and from $a_k\leqslant a_{k+1}$ it follows

$$\varliminf_{k\to\infty}\frac{a_k}{a_{k+1}}>0 \quad \text{iff } \Gamma(X) = \Gamma(X^{(1)})$$

it is sufficient to apply Lemma 6 (§ 8).

The product $X = X_1\times X_2$ of two centers of the same type of Riesz scales is isomorphic to the center of a Riesz scale too. Therefore Theorem 7 combined with Proposition 18 of [18] gives us a complete isomorphic classification of all spaces of the form $X_1\times X_2$, where X_1, X_2 are finite or infinite centers of compact Riesz scales.

The following statement on spaces of analytic functions of several complex variables is an immediate consequence of Theorem 7.

THEOREM 9. *Let $A(E^n)$ be the space of all analytic functions in the unit polycylinder E^n in C^n, and let $A(C^k)$ be the space of all entire functions of k variables. Then*

$$A(E^n)\times A(C^k)\simeq A(E^{n_1})\times A(C^{k_1}) \quad \textit{iff } (n,k)=(n_1,k_1).$$

§ 14. Assume that two binary relations are defined on a class $\mathscr{E}$ of lcs's: the partial-order relation α and the equivalence relation β. Furthermore the relations α,β are *compatible*, i.e. $X_1\beta X$, $Y_1\beta Y$ *and* $X\alpha Y$ *imply* $X_1\alpha Y_1$. Two spaces $X, Y\in\mathscr{E}$ are *comparable* if *either* $X\alpha Y$ *or* $Y\alpha X$ *or* $X\beta Y$. Assume that in every space $X\in\mathscr{E}$ there is a continuous norm (or, equivalently, there is an absolutely convex neighborhood of zero containing no line). Finally, assume that $X\alpha Y$ implies $(X,Y)\in\mathfrak{R}$.

Under these assumpions we have

THEOREM 10. *Let $X=\bigtimes_{\lambda\in L}X_\lambda$, $Y=\bigtimes_{\mu\in M}Y_\mu$ where X_λ, Y_μ are infinite-dimensional lcs's, L, M are linearly ordered sets and $X_\lambda, Y_\mu\in\mathscr{E}$ for $\lambda\in L$ and $\mu\in M$. Furthermore, $X_\lambda\alpha X_{\lambda_1}$, $Y_\mu\alpha Y_{\mu_1}$ for $\lambda<\lambda_1$ and $\mu<\mu_1$ and all the pairs X_λ, Y_μ are comparable. Then:*

If L an infinite set, then $X\simeq Y$ iff

(a) *there is an order-isomorphism $\tau\colon L\to M$ with $X_\lambda\approx Y_{\tau(\lambda)}$.*

If the set L is finite, then $X\simeq Y$ iff the conditions (a) *above and*

(b) *there are integers ν_λ with $X^{(\nu_\lambda)}\simeq Y_{\tau(\lambda)}$ and $\sum\nu_\lambda=0$, are satisfied.*

In the case when L is finite this theorem can easily be reduced to Proposition 3 (§ 6). So we shall consider only infinite sets L.

Proof. Necessity. Let i_λ denote the identity injection of X_λ into X and p_λ — the natural projection of X onto X_λ. For the space Y we use respective notations j_μ, q_μ.

Let $T\colon X\to Y$ be an isomorphism and $S=T^{-1}\colon Y\to X$. Wirte $T_{\mu,\lambda}=q_\mu\cdot T\cdot i_\lambda\colon X_\lambda\to Y_\mu$, $S_{\lambda,\mu}=p_\lambda\cdot S\cdot j_\mu\colon Y_\mu\to X_\lambda$. The statement (a) after Lemma 3b gives: for every $\mu\in M$ there exists a finite set $Q=Q(\mu)\subset L$ such that $T_{\mu,\lambda}=0$ for $\lambda\in L\setminus Q$, and for every $\lambda\in L$ there exists a finite set $R=R(\lambda)\subset M$ such that $S_{\lambda,\mu}=0$ for $\mu\in M\setminus R$. Moreover

$$\sum_{\lambda\in Q(\mu)}T_{\mu,\lambda}\cdot S_{\lambda,\mu}=I_{Y_\mu}, \tag{22}$$

$$\sum_{\mu\in R(\lambda)}S_{\lambda,\mu}T_{\mu,\lambda}=I_{X_\lambda}. \tag{23}$$

We shall show that for every $\lambda\in L$ there exists exactly one $\mu=\tau(\lambda)\in M$ such that $X_\lambda\beta Y_\mu$. Uniqueness of μ follows from compatibility of α and β and monotony of the families $\{X_\lambda\}$ and $\{Y_\mu\}$. Suppose that for some $\lambda\in L$ there is no $\mu\in M$ with $X_\lambda\beta Y_\mu$. Then, for every $\mu\in M$ either $X_\lambda\alpha Y_\mu$ or $Y_\mu\alpha X_\lambda$.

Therefore either $(X_\lambda,Y_\mu)\in\mathfrak{R}$ or $(Y_\mu,X_\lambda)\in\mathfrak{R}$. Hence $S_{\lambda,\mu}T_{\mu,\lambda}\colon X_\lambda\to Y_\mu$ is compact for $\mu\in M$. The last statement contradicts (23) because X_λ is infinite-dimensional.

Thus, the transformation $\tau\colon L\to M$ satisfies the condition $X_\lambda\beta Y_{\tau(\lambda)}$. This transformation is bijective (applying the previous arguments to M and L taken in the reverse order we obtain a transformation $\sigma\colon M\to L$ which is the inverse of τ). Moreover τ is an order-isomorphism. Indeed, let $\lambda<\lambda_1$, then since $X_\lambda\beta Y_{\tau(\lambda)}$, $X_{\lambda_1}\beta Y_{\tau(\lambda_1)}$ and $X_\lambda\alpha X_{\lambda_1}$, the relation $Y_{\tau(\lambda)}\alpha Y_{\tau(\lambda_1)}$ holds and therefore $\tau(\lambda)<\tau(\lambda_1)$.

We shall show that $X_\lambda\approx Y_{\tau(\lambda)}$. To that end rewrite (22) and (23) in the form

$$S_{\lambda,\tau(\lambda)}T_{\tau(\lambda),\lambda}=I_{X_\lambda}-\sum_{\substack{\mu\in R(\lambda)\\ \mu\neq\tau(\lambda)}}S_{\lambda,\mu}T_{\mu,\lambda},$$

$$T_{\tau(\lambda),\lambda}S_{\lambda,\tau(\lambda)}=I_{Y_{\tau(\lambda)}}-\sum_{\substack{\nu\in Q(\tau(\lambda))\\ \nu\neq\lambda}}T_{\tau(\lambda),\nu}S_{\nu,\tau(\lambda)}.$$

The sums on the right-hand side define compact operators in X and Y respectively. Hence (see §5) the operator $T_{\tau(\lambda),\lambda}\colon X_\lambda\to Y_{\tau(\lambda)}$ is a near-isomorphism. This finishes the proof of necessity.

The sufficiency is a consequence of the following lemma.

LEMMA 9. *Let $X=\bigtimes_{\lambda\in L}X_\lambda$, $Y=\bigtimes_{\lambda\in L}Y_\lambda$, $X_\lambda\approx Y_\lambda$, $\lambda\in L$, L an infinite set. Then $X\simeq Y$.*

Proof. Let us consider first the case when L is a countable set. Without loss of generality we may assume that $L=\{1,\ldots,i,\ldots\}$ and $Y_i\simeq X^{(\nu_i)}$, $\nu_i\geqslant 0$, $i=1,2,\ldots$

Let us consider the decompositions: $X_i=\tilde{X}_i\oplus Z_i$, where $\dim Z_i=\sigma_i=\nu_1+\ldots+\nu_i$ and $Y_i=\tilde{Y}_i\oplus W_i$, where $\dim W_i=\sigma_{i-1}$, $\sigma_0=0$, $i=1,2,\ldots$ For every i take any isomorphism $T_i\colon X_i\to Y_i$, $S_i\colon Z_i\to W_{i+1}$. Then the required isomorphism $T\colon X\to Y$ can be defined by $y_1=T_1x_1$, $y_i=T_ix_i+S_{i-1}x_{i-1}$ for $i\geqslant 2$ if $x=(x_i)$, $y=(y_i)$.

In general case we represent L as a union of countable pair-wise disjoint sets: $L=\bigcup_{\gamma\in\Gamma}L_\gamma$. Then $X\simeq\bigtimes_{\gamma\in\Gamma}(\bigtimes_{\lambda\in L_\gamma}X_\lambda)\simeq\bigtimes_{\gamma\in\Gamma}(\bigtimes_{\lambda\in L_\gamma}Y_\lambda)\simeq Y$. Lemma is proved.

It is easy to state an analogous theorem for infinite sums $X=\sum_{\lambda\in L}X_\lambda$, $Y=\sum_{\mu\in M}Y_\mu$. In the proof of such a theorem one needs to use Lemma 3b instead of Lemma 3a.

§ 15. We shall consider the set $\mathfrak{F}$ of all pairs $[f,r]$, where f is a function satisfying the conditions of Definition 3 and $-\infty<r\leqslant\infty$. We are defining an order on $\mathfrak{F}$: $[f_1,r_1]\succ[f_2,r_2]$, if one the conditions holds:

a) $r_1 > 0$, $r_2 \leqslant 0$ and f_2 is a rapidly increasing function,

b) $r_1 \leqslant 0$, $r_2 \leqslant 0$ and $f_1^{-1} \circ f_2$ is a rapidly increasing function,

c) $r_1 > 0$, $r_2 > 0$ and $f_2^{-1} \circ f_1$ is a slowly increasing function,

d) $r_1 < 0$, $r_2 = 0$ and $f_1^{-1} \circ f_2, f_2^{-1} \circ f_1$ are slowly increasing functions and f_1 is a rapidly increasing function,

e) $r_1 = \infty$, $-\infty < r_2 < \infty$ and $f_1^{-1} \circ f_2, f_2^{-1} \circ f_1$ are slowly increasing functions.

We shall say that $[f_1, r_1] \sim [f_2, r_2]$ if both $f_1^{-1} \circ f_2$ and $f_2^{-1} \circ f_1$ are *slowly increasing functions* and one of the following conditions is satisfied: α) $-\infty < r_1, r_2 < 0$, β) $0 < r_1, r_2 < \infty$, γ) $r_1 = r_2 = 0$, δ) $r_1 = r_2 = \infty$, ε) f_1 increases slowly, $-\infty < r_1, r_2 < \infty$.

We shall call two pairs comparable if either $[f_1, r_1] \succ [f_2, r_2]$ or $[f_2, r_2] \succ [f_1, r_1]$ or $[f_1, r_1] \sim [f_2, r_2]$.

Remark that $[f_1, r_1] \sim [f_2, r_2]$ and $[f_1, r_1] \succ [g, s]$ imply $[f_2, r_2] \succ [g, s]$.

The results of Theorems 3, 4 togethe rwith Proposition 6 and Theorem 2 can now be joined in the following statement:

THEOREM 11. *If* $[f, r] \succ [g, s]$ *then* $(L_f(a, r), L_g(b, s)) \in \mathfrak{R}$ *independently of the choice of the sequences* a, b. (8)

From the above observations it follows that the class $\mathscr{E}$ of all spaces $L_f(a, r)$ satisfies conditions of § 14, if we set

$$L_f(a, r)\, \alpha L_g(b, s) \overset{\text{df}}{\equiv} [g, s] \succ [f, r],$$

$$L_f(a, r)\, \beta L_g(b, s) \overset{\text{df}}{\equiv} [f, r] \sim [g, s].$$

In connection with Theorem 9 for this special case it is worth noticing the following

PROPOSITION 8. (Cf. [7], pp. 170-171). *Let* $\varphi = g^{-1} \circ f$. *Then* $L_f(a, r)$ *is near-isomorphic to* $L_g(b, s)$ *iff* $[f, r] \sim [g, s]$ *and there exists an integer* ν *such that either*

$$0 < \varliminf_{k \to \infty} b_k / \varphi(a_{k+\nu}) \leqslant \varlimsup_{k \to \infty} b_k / \varphi(a_{k+\nu}) < \infty,$$

when $r = \infty$ *or* $r = 0$, *or* f *is slowly increasing and*

$$\lim_{k \to \infty} s b_k / \varphi(r a_{k+\nu}) = 1$$

for the other cases.

(8) In the definition of the space $L_f(b, r)$ (§11) it was assumed that $b_k \nearrow \infty$. The theorem is valid without this assumption for the first space.

§ 16. Finally we shall give some applications of the previous results to the problem of quasi-equivalence of bases in nuclear spaces. *The results of this section were obtained jointly with M. M. Dragilev.*

Let X be a nuclear countably-normed space with a basis $\{x_k\}$. Let $\{\|x\|_p,\ p = 1, 2, \ldots\}$ be a system of norms defining the topology of X. According to [9], we denote by $K(X)$ the class of all different Köthe spaces $L(\lambda_k \|x_{n_k}\|_p)$, where $\{n_k\}$ is an arbitrary sequence of natural numbers tending to infinity and λ_k is an arbitrary sequence of positive numbers (two Köthe spaces are different, if they are set-theoretically different).

PROPOSITION 9. (M. M. Dragilev [9])(9) *The class* $K(X)$ *does not depend on the choice of the basis* $\{x_k\}$ *and the system of norms defining the topology of* X, *and it is a linear topological invariant.*

The following statement is a modification of the Theorem 4 of [8].

PROPOSITION 10. *Let* $X = \bigtimes_{\lambda \in L} X_\lambda$ *be a nuclear space,* L — *a finite or countable set. Assume that* $X \approx \bigtimes_{\lambda \in L} Y_\lambda$, *where* $Y_\lambda \in K(X_\lambda)$ *iff* $Y_\lambda \approx X_\lambda$ *for all* $\lambda \in L$. *Moreover, in each space* X_λ *all bases are quasi-equivalent. Then all bases in* X *are quasi-equivalent.*

Proof. Assume that for each $\lambda \in L$, $\{x_k\}_{k \in N_\lambda}$ is a basis of X_λ, the sets N_λ are pair-wise disjoint and $\bigcup_{\lambda \in L} N_\lambda = N = \{1, 2, \ldots\}$. Let $\{x_k\}_{k \in N}$ be the basis of X obtained by joining together the bases $\{x_k\}_{k \in N_\lambda}$. Let $\{y_k\}$ be another basis of X. According to Proposition 9 (see also [4], lemma 2.0), there exist a sequence of natural numbers $\{n_k\}$ tending to infinity and numbers $\lambda_k > 0$ such that the basis $\{y_k\}$ is equivalent to the unit-vector-basis of the Köthe space

$$Z = L(\lambda_k \|x_{n_k}\|_p).$$

Denote $M_\lambda = \{k\colon n_k \in N_\lambda\}$. Let Z_λ be the Köthe space generated by the matrix $[\lambda_k \|x_{n_k}\|_p]_{k \in M_\lambda}$. By the assumptions $X \simeq Z$ implies $X_\lambda \approx Z_\lambda$ for each λ, i. e. there exist integers ν_λ such that $X_\lambda \simeq Z_\lambda^{(\nu_\lambda)}$. If L is finite, we can choose ν_λ such that $\Sigma \nu_\lambda = 0$. Without loss of generality we shall assume that $\nu_\lambda = 0$ for $\lambda \in L$.

Let $Y_\lambda = \overline{\operatorname{span}\{y_k\colon k \in M_\lambda\}}$. Then $Y_\lambda \simeq Z_\lambda$ and therefore $Y_\lambda \simeq X_\lambda$. Since all bases in X_λ are quasi-equivalent there exist isomorphisms T_λ: $X_\lambda \to Y_\lambda$, bijective transformations $\sigma_\lambda\colon M_\lambda \to N_\lambda$ and numbers $\lambda_k > 0$ such that $y_k = \lambda_k T x_{\sigma_\lambda(k)}$ for $k \in M_\lambda$. Consequently, there are defined: an isomorphism $T\colon X \to X$ $(Tx = \{T_\lambda x_\lambda\}$ where $x = \{x_\lambda\})$, a bijective transformation $\sigma\colon N \to N$ $(\sigma(k) = \sigma_\lambda(k)$ for $k \in M_\lambda)$ and numbers $\lambda_k > 0$ such that $y_k = \lambda_k T x_{\sigma(k)}$, $k \in N$. This completes the proof of the proposition.

(9) This result was announced by M. M. Dragilev at the seminar on the theory of functions and functional analysis at the Rostov University in 1966.

THEOREM 12. *Let $X = X_1 \times X_2$ be a nuclear space, $X_1 \in d_2$, $X_2 \in d_1$. Each X_i let admit a regular basis. Then all bases of X are quasi-equivalent.*

Proof. In each space X_i, $i = 1, 2$, all bases are quasi-equivalent (cf. [9], Theorem 6). Since $X \in d_i$ implies $K(X) \subset d_i$ (for the definition of d_i see § 9), by Theorem 6, the assumptions of Proposition 10 are satisfied and, consequently, all bases in X are quasi-equivalent.

THEOREM 13. *Let $X = \bigtimes_{\lambda \in L} X_\lambda$, L-countable, linearly ordered set. Assume that $X_\lambda = L_{f_\lambda}(b_\lambda, r_\lambda)$ are nuclear and $[f_{\lambda'}, r_{\lambda'}] < [f_\lambda, r_\lambda]$ if $\lambda < \lambda'$. Then all bases in X are quasi-equivalent.*

Proof. In every X_λ all bases are quasi-equivalent because of [7], Theorem 6. For $X = L_f(a, r)$ the class $K(X)$ consists of the spaces of the same form $L_f(b, r)$, where $(b_k) = (a_{n_k})$ (up to a diagonal isomorphism). Hence, to complete the proof it suffices to apply Theorem 9, considerations of § 15 and Proposition 10.

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