STUDIA MATHEMATICA, T. XLVI. (1973)

Bochner's theorem, states, and the Fourier transforms of measures

by

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Abstract. We consider the linear functionals on a complex Banach algebra A (with approximate identity), which admit a "Bochner type" representation on the Gelfand (maximal ideal) space, and characterize them in terms of norms invariant with respect to certain groups of transformations of A. Classical and abstract Bochner theorems (on positive-definitness) are then easily derived. Also follows a characterization of the Fourier transforms of complex measures in terms of translation-invariant norms.

Let G be a locally compact abelian group, and denote (a determination of) the Haar measure on G by "dx". The well-known Bochner theorem on positive definite functions asserts that $p \in L^\infty(G) = L^\infty(dx)$ is positive definite if and only if it is the Fourier transform of a finite positive measure.

Here we give a new proof of this theorem, as well as new abstract and $L^1(G)$ results related to the characterization of the Fourier transforms of measures of finite total variation.

Our approach uses simple ideas and techniques, which are at the same time of rather general nature. The point of view differs from the usual one in that, rather than Hilbert-space related (using inner-products, iterated Schwarz inequalities, or relating to spectral theory), it is connected with states in general Banach algebras and the approximation of spectra by numerical ranges (see [1], [3], and Section 17 below).

The paper is self-contained. Let us recall that if A is a unital Banach algebra, i.e. one with a unit element 1 of norm 1 (we shall always use the same symbol for the unit element and the number 1) and if A^* is its dual as a Banach space, then $\varphi \in A^*$ is called a (*normalized*) *state* if $\|\varphi\| = \varphi(1) = 1$. We call a norm on a Banach algebra A "*compatible*" (*with* A) if it is a Banach algebra norm for A equivalent to the original one (and makes A unital if it was originally unital). Bochner's theorem will easily

* This research was supported in part by National Science Foundation grant 22727. Part of the material in this paper was presented recently — November 1971 — in the second of two talks given by the author at the Séminaire d'Analyse Harmonique, Université Paris–Sud (Orsay).

be derived (see Corollaries 2 and 3 below) from the next theorem. Below, by "a transformation of A" we shall simply mean "a mapping of A into A".

1.

THEOREM. *Let A be a complex commutative unital Banach algebra and let Γ be a finite group of norm-preserving transformations of A, satisfying $\gamma ab = \gamma a \gamma b$, for $\gamma \in \Gamma$, $a, b \in A$. Then, a linear functional φ on A is of the form*

$$\varphi(x) = \int_{\mathcal{M}(A)} \hat{x}\, d\mu \quad \text{for } \forall x \in A, \tag{1}$$

where $\mathcal{M}(A)$ is the Gelfand space of A, $\hat{x}$ the Gelfand transform of x, and μ a probability measure on $\mathcal{M}(A)$, if and only if φ is a state with respect to every Γ-invariant compatible norm on A (to say that $\|\ \|$ is Γ-invariant means of course $\|\gamma x\| = \|x\|$, for $\gamma \in \Gamma$, $x \in A$).

Proof. Suppose $x \in A$, $|sp(x)|$ = spectral radius of x is < 1. We shall show that there is a Γ-invariant compatible norm $|||\ |||$ on A such that $|||x||| \leqslant 1$. (We denote the original norm by $\|\ \|$). Let n be the number of elements in Γ. We renorm A as a Banach space by defining $\forall y \in A$,

$$\|y\|' = \sup\{\|(\gamma_1 x^{j_1}\gamma_2 x^{j_2}\ldots\gamma_n x^{j_n})y\|:\ \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma;\ j_1, j_2, \ldots, j_n \text{ integers} \geqslant 0\}. \tag{2}$$

Since $|sp(x)| < 1$, the $\|x^j\|$ for $j = 0, 1, 2, \ldots$, are bounded say by K, and since the γ are norm-preserving, $\|y\| \leqslant \|y\|' \leqslant K^n\|y\|$. Next we set $\forall y \in A$, $|||y||| = \sup\limits_{z\neq 0} \dfrac{\|yz\|'}{\|z\|'}$ (thus $|||\ |||$ is the operator norm of $z \to yz$ on A renormed as in (2)). Clearly $|||\ |||$ is compatible with A. If $\gamma_0 \in \Gamma$, $y \in A$, then

$$\begin{aligned}\|\gamma_1 x^{j_1}\gamma_2 x^{j_2}\ldots\gamma_n x^{j_n}\gamma_0 y\| &= \|\gamma_0^{-1}(\gamma_1 x^{j_1}\gamma_2 x^{j_2}\ldots\gamma_n x^{j_n}\gamma_0 y)\|\\ &= \|(\gamma_0^{-1}\gamma_1 x^{j_1}\gamma_0^{-1}\gamma_2 x^{j_2}\ldots\gamma_0^{-1}\gamma_n x^{j_n})y\|\end{aligned}$$

and hence from (2) we see that $\|\gamma_0 y\|' = \|y\|'$. Thus $\forall \gamma \in \Gamma$, $y \in A$,

$$|||\gamma y||| = \sup_{z\neq 0}\frac{\|(\gamma y)z\|'}{\|z\|'} = \sup_{z\neq 0}\frac{\|\gamma^{-1}((\gamma y)z)\|'}{\|\gamma^{-1}z\|'} = \sup_{z\neq 0}\frac{\|y\gamma^{-1}z\|'}{\|\gamma^{-1}z\|'} = |||y|||.$$

Hence $|||\ |||$ is Γ-invariant. Finally, to check that $|||x||| \leqslant 1$, notice that since any collection $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ contains n distinct elements only if one of them is the identity, it is clear that in any case, $\forall z \in A$, $(\gamma_1 x^{j_1}\gamma_2 x^{j_2}\ldots\gamma_n x^{j_n})xz$ is again of the form $(\gamma_1' x^{j_1'}\gamma_2' x^{j_2'}\ldots\gamma_n' x^{j_n'})z$ with the γ' in Γ and the j' integers $\geqslant 0$, so that $\|xz\|' \leqslant \|z\|'$, and therefore $|||x||| \leqslant 1$.

Now, if φ is a state for every Γ-invariant compatible norm on A, and x, $|||\ |||$, are as above, then $|\varphi(x)| \leqslant |||x||| \leqslant 1$. This shows that $\forall y \in A$ $|\varphi(y)| \leqslant |sp(y)|$ = sup-norm of $\hat{y}$ on $\mathcal{M}(A)$. It follows that $\varphi(\hat{y}) = \varphi(y)$

is well-defined and (via Hahn–Banach) we see that $\hat{\varphi}(\hat{y}) = \int_{\mathcal{M}(A)} \hat{y}\, d\mu$, μ a measure of total variation $\|\mu\| \leqslant 1$. Since $\hat{\varphi}(\hat{1}) = 1 = \|\mu\|$, μ must be a probability measure; therefore (1) holds and one half of the theorem is proved; the other half is immediate. QED.

As a consequence we derive the known Bochner–Weil–Raikov theorem for $*$-algebras (see [2], p. 261; our terminology in the corollary below is that of [2]).

2.

COROLLARY. *Let A denote a complex commutative Banach algebra with an isometric (or merely continuous) involution $x \to x^*$, satisfying $(x^*)\hat{} = (\hat{x})^-$. Then the linear functional φ on A is positive and extendable (onto A extended by adjoining a unit element) if and only if it is of the form*

$$\varphi(x) = \int_{\mathcal{M}(A)} \hat{x}\, d\mu, \tag{1'}$$

where μ is a finite positive measure on $\mathcal{M}(A)$. [1]

Proof. We assume the involution is isometric (if it is merely continuous, the simple renorming $|||x||| = \sup\{\|x\|, \|x^*\|\}$ will bring us back to the isometric case), and we shall apply theorem 1 taking Γ to be the group of two elements generated by γ, $\gamma x = x^*$ $\forall x \in A$. Merely the "only if" part of the statement needs proof, and in that case we can assume A to be unital; then all we must show is that "φ positive, $\varphi(1) = 1$" implies "φ is a state with respect to any $*$-invariant compatible norm on A". Let $S = \{u \in A:\ u = u^*\}$, then for any compatible norm $|||\ |||$ it is simple and routine to see that $|\varphi(u)| \leqslant |||u|||$ for $u \in S$ (elements in S with positive spectra have square roots in S). Now $\forall x \in A$, $x = u + iv$, $u, v \in S$, and if $|||\ |||$ is compatible and $*$-invariant, $|||u||| \leqslant |||x|||$, so that $|\mathrm{Re}\,\varphi(x)| = |\varphi(u)| \leqslant |||x|||$. Thus $\forall x \in A$, $|\varphi(x)| = |\mathrm{Re}\,\varphi(e^{i\theta}x)| \leqslant |||e^{i\theta}x||| = |||x|||$ (for some $e^{i\theta}$ complex, $|e^{i\theta}| = 1$). QED.

3.

COROLLARY (BOCHNER'S THEOREM FOR LOCALLY COMPACT ABELIAN GROUPS). *A function $p \in L^\infty(G)$ is positive definite if and only if it is the Fourier transform of a finite positive measure.*

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[1] In 26 I one actually assumes A semi-simple and self-adjoint, which we do not assume here since it is stronger than what we need and makes the result less useful. It should also be noticed that the Schwarz inequality for positive functionals $|\varphi(y^*x)|^2 \leqslant \varphi(y^*y)\varphi(x^*x)$, or $|\varphi(x)|^2 \leqslant k\varphi(x^*x)$ for positive extendable ones (k a constant), is never needed here not even at the start to see that a positive continuous φ is extendable in the $L^1(G)$ context (the existence of an approximate identity permits to verify this by an easy direct computation), nor in the abstract context where one can simply define "extendable" by "admitting a positive extension to the algebra with a unit element added".

Proof. It is standard that this follows with little effort from Corollary 2 by considering the positive functional φ_p on $A = L^1(G)$ associated to p via $\varphi_p(f) = \int_G fp\,dx$ $\forall f \in A$. (See [2], p. 141).

4. Comments. In what precedes the basic issue is that one can construct an appropriate Γ-invariant renorming, with respect to a group Γ of norm-preserving transformations of A relevant to the situation at hand. In the above context Γ was a finite group akin to involution. One may want to consider invariant renormings with respect to other groups or families of transformations of A. Certainly in the $L^1(G)$ context, there is another group of isometric transformations of different nature than the above group Γ: the one coming from "translation". The simultaneous action of both types of groups is taken up in the next theorem, in the abstract context. The latter applied to the $L^\infty(G)$ context yields a characterization of the Fourier transforms of (complex) measures of finite total variation related to translation-invariant norms on $L^1(G)$.

5.

THEOREM. *Let A denote a complex commutative Banach algebra with an approximate identity (we mean a bounded net $\{e_a\}$ such that $\forall x \in A$ $\|xe_a - x\|$ tends to 0). Suppose that Δ is a family of norm-preserving transformations of A, and Γ a finite group of norm-preserving transformations of A, connected with the multiplication in A in the following way: $\forall x, y \in A$ and $\gamma \in \Gamma$, $\delta \in \Delta$,*

$$\gamma xy = \gamma x \gamma y,$$
$$\delta xy = x\delta y.$$

Then a linear functional φ on A admits the representation

$$\varphi(x) = \int_{\mathscr{M}(A)} \hat{x}\,d\nu, \tag{1''}$$

where ν is a (complex) measure of finite total variation $\|\nu\| \leqslant c$, if and only if φ is bounded with norm $\leqslant c$ with respect to every simultaneously Γ and Δ-invariant compatible norm on A.

Proof. As in Theorem 1 the main point is to show that for any $x \in A$ with $|\mathrm{sp}(x)| < 1$, we can find a Γ and Δ-invariant compatible norm on A, $|||\ |||$, for which $|||x||| \leqslant 1$. Now, for such an x, we shall construct a Γ-invariant norm with $|||x||| \leqslant 1$ in essentially the same way as in the proof of Theorem 1 (a slight modification being needed because of the lack of a unit element), and under our assumptions this norm will also be Δ-invariant. Again we renorm in two steps as follows. For $y \in A$, we define (letting n be the number of elements in Γ),

$$\|y\|' = \sup\{\|(\gamma_1 x^{j_1}\gamma_2 x^{j_2}\dots\gamma_n x^{j_n})y\|, \|y\|:\ \gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma;\ j_1, j_2, \dots, j_n \text{ integers} > 0\}. \tag{3}$$

Defined in this way $\|\ \|'$ is again an equivalent Banach space norm on A, and $|||y||| = \sup_{z\neq 0} \frac{\|yz\|'}{\|z\|'}$ defines a compatible norm on A (we use the existence of the approximate identity at this point to see that $\|\ \|$ and $|||\ |||$ are equivalent). As before we see that $|||\ |||$ is Γ-invariant and satisfies $|||x||| \leqslant 1$. Now for any $\delta \in \Delta$,

$$\|\gamma_1 x^{j_1}\gamma_2 x^{j_2}\dots\gamma_n x^{j_n}\delta y\| = \|\delta((\gamma_1 x^{j_1}\gamma_2 x^{j_2}\dots\gamma_n x^{j_n})y)\| = \|(\gamma_1 x^{j_1}\gamma_2 x^{j_2}\dots\gamma_n x^{j_n})y\|,$$

and $\|\delta y\| = \|y\|$, hence from (3) we have $\|\delta y\|' = \|y\|'$ for $y \in A$, $\delta \in \Delta$. Therefore

$$|||\delta y||| = \sup_{z\neq 0}\frac{\|(\delta y)z\|'}{\|z\|'} = \sup_{z\neq 0}\frac{\|\delta(yz)\|'}{\|z\|'} = \sup_{z\neq 0}\frac{\|yz\|'}{\|z\|'} = |||y|||.$$

So $|||\ |||$ is Γ and Δ-invariant. Now if φ is bounded in the way described in the statement of the present theorem, then it follows from what we have just established that $\forall x \in A$,

$$|\varphi(x)| \leqslant c\,|\mathrm{sp}(x)| = c\,\|\hat{x}\|_\infty,$$

$\|\ \|_\infty$ = sup-norm on $\mathscr{M}(A)$. Thus (using first Hahn–Banach to extend $\hat{\varphi}$, $\hat{\varphi}(\hat{x}) = \varphi(x)$) it follows that (1'') holds with ν a complex measure on $\mathscr{M}(A)$ of total variation $\leqslant c$.

The converse part of the theorem is obvious. QED.

For any $h \in L^\infty(G)$, let us denote by φ_h the functional on $L^1(G)$ defined by $\varphi_h(f) = \int_G fh\,dx$. We consider $L^1(G)$, as usual, a Banach algebra under convolution. Below, "translation-invariant" has the obvious meaning in the $L^1(G)$ context, "involution-invariant" refers to the usual involution $f \to f^*$, $(f^*)(t) = \overline{f(-t)}$.

6.

COROLLARY. *The function $h \in L^\infty(G)$ is the Fourier transform of a (complex) measure of finite total variation $\leqslant c$, if and only if φ_h is bounded with norm $\leqslant c$ with respect to every translation-invariant compatible norm on $L^1(G)$, or equivalently, with respect to every simultaneously translation and involution-invariant compatible norm on $L^1(G)$.*

Proof. Apply Theorem 5 with $\Delta = \{\delta_g\colon g \in G\}$ where δ_g is defined by $(\delta_g f)(t) = f(t+g)$ for any $t \in G$, and Γ is the group of two elements generated by γ where $\gamma f = f^*$ $\forall f \in L^1(G)$.

7. Comments. In [3] spectral states (called there topological states) were considered, and it was observed that the integral representation which these states admit has the form one knows by Bochner's theorem to hold for continuous positive functionals of norm 1 on $L^1(G)$. This left open and suggested naturally the compelling question of whether the classical Bochner theorem (and perhaps some generalized form of it) could be indeed recovered—proved—from a numerical range/states approach. This meant one should try to obtain some result(s) easily applicable given the hypothesis of the classical Bochner theorem and asserting that (under the appropriate conditions) a given linear functional is a spectral state. Theorem 1 above does exactly that, while in Theorem 5 one does not insist on having states nor a unit element since one has in mind the Fourier transforms of complex measures (and of course one also considers in the latter theorem different transformations of A than in Theorem 1). Finally, notice that while in our applications the transformations of A were linear or conjugate linear, no such thing is needed nor was assumed in the theorems given above in the abstract context, (Theorems 1 and 5).

References

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UNIVERSITÉ PARIS-SUD
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Received January 26, 1972 (468)

STUDIA MATHEMATICA, T. XLVI. (1973)

Un critère de compacité dans les espaces vectoriels topologiques

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Sommaire. Dans les espaces vectoriels topologiques les plus usuels, notamment l'espace des fonctions mesurables sur (0, 1) muni de la topologie de la convergence en mesure, un ensemble borné B est précompact si et seulement s'il existe pour tout voisinage U de 0 un sous-espace vectoriel L de dimension finie tel que $B \subset L+U$.

1. Soit E un espace vectoriel topologique, réel ou complexe. Une partie B de E est dite mince quand pour tout voisinage U de 0 dans E il existe un sous-espace vectoriel L de E de dimension finie tel que $B \subset L+U$.

Par exemple tout précompact de E est mince.

On rappelle que $B \subset E$ est dit borné quand tout voisinage de 0 absorbe B (c'est à dire contient uB pour quelque $u > 0$).

Tout borné mince d'un espace localement convexe, ou ([6]), plus généralement, d'un espace localement pseudo-convexe (c'est à dire dont tout voisinage de 0 contient un voisinage U de 0 qui absorbe $U+U$) est précompact. S. Rolewicz pose le problème de savoir si, dans un espace vectoriel topologique quelconque, tout borné mince est précompact. ([6], p. 165).

On donne ici, sans résoudre ce problème, quelques classes d'espaces vectoriels topologiques dont tout borné mince est précompact et on obtient notamment le résultat suivant.

THÉORÈME 1. *Dans l'espace $L^0(0,1)$ de toutes les (classes de) fonctions mesurables au sens de Lebesgue sur l'intervalle $(0,1)$ muni de la topologie de la convergence en mesure, tout borné mince est relativement compact.*

On démontre en fait (Propositions 3 et 4), plus généralement, le Théorème 2 ci-dessous.

Soit m une mesure positive sur une tribu de parties d'un ensemble T. Soit $\varphi(x,t)$ une fonction numérique de $x \geqslant 0$, $t \in T$, $\varphi(x,\cdot)$ étant m-mesu-

* Ce travail était presenté pendant la "Conference sur les espaces de fonctions et espaces modulaires", Poznań, 1–5, Octobre, 1971.