Directional contractors and equations in Banach spaces

by

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Abstract. Directional contractors extend the notion of contractors introduced in [1]. Hypotheses involving directional contractors are less restrictive than those using Gâteaux differentiability. A global existence theorem for operator equations in Banach spaces as well as a generalization of a local existence theorem by Gavurin are presented. An application to evolution equations and a generalization of the Banach fixed point theorem are also included.

1. Introduction. Let \( P : X \to Y \) be a nonlinear operator from a Banach space \( X \) to a Banach space \( Y \). A bounded linear operator \( \Gamma(x) : Y \to X \) is called an inverse Fréchet derivative of \( P \) at \( x \in X \) if

\[
P(x + \Gamma(x)y) - P(x) - y = o(|y|),
\]

where \( o(|y|)/|y| \to 0 \) as \( y \to 0 \), and \( \Gamma(x) \) is said to be a contractor, if there are positive numbers \( \eta \) and \( q < 1 \) such that

\[
\|P(x + \Gamma(x)y) - P_x - y\| \leq q|y| \quad \text{for} \quad |y| \leq \eta, \ y \in Y.
\]

Inequality (1.2) is called the contractor inequality. Put \( S(x_0, r) = \{x : \|x - x_0\| < r, x, x_0 \in X\} \). Suppose that \( \Gamma(x_0) : Y \to X \) and

\[
\|P(x + \Gamma(x_0)y) - P_x - y\| \leq q|y|
\]

for all \( x \in S(x_0, r) \) and all \( y \in Y \) such that \( x + \Gamma(x_0)y \in S(x_0, r) \). Then \( \Gamma(x_0) \) is called a strong contractor of \( P \) at \( x_0 \). If, for instance, the Fréchet derivative \( F'(x) \) is continuous at \( x_0 \) and \( F'(x_0) \) is nonsingular, then \( \Gamma(x_0) = F'(x_0)^{-1} \) is a strong contractor. Conditions (1.2) or (1.3) can be applied to prove local existence theorems for the equation \( P = 0 \) and to construct iterative procedures convergent toward such a solution. However, for global theorems the contractor inequality (1.2) is required to be satisfied for all \( y \in Y \). Such a contractor may be called a global contractor. To prove only existence theorems it is sufficient to define a weaker kind of a contractor. The method used in this paper combines Gavurin's [2] method of transfinite induction and the notion of a directional contractor.
2. Let \( P : D(P) \subset X \rightarrow X \) be a nonlinear operator and let \( \Gamma(x) : Y \rightarrow X \) be a bounded linear operator, where \( X \) and \( Y \) are Banach spaces, and \( D(P) \) is a linear subset of \( X \). Suppose that \( \Gamma(x)(Y) \subset D(P) \) and

\[
P(x + t\Gamma(x)y) - Py - ty = o(t)
\]

for every \( y \in Y \), where \( \|o(t)\|t^{-1} \rightarrow 0 \) as \( t \rightarrow 0 \).

Then \( \Gamma(x) \) is called the \textit{Gâteaux inverse derivative} of \( P \) at \( x \in D(P) \).

It follows from this definition that \( \Gamma(x) \) is one-to-one, and if \( P \) has a nonsingular Gâteaux derivative \( \Gamma'(x) \), then \( \Gamma'(x) = \Gamma'(x)^{-1} \) is an inverse Gâteaux derivative.

Suppose now that there exists a positive number \( q < 1 \) such that

\[
\|P(x + t\Gamma(x)y) - Py - ty\| \leq q\|y\| \quad \text{for} \quad 0 \leq t \leq \delta(x, y).
\]

Then we say that \( \Gamma(x) \) is a directional contracator of \( P \) at \( x \). It follows from this definition that if \( \Gamma(x)y = 0 \) implies \( y = 0 \), i.e., \( \Gamma(x) \) is one-to-one, and an inverse Gâteaux derivative is obviously a directional contracator.

In order to apply the transfinite induction method of Gavrugin [2], [3] we shall make use of the following two lemmas of Gavrugin (see [2], [3]).

**Lemma 2.1.** Let \( a \) be an ordinal number of first or second class and let \( \{t_\alpha\}_{\alpha \in \delta_0} \) be a naturally well-ordered sequence of real numbers provided that for numbers \( \beta \) of second kind we have

\[
t_\beta = \lim_{\tau \uparrow \beta} t_\tau.
\]

Then the following equality holds.

\[
t_\beta = t_\alpha + \sum_{\tau < \beta} (t_\beta - t_\tau).
\]

**Lemma 2.2.** Let \( a \) be an ordinal number of first or second class and let \( \{x_\alpha\}_{\alpha \in \delta_0} \) be a well-ordered sequence of elements of \( X \) provided that

\[
x_\beta = \lim_{\tau \uparrow \beta} x_\tau.
\]

Then

\[
\|x_\beta - x_\alpha\| \leq \sum_{\tau < \beta} (x_\beta - x_\tau).
\]

3. An operator \( P : D(P) \subset X \rightarrow Y \) is said to be \textit{closed} if \( x_\alpha \rightarrow x \) and \( Px_\alpha \rightarrow y \) imply \( x \in D(P) \) and \( y = Px \).

We say that the nonlinear operator \( P \) has a \textit{bounded directional contracator} \( \Gamma(x) \) if (2.2) is satisfied and, in addition, \( \|\Gamma(x)\| \leq B \) for all \( x \in D(P) \) and some constant \( B \). It is also assumed that \( D(P) \subset X \) is linear and \( \Gamma(x)(Y) \subset D(P) \) for all \( x \in D(P) \).

**Theorem 3.1.** A closed nonlinear operator \( P : D(P) \subset X \rightarrow Y \) which has a bounded directional contracator \( \Gamma(x) \) is a mapping onto \( Y \).

**Proof.** Since for arbitrary fixed \( y \in Y \) the operators \( Px \) and \( Px - y \) have the same bounded directional contracator \( \Gamma(x) \), it is sufficient to prove that the equation \( Px = 0 \) has a solution. To prove this we shall construct well-ordered sequences of numbers \( t_\alpha \) and elements \( x_\alpha \in D(P) \) as follows. Put \( t_0 = 0 \) and let \( x_0 \) be an arbitrary element of \( D(P) \). Suppose that \( t_\alpha \) and \( x_\alpha \) have been constructed for all \( \gamma < \alpha \), provided that: for arbitrary number \( \gamma < \alpha \) inequality (3.1) is satisfied.

\[
\|Px_\gamma\| \leq e^{0 - \gamma d}(\|Px_\alpha\|)
\]

for first kind numbers \( \gamma + 1 < \alpha \) the following inequalities are satisfied:

\[
\|x_\gamma + 1 - x_\alpha\| \leq B\|Px_\alpha\|e^{0 - \gamma d}(t_\gamma + t_\alpha)
\]

and for second kind numbers \( \gamma < \alpha \) the following relations hold:

\[
t_\gamma = \lim_{\tau \uparrow \gamma} t_\tau, \quad x_\gamma = \lim_{\tau \uparrow \gamma} x_\gamma, \quad Px_\gamma = \lim_{\tau \uparrow \gamma} Px_\gamma.
\]

Then it follows from (3.2) and (3.4), Lemmas 3.1 and 3.2 that for arbitrary \( \gamma < \alpha \) and \( \lambda < \alpha \) we have

\[
\|x_\gamma - x_\alpha\| \leq \sum_{\lambda < \gamma < \alpha} \|x_{\lambda + 1} - x_\alpha\| \leq B\|Px_\alpha\|\sum_{\lambda < \gamma < \alpha} e^{0 - \lambda d}(t_{\lambda + 1} - t_\alpha)
\]

\[
< B\|Px_\alpha\| \sum_{\lambda < \gamma < \alpha} t_{\lambda + 1} e^{0 - \lambda d} < B\|Px_\alpha\| \sum_{\lambda < \gamma < \alpha} e^{0 - \lambda d}.
\]

In the same way we obtain from (3.3), (3.4), Lemmas 3.1 and 3.2

\[
\|Px_\gamma - Px_\alpha\| \leq \sum_{\lambda < \gamma < \alpha} \|Px_{\lambda + 1} - Px_\alpha\| \leq (1 + g)\|Px_\alpha\|\sum_{\lambda < \gamma < \alpha} e^{0 - \lambda d}(t_{\lambda + 1} - t_\alpha)
\]

\[
< (1 + g)\|Px_\alpha\| \sum_{\lambda < \gamma < \alpha} t_{\lambda + 1} e^{0 - \lambda d} \leq (1 + g)\|Px_\alpha\| \sum_{\lambda < \gamma < \alpha} e^{0 - \lambda d}.
\]

Suppose that \( \alpha + 1 \) is a first kind number. If \( Px_\alpha = 0 \), then the proof of the theorem is completed. If \( Px_\alpha \neq 0 \), then we put

\[
t_{\alpha + 1} = t_\alpha + t_\alpha, \quad x_{\alpha + 1} = x_\alpha - t_\alpha \Gamma(x_\alpha) Px_\alpha
\]

where \( t_\alpha \) is chosen so as to satisfy (2.2) with \( y = -Px_\alpha \), i.e.

\[
\|Px_\alpha - t_\alpha \Gamma(x_\alpha) Px_\alpha\| \leq q_\alpha\|Px_\alpha\|.
\]
Then we obtain by using the induction assumption (3.14) and (3.8)
\[ ||P_{x_{n+1}}|| \leq (1 + \tau_0) ||P_{x_0}|| + g \tau_0 \leq (1 + (1 - \gamma) g) ||P_{x_0}|| < e^{-\gamma g} ||P_{x_0}|| \leq e^{-\gamma g} ||P_{x_0}||, \]
by (3.7). It follows from (3.7) and (3.14) that
\[ ||x_{n+1} - x_0|| \leq B \tau_0 ||P_{x_0}|| < B \tau_0 e^{-\gamma g} ||P_{x_0}||. \]
In virtue of (3.7), (3.8) and (3.14) we obtain
\[ ||P_{x_{n+1}} - P_{x_0}|| < (1 + g) \tau_0 ||P_{x_0}|| < (1 + g) e^{-\gamma g} ||P_{x_0}||. \]
Thus, conditions (3.14), (3.10) and (3.11) are satisfied for \( x_{n+1} \) and \( x_{n+1}. \)

Now, suppose that \( a \) is a number of second kind and put \( t_\ast = \lim_{n \to \infty} t_n. \)
Let \( \{a_n\} \) be an increasing sequence converging to \( a \). It follows from (3.8) that
\[ ||P_{a_n} - P_{a_0}|| \to 0 \quad \text{as} \quad n \to \infty. \]
Hence, the sequence \( \{a_n\} \) has a limit \( a_0 \) and does so does \( \{x_n\} \). It follows from (3.6) that the sequence \( \{P_{a_n} \} \) has a limit \( P_{a_0} \) and does so does \( \{P_{x_n} \} \). Since \( F \)
is closed we infer that \( a_0 \in D(F) \) and \( y_0 = P_{a_0}. \) If \( t_\ast < \infty, \) then the limit passage in (3.14) yields (3.1). The relationships (3.4) are satisfied by the definition of \( t_\ast \) and \( a_0 \), since \( y_0 = P_{a_0}. \) The process will terminate if \( t_\ast = \infty, \) where \( a \) is of second kind. In this case (3.1) yields \( P_{a_0} = 0 \) and the proof is completed.

For operators \( P = I - F \), where \( X = X \) and \( I \) is the identity mapping of \( X \), it is convenient to have contracts of the form \( I + F(x). \)
Then the contractor inequality (2.2) becomes
\[ \|F(x + tf(y + F(x)y)) - F(x + tF(x)y)\| \leq gt\|y\| \]
for \( 0 \leq t \leq h(x, y), \) \( x \in D(F), \) \( y \in X. \)
Thus, Theorem 3.1 yields immediately

**Theorem 4.1.** A closed nonlinear operator \( F: D(F) \subset X \to X \) which has a bounded directional contractor satisfying condition (4.1) and \( ||F(x)|| \leq B, \) \( x \in D(F), \) has a fixed point \( x^*, \) i.e. \( x^* = F(x^*). \) Moreover, \( I - F \) is a mapping onto \( X. \)

This theorem generalizes the well-known Banach fixed point theorem. In fact, if \( F: X \to X \) is a contraction with Lipschitz constant \( \gamma < 1, \) then \( I + \lambda F(x) \) with \( \lambda = 0 \) is obviously a bounded contractor (see [1]) and this notion is much smaller than a directional contractor. However, since the hypotheses of Theorem 4.1 are rather weak, we cannot prove the existence of the inverse mapping of \( I - F. \)

5. We shall apply Theorem 4.1 to prove an existence theorem for nonlinear evolution equations.

Consider the initial value problem
\[ \frac{dx}{dt} = F(t, x), \quad 0 \leq t \leq T, \quad x(0) = x_0, \]
where \( x = x(t) \) is a function defined on the real interval \([0, T]\) with values in the Banach space \( X, \) and \( F: [0, T] \times \mathbb{R} \to \mathbb{R}. \) Denote by \( X_T \) the space of all continuous functions \( x = x(t) \) defined on \([0, T]\) with values in \( X \) and with the norm \( ||x||_T = \max \{|x(t)| : 0 \leq t \leq T\}. \) Instead of (5.1) we consider the integral equation
\[ x(t) = \int_0^t F(s, x(s))ds + \xi \]
as an operator equation in \( X_T \) and we assume that the integral operator is closed in \( X_T. \)

For arbitrary fixed \( x \in X \) and \( t \in [0, T] \) let \( \Gamma(t, x): X \to X \) be a bounded linear operator, strongly continuous with respect to \( (t, x) \) in the sense of the operator norm. Suppose that there exist positive numbers \( K \) and \( B \) such that the inequality
\[ \max_{t \in [0, T]} \left\| \int_0^t \Gamma(s, x(s))y(s)ds - F(t, x(t)) \right\| \leq K \|y\|_2 \]
is satisfied for arbitrary continuous functions \( x = x(t), \) \( y = y(t) \in X_T, \)
\( 0 \leq \xi \leq \delta(x, y), \) where \( ||\Gamma(t, x)|| \leq B \) for all \( x \in X \) and \( t \in [0, T]. \) Then we say that \( F(t, x) \) has a bounded directional contractor \( I + \frac{d}{dt} \) of integral type.

**Theorem 5.1.** Suppose that \( F(t, x) \) has a bounded directional contractor satisfying (5.3) and \( T \) is such that \( TK < 1. \) Then for arbitrary \( \xi \in X \) equation (5.1) has a continuous solution \( x(t). \)

Proof. Following the method of proof of Theorem 3.1 we shall construct well-ordered sequences of numbers \( a_n \) and elements \( x_n, \)
\( y_n \in X. \) Put \( a_0 = 0, \) \( x_0 = x_0(t) - \xi \) for \( t \in [0, T] \) and \( y_0 = y_0(t) = x_0(t) - \frac{1}{T} \int_0^t F(s, x_0(s))ds - \xi. \) Suppose that \( a_n, x_n, \) and \( y_n \) have been constructed for all \( n \geq a, \) provided that: for arbitrary \( \gamma < \eta \) an inequality (5.4) is satisfied.

\[ \|y_n\|_2 \leq e^{-\gamma t} \|y_0\|_2 \]
for first kind numbers \( \gamma + 1 < \alpha \) the following inequalities are satisfied:

\[
(5.5.\gamma+1) \quad \| x_{\gamma+1} - x_{\alpha} \|_0 \leq (1 + BT) e^{-\alpha} (s_{\gamma+1} - s_{\gamma}) \| y \|_0,
\]

\[
(5.5.\gamma+2) \quad \| y_{\gamma+1} - y_{\gamma} \|_0 \leq (1 + y) e^{-\alpha} (s_{\gamma+1} - s_{\gamma}) \| y \|_0,
\]

and for second kind numbers \( \gamma < \alpha \) the following relations hold:

\[
(5.7) \quad \sigma_\gamma = \lim_{\gamma \to \gamma} \sigma_\gamma, \quad \pi_\gamma = \lim_{\gamma \to \gamma} \pi_\gamma, \quad y_\gamma = \lim_{\gamma \to \gamma} y_\gamma.
\]

Then, in the same way is in the proof of Theorem 3.1, it follows from (5.5)-(5.7), Lemmas 3.1 and 3.2 that for arbitrary \( \gamma < \alpha \) and \( \lambda < \alpha \) we have

\[
\| x_{\gamma} - x_{\lambda} \|_0 \leq (1 + BT) \int_{\sigma_\gamma}^{\sigma_\lambda} e^{-\alpha} d\sigma \| y \|_0,
\]

\[
\| y_{\gamma} - y_{\lambda} \|_0 \leq (1 + BT) \int_{\pi_\gamma}^{\pi_\lambda} e^{-\alpha} d\pi \| y \|_0.
\]

Suppose that \( \alpha + 1 \) is a first kind number. Then we put

\[
(5.10) \quad \sigma_{\alpha+1} = \omega_\alpha + \omega_\alpha, \quad x_{\alpha+1}(t) = x_{\alpha}(t) - \omega_\alpha[y_{\alpha}(t) + \int_0^t \Gamma(s, x_{\alpha}(s)) \, ds],
\]

\[
(5.11) \quad y_{\alpha}(t) = x_{\alpha}(t) - \int_0^t F(s, x_{\alpha}(s)) \, ds, \quad t \in [0, T],
\]

where \( \omega_\alpha \) is chosen so as to satisfy (5.3) with \( \beta = \omega_\alpha, x = x_{\alpha} \) and \( y = y_{\alpha} \), i.e.

\[
\max_{s \in [0, T]} \left\| F \left( t, x_{\alpha}(t) + \omega_\alpha \int_0^t \Gamma(s, x_{\alpha}(s)) \, ds \right) - F(t, x_{\alpha}(t)) \right\| \leq K \omega_\alpha \| y \|_0.
\]

If \( y_{\alpha} \to 0 \), then the proof of the theorem is completed. Otherwise, we have

\[
\| y_{\alpha+1}(t) - y_{\alpha}(t) + \omega_\alpha [y_{\alpha}(t) - y_{\alpha}(t)] \|_0 \leq K \omega_\alpha \| y \|_0.
\]

Hence, it follows, by (5.11) and (5.12) with \( -y_{\alpha} \) replacing \( y_{\alpha} \), that

\[
\| y_{\alpha+1} - y_{\alpha} + \omega_\alpha [y_{\alpha} - y_{\alpha}] \|_0 \leq K \omega_\alpha \| y \|_0.
\]

Hence,

\[
\| y_{\alpha+1} \|_0 \leq (1 - (1 - q) \omega_\alpha) \| y \|_0 < e^{-\alpha} \| y \|_0 < e^{-\alpha} \| y \|_0 \leq e^{-\alpha} \| y \|_0 \| y \|_0,
\]

in virtue of (5.4) and (5.10). Thus, (5.4) implies (5.4.\alpha). Further, we obtain from (5.10) and (5.4),

\[
\| y_{\alpha+1} - y_{\alpha} \|_0 \leq (1 + BT) \| y \|_0 \leq (1 + BT) e^{-\alpha} \| y \|_0,
\]

that is, (5.4.\alpha) is satisfied. Now, we have

\[
\| y_{\alpha+1} - y_{\alpha} \|_0 \leq (1 + y) e^{-\alpha} \| y \|_0 \leq (1 + y) e^{-\alpha} \| y \|_0,
\]

by (5.13) and (5.4). Thus, conditions (5.4.\alpha), (5.4.\alpha+1) and (5.6.\alpha+1) are satisfied for \( \alpha+1, \alpha+1 \) and \( \alpha+1 \).

Now, suppose that \( \alpha \) is a number of some kind and put \( t_\alpha = \lim_{\gamma \to \gamma} t_\gamma \).

Let \( \{ y_\gamma \} \) be an increasing sequence convergent toward \( y_\alpha \). It follows from (5.8) that \( \| x_{\alpha+1} - x_{\alpha} \|_0 \to \infty \) as \( \gamma \to \infty \). Hence, the sequence \( \{ x_\gamma \} \) has a limit \( x_\alpha \) and so does \( \{ y_\gamma \} \). It follows from (5.9) that the sequence \( \{ y_\gamma \} \) has a limit \( y_\alpha \). Since the integral operator in (5.2) is closed in \( X_\alpha \) by assumption, we conclude that \( y_\alpha \) satisfies (5.11). If \( t_\alpha > \infty \), then the limit passage in (5.4.\alpha) yields (5.4). The relations (3.7) are also satisfied by the definition of \( t_\alpha, x_\alpha \) and \( y_\alpha \) and since we proved that (5.11) holds for \( y_\alpha \). The process will terminate if \( \alpha_\alpha = + \infty \), where \( \alpha_\alpha \) is a kind of this case. In this case (5.4) yields \( y_\alpha = 0 \), i.e. \( x_\alpha \) is a solution of (5.2) and the proof is completed.

Remark 5.1. It is not necessary that \( F(t, x) \) be defined on the whole of \( X \). It is sufficient to assume that \( F(t, x) \) is defined for \( x \in D \), where \( D \) is a linear subset of \( X \). Then we assume in addition that \( \| F(t, x) \|_X \leq D \) for each \( x \in D \) and \( t \in [0, T] \).

6. Using the directional contraction method we shall prove a local existence theorem for solving nonlinear equations. Let \( X_\alpha \) be a linear subset of the Banach space \( X \). Put \( S = S(x_\alpha, r) = \{ x \; | \; \| x - x_\alpha \|_\alpha < r \} \) for a given \( x_\alpha \in X_\alpha \) and \( U = X_\alpha \cap \overline{S} \), where \( S \) is the closure of \( S \). Let \( P : U \to X \) be a nonlinear operator closed on \( U \), i.e. \( x_\alpha \in U, x_\alpha \to x \) and \( P_{x_\alpha} \to y \) imply \( x \in U \) and \( y = P x \). Suppose that \( P \) has a directional contraction \( \Gamma(x) : X \to X \) for \( x \in U_\alpha = X_\alpha \cap \overline{S} \), i.e. there exists a positive constant \( q < 1 \) such that

\[
\| P(x + \Gamma(x) y) - P x - y \| \leq q \| y \|
\]

for all \( x \in U_\alpha \) and \( 0 \leq t \leq \sigma(x, y) \), \( y \) being an arbitrary element of the Banach space \( X \).

Theorem 6.1. Suppose that the following hypotheses are satisfied:

1) \( P \) is closed on \( U \),
2) \( P \) has a bounded directional contraction \( \Gamma(x) \), \( x \in U_\alpha \) satisfying condition (6.1) and
3) \( \| \Gamma(x) \| \leq B \) for all \( x \in U_\alpha \),
4) \( r \geq B (1-2q)^{-1} \| P x \|, \quad 0 \leq q < 1/2.\)
Then equation $Pw = 0$ has a solution in $U$.

Proof. In the same way as in the proof of Theorem 3.1 we construct the sequences of numbers $t_n/(1 - 2^g) > 0$ and elements $x_n$ satisfying conditions (3.1), (3.2), (3.4), and additionally (6.2) for first kind numbers $a > 1$.

\[(8.2)\quad 0 < t_{n+1} - t_n < (1 - 2g)^{-1} \ln (1 - g) (1 - 2g)^{-1}.
\]

Then using the same argument as in the proof of Theorem 3.1 we obtain for arbitrary $y < a$ and $\lambda < a$:

\[
\|x - x_n\| \leq \sum_{k \geq 0} \|x_{p_{k+1}} - x_n\|
\]

\[
< B \|Pw_0\| \sum_{k \geq 0} e^{-(1 - \lambda)(t_{k+1} - t_k)}
\]

\[
= B \|Pw_0\| \sum_{k \geq 0} e^{-(1 - \lambda)(t_{k+1} - t_k)} e^{-(1 - \lambda)(t_{k+1} - t_k)}
\]

\[
< (1 - g) (1 - 2g)^{-1} B \|Pw_0\| \sum_{k \geq 0} e^{-(1 - \lambda)(t_{k+1} - t_k)}
\]

\[
= (1 - g) (1 - 2g)^{-1} B \|Pw_0\| e^{-(1 - \lambda)(t_{k+1} - t_k)}
\]

Thus, all elements $x_n$ belong to $U^*$. Also in (3.7) $t_n$ should satisfy additionally $a < (1 - g) \ln (1 - g) (1 - 2g)^{-1}$. The further reasoning is exactly the same as in the proof of Theorem 3.1. As a particular case of Theorem 6.1 we obtain

**Theorem 6.2.** If $P$ is bounded and $\lambda < a$ and $P$ has an inverse $G\Theta$-contraction $\Gamma(x, s, z)$ satisfying conditions (3) and 4), then $Pw = 0$ has a solution in $U$.

The proof follows from the fact that an inverse $G\Theta$-contraction is a directional derivative and there always exists $0 < g < 1/2$ satisfying condition (6.1).

Thus, $\Gamma(x, s, z)$ is a directional derivative and there always exists $0 < g < 1/2$ satisfying condition (6.1).

This theorem has been proved by GÂºurin [3], where $\Gamma(x, s, z) = [P'(x, s)]^{-1}$, $w \in U^*$, $P'(x, s)$ being the linear $G\Theta$-contraction defined on $X$ and such that its inverse exists and is continuous on $Y$. In this case $\Gamma(x, s)$ is an inverse $G\Theta$-contraction.

7. Using Theorem 6.1 as a basis, an implicit function theorem can be proved. Let $\chi, z, y$ be Banach spaces and put

\[
S = \{x : \|x - x_n\| \leq \varepsilon, \|x - x_n\| \leq \varepsilon, x \in Y, \chi \in X\}
\]

for given $x_n \in X, x_n \in Z$ and $y$. Let $P : S \rightarrow X$ be a continuous linear operator and suppose that for every $x$ such that $(x, s) \in S$ $P$ has a directional derivative $\Gamma(x, s, z) : Y \rightarrow X$ which is $s$-uniform, i.e. there exists a positive $\eta \in (1, l]$. Then $P(x, s) - ty \in S$ for $0 \leq t \leq \sigma(x, s)$ and $y$ is an arbitrary element of $X$. We assume, in addition, that the directional derivative $\Gamma(x, s)$ is bounded, i.e. there exists a constant $B$ such that

\[
\|\Gamma(x, s)\| \leq B, \quad \text{for } (x, s) \in S.
\]

Finally, we suppose that the directional derivative $\Gamma(x, s)$ is strongly continuous in $(x, s)$, i.e. in the sense of the operator norm.

**Theorem 7.1.** Suppose that $P : S \rightarrow Y$ is a continuous operator satisfying the following conditions:

- 1) $P(x_n, s) = 0$.
- 2) $P$ has a bounded directional $s$-uniform satisfying conditions (7.1) and (7.2) and being strongly continuous with respect to $(x, s)$.
- 3) There exists a continuous function $g(x)$ defined in some neighborhood of $x_n$ with values in $X$ and such that $P(g(x), s) = 0$.

Proof. First of all we choose $\eta$ and $\varepsilon_1$ such that

\[
B(1 - 2g)^{-1} \eta \leq \sigma, \quad \text{and } \|P(x_n, s)\| \leq \eta, \quad \text{for } \|x - x_n\| \leq \varepsilon_1.
\]

Now, in the same way as in the proof of Theorem 6.1 we construct sequences of numbers $t_n/(1 - 2^g) > 0$ and continuous functions $x_n(x)$ (replacing $x_n$), where $(x_n, s) \in S$, $S = \{x : \|x - x_n\| \leq \varepsilon, \|x - x_n\| \leq \varepsilon, x \in X\}$. The values of $x_n(x)$ are in $X$ and $x_n(x) = x_n$. These sequences are to satisfy conditions (3.1), (3.2), (3.4), and (6.2) provided that $x_{n+1}, x_n, x_n(x), P(x_{n+1}(x), s), P(x_n(x), s), P(x_n(x), s)$ and $P(x_n(x), s)$, respectively. Thus, we obtain in place of (3.5) and (3.6), by (7.3),

\[
\|x_n(x) - x_n(x)\| \leq B \int_{t_0}^{t} e^{-(1 + \lambda) t} dt.
\]

\[
\|P(x_n(x), s) - P(x_n(x), s)\| \leq (1 + g) \int_{t_0}^{t} e^{-(1 + \lambda) t} dt.
\]
Inequalities (7.4) and (7.5) show the $\varepsilon$-uniform convergence in the analogous relations (3.4), yielding the continuity of the limit functions. An equivalent of (6.3) will be,

$$|x_\varepsilon(s) - x_\varepsilon| \leq B(1 - 2\eta)^{-1} \eta \leq r,$$

in virtue of (7.3). Hence,

$$[x_\varepsilon(s), s] \in \mathcal{K}_1.$$

Thus, all constructed functions $x_\varepsilon(s)$ are continuous and well-defined. The further reasoning is exactly the same as in the proof of Theorem 6.1.

Since the assumption of Theorem 7.1 are rather weak, we cannot prove the uniqueness of the function $g(\varepsilon)$.

References


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L’analogue dans $\varphi^p$ des theoremes de convexite
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La méthode utilisée dans cet article est celle de l’interpolation holomorphe.

§ 1. Les espaces vectoriels considérés sont complexes, sauf mention du contraire. Nous notons $||$ les différentes normes rencontrées.

Soit $H$ un espace de Hilbert; $\varphi^{p+1}(H)$ (resp. $\varphi^p(H)$; resp. $\mathbb{B}(H)$) désignera l’espace vectoriel des opérateurs linéaires et continus de $H$ dans $H$ (resp. l’idéal des opérateurs compacts; resp. l’idéal des opérateurs de rang fini) $\varphi^{p+1}(H)$ et $\varphi^p(H)$ seront munis de la norme des opérateurs; ce sont des espaces de Banach. $\mathbb{B}(H)$ est dense dans $\varphi^p(H)$.

Soit $p$ une valeur numérique réelle (finie), $p \geq 1$. Nous désignons par $\varphi^p(H)$ l’ensemble des $T \in \varphi^p(H)$ tels que la suite $(\mathbb{I}^p(T))_{1 \leq q \leq p}$ décroissante, tendant vers zéro (avec répétition éventuelle suivant la multiplicité) des valeurs propres de $(T^*T)^{1/p}$, la racine carrée de $T^*T$, soit dans $\mathbb{P}$, espaces des suites de puissance $p$-ième sommable.

Si $T \in \varphi^p(H)$, notons $||T|| = \left\{ \sum_{1 \leq q \leq p} |\mathbb{I}^p(T)|^q \right\}^{1/p}$.

On sait que (cf [2], par exemple), pour tout $p > 1$, $\varphi^p(H)$ est un espace vectoriel et $||T||$ est une norme sur $\varphi^p(H)$; munie de cette norme $\varphi^p(H)$ est un espace de Banach et $\mathbb{B}(H)$ est dense dans $\varphi^p(H)$. De plus, pour $\forall p, q$ tels que $1 \leq p \leq q$

$\varphi^p(H) = \varphi^q(H) = \varphi^p(H)$

avec injections continues.

Nous allons prouver dans ce travail que les analogues des théorèmes de convexité bien connus des espaces de fonctions $L^p$ sont vrais pour les espaces d’opérateurs $\varphi^p(H)$.

§ 2. Soit $H$ un espace de Hilbert.

Si $T \in \varphi^p(H)$, la suite $(\mu(T))_{1 \leq q \leq p}$ des valeurs propres de $T$ étant rangée par ordre de module décroissant (avec répétition éventuelle suivant la multiplicité), la série $\sum_j \mu_j(T)$ est absolument convergente (cf [2]).