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Directional contractors and equations in Banach spaces

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Abstract. Directional contractors extend the notion of contractors introduced in [1]. Hypotheses involving directional contractors are less restrictive than those using Gâteaux differentiability. A global existence theorem for operator equations in Banach spaces as well as a generalization of a local existence theorem by Gavurin are presented. An application to evolution equations and a generalization of the Banach fixed point theorem are also included.

1. Introduction. Let $P\colon X\to Y$ be a nonlinear operator from a Banach space X to a Banach space Y. A bounded linear operator $\Gamma(x)\colon Y\to X$ is called an *inverse Fréchet derivative* of P at $x\in X$ (see [1]) if

$$P\big(x+\Gamma(x)y\big)-P(x)-y=o(\|y\|), \tag{1.1}$$

where $o(\|y\|)/\|y\|\to 0$ as $y\to 0$, and $\Gamma(x)$ is said to be a *contractor*, if there are positive numbers η and $q<1$ such that

$$\big\|P\big(x+\Gamma(x)y\big)-Px-y\big\|\leqslant q\|y\| \quad \text{for } \|y\|\leqslant\eta,\ y\in Y. \tag{1.2}$$

Inequality (1.2) is called the *contractor inequality*. Put $S(x_0,r)=[x\colon \|x-x_0\|\leqslant r,\ x,x_0\in X]$. Suppose that $\Gamma(x_0)\colon Y\to X$ and

$$\big\|P\big(x+\Gamma(x_0)y\big)-Px-y\big\|\leqslant q\|y\| \tag{1.3}$$

for all $x\in S(x_0r)$ and all $y\in Y$ such that $x+\Gamma(x_0)y\in S(x_0,r)$. Then $\Gamma(x_0)$ is called a *strong contractor* of P at x_0. If, for instance, the Fréchet derivative $P'(x)$, $x\in S(x_0,r)$, is continuous at x_0 and $P'(x_0)$ is nonsingular, then $\Gamma(x_0)=P'(x_0)^{-1}$ is a strong contractor. Conditions (1.2) or (1.3) can be applied to prove local existence theorems for the equation $Px=0$ and to construct iterative procedures convergent toward such a solution. However, for global theorems the contractor inequality (1.2) is required to be satisfied for all $y\in Y$. Such a contractor may be called a *global contractor*. To prove only existence theorems it is sufficient to define a weaker kind of a contractor. The method used in this paper combines Gavurin's [2] method of transfinite induction and the notion of a directional contractor.

2. Let $P: D(P) \subset X \to Y$ be a nonlinear operator and let $\Gamma(x): Y \to X$ be a bounded linear operator, where X and Y are Banach spaces, and $D(P)$ is a linear subset of X. Suppose that $\Gamma(x)(Y) \subset D(P)$ and

$$P\big(x+t\Gamma(x)y\big)-Px-ty = o(t) \quad \text{for every } y \in Y, \tag{2.1}$$

where $\|o(t)\|t^{-1} \to 0$ as $t \to 0$.

Then $\Gamma(x)$ is called the *Gâteaux inverse derivative* of P at $x \in D(P)$.

It follows from this definition that $\Gamma(x)$ is one-to-one, and if P has a nonsingular Gâteaux derivative $P'(x)$, then $\Gamma(x) = P'(x)^{-1}$ is an inverse Gâteaux derivative.

Suppose now that there exists a positive number $q < 1$ such that

$$\big\|P\big(x+t\Gamma(x)y\big)-Px-ty\big\| \leqslant qt\|y\| \quad \text{for } 0 \leqslant t \leqslant \delta(x, y). \tag{2.2}$$

Then we say that $\Gamma(x)$ is a directional contractor of P at x. It follows from this definition that $\Gamma(x)y = 0$ implies $y = 0$, i.e. $\Gamma(x)$ is one-to-one, and an inverse Gateaux derivative is obviously a directional contractor.

In order to apply the transfinite induction method of Gavurin [2], [3] we shall make use of the following two lemmas of Gavurin (see [2], [3]).

LEMMA 2.1. *Let a be an ordinal number of first or second class and let $\{t_\gamma\}_{0 \leqslant \gamma \leqslant a}$ be a naturally well-ordered sequence of real numbers provided that for numbers β of second kind we have*

$$t_\beta = \lim_{\gamma \uparrow \beta} t_\gamma .$$

Then the following equality holds.

$$t_a = t_0 + \sum_{0 \leqslant \gamma < a} (t_{\gamma+1} - t_\gamma).$$

LEMMA 2.2. *Let a be an ordinal number of first or second class and let $\{x_\gamma\}_{0 \leqslant \gamma \leqslant a}$ be a well-ordered sequence of elements of X provided that*

$$x_\beta = \lim_{\gamma \uparrow \beta} x_\gamma .$$

Then

$$\|x_a - x_0\| \leqslant \sum_{0 \leqslant \gamma < a} \|x_{\gamma+1} - x_\gamma\|.$$

3. An operator $P: D(P) \subset X \to Y$ is said to be *closed* if $x_n \to x$ and $Px_n \to y$ imply $x \in D(P)$ and $y = Px$.

We say that the nonlinear operator P has a *bounded directional contractor* $\Gamma(x)$ if (2.2) is satisfied and, in addition, $\|\Gamma(x)\| \leqslant B$ for all $x \in D(P)$ and some constant B. It is also assumed that $D(P) \subset X$ is linear and $\Gamma(x)(Y) \subset D(P)$ for all $x \in D(P)$.

THEOREM 3.1. *A closed nonlinear operator $P: D(P) \subset X \to Y$ which has a bounded directional contractor $\Gamma(x)$ is a mapping onto Y.*

Proof. Since for arbitrary fixed $y \in Y$ the operators Px and $Px - y$ have the same bounded directional contractor $\Gamma(x)$, it is sufficient to prove that the equation $Px = 0$ has a solution. To prove this we shall construct well-ordered sequences of numbers t_a and elements $x_a \in D(P)$ as follows. Put $t_0 = 0$ and let x_0 be an arbitrary element of $D(P)$. Suppose that t_γ and x_γ have been constructed for all $\gamma < a$, provided that: for arbitrary number $\gamma < a$ inequality (3.1_γ) is satisfied.

$$\|Px_\gamma\| \leqslant e^{-(1-q)t_\gamma}\|Px_0\|; \tag{3.1_γ}$$

for first kind numbers $\gamma+1 < a$ the following inequalities are satisfied:

$$\|x_{\gamma+1} - x_\gamma\| \leqslant B\|Px_0\|e^{-(1-q)t_\gamma}(t_{\gamma+1}-t_\gamma), \tag{$3.2_{\gamma+1}$}$$

$$\|Px_{\gamma+1} - Px_\gamma\| \leqslant (1+q)\|Px_0\|e^{-(1-q)t_\gamma}(t_{\gamma+1}-t_\gamma); \tag{$3.3_{\gamma+1}$}$$

and for second kind numbers $\gamma < a$ the following relations hold:

$$t_\gamma = \lim_{\beta \uparrow \gamma} t_\beta, \quad x_\gamma = \lim_{\beta \uparrow \gamma} x_\beta, \quad Px_\gamma = \lim_{\beta \uparrow \gamma} Px_\beta. \tag{3.4_γ}$$

Then it follows from (3.2), (3.4), Lemmas 3.1 and 3.2 that for arbitrary $\gamma < a$ and $\lambda < a$ we have

$$\begin{aligned}\|x_\gamma - x_\lambda\| &\leqslant \sum_{\lambda \leqslant \beta < \gamma} \|x_{\beta+1} - x_\beta\| < B\|Px_0\| \sum_{\lambda \leqslant \beta < \gamma} e^{-(1-q)t_\beta}(t_{\beta+1}-t_\beta) \\ &< B\|Px_0\| \sum_{\lambda \leqslant \beta < \gamma} \int_{t_\beta}^{t_{\beta+1}} e^{-(1-q)t}\,dt = B\|Px_0\| \int_{t_\lambda}^{t_\gamma} e^{-(1-q)t}\,dt.\end{aligned} \tag{3.5}$$

In the same way we obtain from (3.3), (3.4), Lemmas 3.1 and 3.2

(3.6)
$$\begin{aligned}\|Px_\gamma - Px_\lambda\| &\leqslant \sum_{\lambda \leqslant \beta < \gamma} \|Px_{\beta+1} - Px_\beta\| \leqslant (1+q)\|Px_0\| \sum_{\lambda \leqslant \beta < \gamma} e^{-(1-q)t_\beta}(t_{\beta+1}-t_\beta) \\ &< (1+q)\|Px_0\| \sum_{\lambda \leqslant \beta < \gamma} \int_{t_\beta}^{t_{\beta+1}} e^{-(1-q)t}\,dt = (1+q)\|Px_0\| \int_{t_\lambda}^{t_\gamma} e^{-(1-q)t}\,dt.\end{aligned}$$

Suppose that $a+1$ is a first kind number. If $Px_a = 0$, then the proof of the theorem is completed.

If $px_a \neq 0$, then we put

$$t_{a+1} = t_a + \tau_a, \quad x_{a+1} = x_a - \tau_a\Gamma(x_a)Px_a, \tag{3.7}$$

where τ_a is chosen so as to satisfy (2.2) with $y = -Px_a$, i.e.

$$\big\|P\big(x_a - \tau_a\Gamma(x_a)Px_a\big) - Px_a + \tau_a Px_a\big\| \leqslant q\tau_a\|Px_a\|. \tag{3.8}$$

Then we obtain by using the induction assumption (3.1_α) and (3.8)

$$(3.9_{\alpha+1})\qquad \|Px_{\alpha+1}\| \leqslant (1-\tau_\alpha)\|Px_\alpha\| + q\tau_\alpha\|Px_\alpha\| = \big(1-(1-q)\tau_\alpha\big)\|Px_\alpha\| < e^{-(1-q)\tau_\alpha}\|Px_\alpha\| \leqslant e^{-(1-q)t_{\alpha+1}}\|Px_0\|,$$

by (3.7). It follows from (3.7) and (3.1_α) that

$$(3.10_{\alpha+1})\qquad \|x_{\alpha+1}-x_\alpha\| \leqslant B\tau_\alpha\|Px_\alpha\| < B(t_{\alpha+1}-t_\alpha)e^{-(1-q)t_\alpha}\|Px_0\|.$$

In virtue of (3.7), (3.8) and (3.1_α) we obtain

$$(3.11_{\alpha+1})\qquad \|Px_{\alpha+1}-Px_\alpha\| \leqslant (1+q)\tau_\alpha\|Px_\alpha\| \leqslant (1+q)\|Px_0\|e^{-(1-q)t_\alpha}(t_{\alpha+1}-t_\alpha).$$

Thus, conditions $(3.9_{\alpha+1})$, $(3.10_{\alpha+1})$ and $(3.11_{\alpha+1})$ are satisfied for $t_{\alpha+1}$ and $x_{\alpha+1}$.

Now, suppose that α is a number of second kind and put $t_\alpha = \lim_{\gamma\uparrow\alpha} t_\gamma$. Let $\{\gamma_n\}$ be an increasing sequence convergent toward α. It follows from (3.5) that

$$\|x_{\gamma_{n+\varrho}}-x_{\gamma_n}\| \to 0 \quad \text{as } n\to\infty.$$

Hence, the sequence $\{x_{\gamma_n}\}$ has a limit x_α and so does $\{x_\gamma\}$. It follows from (3.6) that the sequence $\{Px_{\gamma_n}\}$ has a limit y_α and so does $\{Px_\gamma\}$. Since P is closed we infer that $x_\alpha \in D(P)$ and $y_\alpha = Px_\alpha$. If $t_\alpha < +\infty$, then the limit passage in (3.1_{γ_n}) yields (3.1_α). The relationships (3.4_α) are satisfied by the definition of t_α and x_α, since $y_\alpha = Px_\alpha$. The process will terminate if $t_\alpha = +\infty$, where α is of second kind. In this case (3.1_α) yields $Px_\alpha = 0$ and the proof is completed.

4. For operators $P = I-F$, where $X = Y$ and I is the identity mapping of X, it is convenient to have contractors of the form $I+\Gamma(x)$. Then the contractor inequality (2.2) becomes

$$(4.1)\qquad \big\|F\big(x+t(y+\Gamma(x)y)\big)-Fx-t\Gamma(x)y\big\| \leqslant qt\|y\|$$

for $0\leqslant t\leqslant \delta(x,y)$, $x\in D(F)$, $y\in X$.

Thus, Theorem 3.1 yields immediately

THEOREM 4.1. *A closed nonlinear operator F: $D(F)\subset X\to X$ which has a bounded directional contractor satisfying condition* (4.1) *and* $\|\Gamma(x)\| \leqslant B$, $x\in D(F)$, *has a fixed point x^*, i.e. $x^* = Fx^*$. Moreover, $I-F$ is a mapping onto X.*

This theorem generalizes the well-known Banach fixed point theorem. In fact, if F: $X\to X$ is a contraction with Lipschitz constant $q<1$, then $I+\Gamma(x)$ with $\Gamma(x)\equiv 0$ is obviously a bounded contractor (see [1]) and this notion is much stronger than a directional contractor. However, since the hypotheses of Theorem 4.1 are rather weak, we cannot prove the existence of the inverse mapping of $I-F$.

5. We shall apply Theorem 4.1 to prove an existence theorem for nonlinear evolution equations.

Consider the initial value problem

$$(5.1)\qquad \frac{dx}{dt} = F(t,x),\quad 0\leqslant t\leqslant T,\quad x(0)=\xi,$$

where $x = x(t)$ is a function defined on the real interval $[0,T]$ with values in the Banach space X, and F: $[0,T]\times X\to X$. Denote by X_T the space of all continuous functions $x = x(t)$ defined on $[0,T]$ with values in X and with the norm $\|x\|_C = \max[\|x(t)\|\colon 0\leqslant t\leqslant T]$. Instead of (5.1) we consider the integral equation

$$(5.2)\qquad x(t)-\int_0^t F\big(s,x(s)\big)ds = \xi$$

as an operator equation in X_T and we assume that the integral operator is closed in X_T.

For arbitrary fixed $x\in X$ and $t\in[0,T]$ let $\Gamma(t,x)$: $X\to X$ be a bounded linear operator, strongly continuous with respect to (t,x) in the sense of the operator norm. Suppose that there exist positive numbers K and B such that the inequality

$$(5.3)\qquad \max_{0\leqslant t\leqslant T}\Big\|F\Big(t,x(t)+\varrho\int_0^t\Gamma\big(s,x(s)\big)y(s)ds\Big)-F\big(t,x(t)\big) - \varrho\Gamma\big(t,x(t)\big)y(t)\Big\| \leqslant K\varrho\|y\|_C$$

is satisfied for arbitrary continuous functions $x = x(t)$, $y = y(t)\in X_T$, $0\leqslant\varrho\leqslant\delta(x,y)$, where $\|\Gamma(t,x)\|\leqslant B$ for all $x\in X$ and $t\in[0,T]$. Then we say that $F(t,x)$ has a bounded directional contractor $\{I+\int_0^t\Gamma\}$ of integral type.

THEOREM 5.1. *Suppose that $F(t,x)$ has a bounded directional contractor satisfying* (5.3) *and T is such that $TK = q<1$. Then for arbitrary $\xi\in X$ equation* (5.2) *has a continuous solution $x(t)$.*

Proof. Following the method of proof of Theorem 3.1 we shall construct well-ordered sequences of numbers σ_α and elements x_α, $y_\alpha\in X_T$. Put $\sigma_0 = 0$, $x_0 = x_0(t) = \xi$ for $t\in[0,T]$ and $y_0 = y_0(t) = x_0(t) - \int_0^t F\big(s,x_0(s)\big)ds - \xi$. Suppose that σ_γ, x_γ and y_γ have been constructed for all $\gamma<\alpha$, provided that: for arbitrary $\gamma<\alpha$ inequality (5.4_γ) is satisfied.

$$(5.4_\gamma)\qquad \|y_\gamma\|_C \leqslant e^{-(1-q)\sigma_\gamma}\|y_0\|_C;$$

for first kind numbers $\gamma+1 < \alpha$ the following inequalities are satisfied:

$$(5.5_{\gamma+1}) \qquad \|x_{\gamma+1}-x_\gamma\|_C \leqslant (1+BT)\, e^{-(1-q)\sigma_\gamma}(\sigma_{\gamma+1}-\sigma_\gamma)\,\|y_0\|_C,$$

$$(5.6_{\gamma+1}) \qquad \|y_{\gamma+1}-y_\gamma\|_C \leqslant (1+q)\, e^{-(1-q)\sigma_\gamma}(\sigma_{\gamma+1}-\sigma_\gamma)\,\|y_0\|_C;$$

and for second kind numbers $\gamma < \alpha$ the following relations hold:

$$(5.7_\gamma) \qquad \sigma_\gamma = \lim_{\beta\uparrow\gamma} \sigma_\beta, \qquad x_\gamma = \lim_{\beta\uparrow\gamma} x_\beta, \qquad y_\gamma = \lim_{\beta\uparrow\gamma} y_\beta.$$

Then, in the same way is in the proof of Theorem 3.1, it follows from (5.5)–(5.7), Lemmas 3.1 and 3.2 that for arbitrary $\gamma < \alpha$ and $\lambda < \alpha$ we have

$$(5.8) \qquad \|x_\gamma - x_\lambda\|_C \leqslant (1+BT) \int_{\sigma_\lambda}^{\sigma_\gamma} e^{-(1-q)\sigma}\, d\sigma\, \|y_0\|_C,$$

$$(5.9) \qquad \|y_\gamma - y_\lambda\|_C \leqslant (1+BT) \int_{\sigma_\lambda}^{\sigma_\gamma} e^{-(1-q)\sigma}\, d\sigma\, \|y_0\|_C.$$

Suppose that $\alpha+1$ is a first kind number. Then we put

$$(5.10) \qquad \sigma_{\alpha+1} = \sigma_\alpha + \varrho_\alpha, \qquad x_{\alpha+1}(t) = x_\alpha(t) - \varrho_\alpha \Big[y_\alpha(t) + \int_0^t \Gamma\big(s, x_\alpha(s)\big) y_\alpha(s)\, ds\Big],$$

$$(5.11) \qquad y_\alpha(t) = x_\alpha(t) - \int_0^t F\big(s, x_\alpha(s)\big)\, ds - \xi, \qquad t \in [0, T],$$

where ϱ_α is chosen so as to satisfy (5.3) with $\varrho = \varrho_\alpha$, $x = x_\alpha$ and $y = y_\alpha$, i.e.

$$(5.12) \qquad \max_{0\leqslant t\leqslant T} \Big\| F\Big(t, x_\alpha(t) + \varrho_\alpha \int_0^t \Gamma\big(s, x_\alpha(s)\big) y_\alpha(s)\, ds\Big) - $$
$$ - F\big(t, x_\alpha(t)\big) - \varrho_\alpha \Gamma\big(t, x_\alpha(t)\big) y_\alpha(t)\Big\| \leqslant K\varrho_\alpha \|y_\alpha\|_C.$$

If $y_\alpha = 0$, then the proof of the theorem is completed. Otherwise, we have

$$y_{\alpha+1}(t) - y_\alpha(t) + \varrho_\alpha y_\alpha(t)$$
$$= -\int_0^t \big[F\big(s, x_{\alpha+1}(s)\big) - F\big(s, x_\alpha(s)\big) + \varrho_\alpha \Gamma\big(s, x_\alpha(s)\big) y_\alpha(s)\big]\, ds.$$

Hence, it follows, by (5.11) and (5.12) with $-y_\alpha$ replacing y_α, that

$$(5.13) \qquad \|y_{\alpha+1} - y_\alpha + \varrho_\alpha y_\alpha\|_C \leqslant q\varrho_\alpha \|y_\alpha\|_C.$$

Hence,

$$\|y_{\alpha+1}\|_C \leqslant \big(1-(1-q)\varrho_\alpha\big)\|y_\alpha\|_C < e^{-(1-q)\varrho_\alpha}\|y_\alpha\|_C \leqslant e^{-(1-q)\sigma_{\alpha+1}}\|y_0\|_C,$$

in virtue of (5.4_α) and (5.10). Thus, (5.4_α) implies $(5.4_{\alpha+1})$. Further, we obtain from (5.10) and (5.4_α)

$$\|x_{\alpha+1} - x_\alpha\| \leqslant (1+BT)\,\varrho_\alpha \|y_\alpha\|_C \leqslant (1+BT)\, e^{-(1-q)\sigma_\alpha}(\sigma_{\alpha+1}-\sigma_\alpha)\,\|y_\alpha\|_C,$$

that is $(5.5_{\alpha+1})$ is satisfied. Now, we have

$$\|y_{\alpha+1} - y_\alpha\| \leqslant (1+q)\,\varrho_\alpha \|y_\alpha\|_C \leqslant (1+q)\, e^{-(1-q)\sigma_\alpha}(\sigma_{\alpha+1}-\sigma_\alpha)\,\|y_0\|_C,$$

by (5.13) and (5.4_α). Thus, conditions $(5.4_{\alpha+1})$, $(5.5_{\alpha+1})$ and $(5.6_{\alpha+1})$ are satisfied for $\sigma_{\alpha+1}$, $x_{\alpha+1}$ and $y_{\alpha+1}$.

Now, suppose that α is a number of second kind and put $t_\alpha = \lim_{\gamma\uparrow\alpha} t_\gamma$. Let $\{\gamma_n\}$ be an increasing sequence convergent toward α. It follows from (5.8) that $\|x_{\gamma_{n+\gamma}} - x_{\gamma_n}\|_C \to 0$ as $n \to \infty$. Hence, the sequence $\{x_{\gamma_n}\}$ has a limit x_α and so does $\{x_\gamma\}$. It follows from (5.9) that the sequence $\{y_{\gamma_n}\}$ has a limit y_α and so does $\{y_\gamma\}$. Since the integral operator in (5.2) is closed in X_T by assumption, we conclude that y_α satisfies (5.11). If $t_\alpha < +\infty$, then the limit passage in (5.4_{γ_n}) yields (5.4_α). The relations (3.7_α) are also satisfied by the definition of t_α, x_α and y_α and since we proved that (5.11) holds for y_α. The process will terminate if $\sigma_\alpha = +\infty$, where α is of second kind. In this case (5.4_α) yields $y_\alpha = 0$, i.e. x_α is a solution of (5.2) and the proof is completed.

Remark 5.1. It is not necessary that $F(t, x)$ be defined on the whole of X. It is sufficient to assume that $F(t, x)$ is defined for $x \in D$, where D is a linear subset of X. Then we assume in addition that $\Gamma(t, x)(X) \subset D$ for each $x \in D$ and $t \in [0, T]$.

6. Using the directional contractor method we shall prove a local existence theorem for solving nonlinear equations. Let X_0 be a linear subset of the Banach space X. Put $S = S(x_0, r) = [x\colon \|x - x_0\| < r]$ for a given $x_0 \in X_0$ and $U = X_0 \cap \bar{S}$, where $\bar{S}$ is the closure of S. Let $P\colon U \to Y$ be a nonlinear operator closed on U, i.e. $x_n \in U$, $x_n \to x$ and $Px_n \to y \in Y$ imply $x \in U$ and $y = Px$. Suppose that P has a directional contractor $\Gamma(x)\colon Y \to X$ for $x \in U_0 = X_0 \cap S$, i.e. there exists a positive constant $q < 1$ such that

$$(6.1) \qquad \big\|P\big(x + t\Gamma(x)y\big) - Px - ty\big\| \leqslant qt\|y\|$$

for $x \in U_0$ and $0 \leqslant t \leqslant \sigma(x, y)$, y being an arbitrary element of the Banach space Y.

THEOREM 6.1. *Suppose that the following hypotheses are satisfied:*

1) *P is closed on U,*

2) *P has a bounded directional contractor $\Gamma(x)$, $x \in U_0$ satisfying condition (6.1) and*

3) *$\|\Gamma(x)\| \leqslant B$ for $x \in U_0$,*

4) *$r \geqslant B(1-2q)^{-1}\|Px_0\|$, $0 < q < 1/2$.*

Then equation $Px = 0$ has a solution in U.

Proof. In the same way as in the proof of Theorem 3.1 we construct the sequences of numbers $t_\alpha (t_0 = 0)$ and elements $x_\alpha \in U_0$ satisfying conditions (3.1_γ), $(3.2_{\gamma+1})$–$(3.4_{\gamma+1})$, and additionally $(6.2_{\gamma+1})$ for first kind numbers $\gamma+1 < \alpha$

$$(6.2_{\gamma+1}) \qquad 0 < t_{\gamma+1} - t_\gamma < (1-q)^{-1}\ln(1-q)\,(1-2q)^{-1}.$$

Then using the same argument as in the proof of Theorem 3.1 we obtain for arbitrary $\gamma < \alpha$ and $\lambda < \alpha$

$$\begin{aligned}
\|x_\gamma - x_\lambda\| &\leqslant \sum_{\lambda\leqslant\beta<\gamma} \|x_{\beta+1} - x_\beta\| \\
&< B\|Px_0\| \sum_{\lambda\leqslant\beta<\gamma} e^{-(1-q)t_{\beta+1}}(t_{\beta+1} - t_\beta) \\
&= B\|Px_0\| \sum_{\lambda\leqslant\beta<\gamma} e^{-(1-q)(t_{\beta+1}-t_\beta)} e^{-(1-q)t_{\beta+1}}(t_{\beta+1} - t_\beta) \\
&< (1-q)\,(1-2q)^{-1}B\|Px_0\| \sum_{\lambda\leqslant\beta<\gamma} e^{-(1-q)t_{\beta+1}}(t_{\beta+1} - t_\beta) \\
&< (1-q)\,(1-2q)^{-1}B\|Px_0\| \sum_{\lambda\leqslant\beta<\gamma} \int_{t_\beta}^{t_{\beta+1}} e^{-(1-q)t}\,dt \\
&= (1-q)\,(1-2q)^{-1}B\|Px_0\| \int_{t_\lambda}^{t_\gamma} e^{-(1-q)t}\,dt.
\end{aligned}$$

Hence,

$$(6.3) \qquad \|x_\gamma - x_0\| < (1-q)\,(1-2q)^{-1}B\|Px_0\| \int_0^\infty e^{-(1-q)t}\,dt = B(1-2q)^{-1}\|Px_0\|.$$

Thus, all elements x_γ belong to U_0. Also in (3.7) t_α should satisfy additionally $\sigma_\alpha < (1-q)\ln(1-q)\,(1-2q)^{-1}$. The further reasoning is exactly the same as in the proof of Theorem 3.1. As a particular case of Theorem 6.1 we obtain

THEOREM 6.2. *If P is closed on U and has an inverse Gâteaux derivative $\Gamma(x)$, $x \in U_0$, satisfying conditions 3) and 4), then $Px = 0$ has a solution in U.*

The proof follows from the fact that an inverse Gâteaux derivative is a directional contractor and there always exists $0 < q < 1/2$ satisfying condition (6.1).

This theorem has been proved by Gâvurin [3], where $\Gamma(x) = [P'(x)]^{-1}$, $x \in U_0$, $P'(x)$ being the linear Gâteaux derivative defined on X_0 and such that its inverse exists and is continuous on Y. In this case $\Gamma(x)$ is an inverse Gâteaux derivative.

7. Using Theorem 6.1 as a basis, an implicit function theorem can be proved. Let X, Z and Y be Banach spaces and put

$$S = [(x, z)\colon\ \|x - x_0\| \leqslant r,\ \ \|z - z_0\| \leqslant \varrho,\ \ x \in X,\ \ z \in Z]$$

for given $x_0 \in X$, $z_0 \in Z$ r and ϱ. Let $P\colon S \to Y$ be a continuous nonlinear operator and suppose that for every z such that $(x, z) \in S$ P has a directional contractor $\Gamma(x, z)\colon Y \to X$ which is z-uniform, i.e. there exists a positive $q < 1$ such that

$$(7.1) \qquad \big\|P\big(x + t\Gamma(x, z)y, z\big) - P(x, z) - ty\big\| \leqslant qt\|y\|$$

for $0 \leqslant t \leqslant \sigma(x, y)$ and y is an arbitrary element of Y. We assume, in addition, that the directional contractor $\Gamma(x, z)$ is bounded, i.e. there exists a const. B such that

$$(7.2) \qquad \|\Gamma(x, z)\| \leqslant B \quad \text{for } (x, z) \in S.$$

Finally, we suppose that the directional contractor $\Gamma(x, z)$ is strongly continuous in (x, z), i.e. in the sense of the operator norm.

THEOREM 7.1. *Suppose that $P\colon S \to Y$ is a continuous operator satisfying the following conditions:*

1) $P(x_0, z_0) = 0$.

2) *P has a bounded directional z-uniform contractor satisfying conditions* (7.1) *and* (7.2) *and being strongly continuous with respect to (x, z). Then there exists a continuous function $g(z)$ defined in some neighborhood of z_0, with values in X, and such that $P\big(g(z), z\big) = 0$.*

Proof. First of all we choose η and ϱ_1 such that

$$(7.3) \qquad B(1-2q)^{-1}\eta \leqslant r \quad \text{and} \quad \|P(x_0, z)\| \leqslant \eta \ \text{ for } \|z - z_0\| \leqslant \varrho_1.$$

Now, in the same way as in the proof of Theorem 6.1 we construct sequences of numbers $t_\alpha (t_0 = 0)$ and continuous functions $x_\alpha(z)$ (replacing x_α), where $(x_\alpha, z) \in S_1$, $S_1 = [(x, z)\colon \|x - x_0\| \leqslant r,\ \|z - z_0\| \leqslant \varrho_1] \subset X \times Z$. The values of $x_\alpha(z)$ are in X and $x_0(z) \equiv x_0$. These sequences are to satisfy conditions (3.1_γ), $(3.2_{\gamma+1})$–$(3.4_{\gamma+1})$ and $(6.2_{\gamma+1})$ provided that $x_{\gamma+1}$, x_γ, $Px_{\gamma+1}$, Px_γ and Px_0 are replaced by $x_{\gamma+1}(z)$, $x_\gamma(z)$, $P\big(x_{\gamma+1}(z), z\big)$, $P\big(x_\gamma(z), z\big)$ and $P(x_0, z)$, respectively. Thus, we obtain in place of (3.5) and (3.6), by (7.3),

$$(7.4) \qquad \|x_\gamma(z) - x_\lambda(z)\| \leqslant B\eta \int_{t_\lambda}^{t_\gamma} e^{-(1-q)t}\,dt.$$

$$(7.5) \qquad \big\|P\big(x_\gamma(z), z\big) - P\big(x_\lambda(z), z\big)\big\| \leqslant (1+q)\eta \int_{t_\lambda}^{t_\gamma} e^{-(1-q)t}\,dt.$$

Inequalities (7.4) and (7.5) show the z-uniform convergence in the analogous relations (3.4_γ) yielding the continuity of the limit functions. An equivalent of (6.3) will be,

$$\|x_\gamma(z)-x_0\| \leqslant B(1-2q)^{-1}\eta \leqslant r,$$

in virtue of (7.3). Hence,

$$(x_\gamma(z), z) \in S_1.$$

Thus, all constructed functions $x_\gamma(z)$ are continuous and well-defined. The further reasoning is exactly the same as in the proof of Theorem 6.1.

Since the assumption of Theorem 7.1 are rather weak, we cannot prove the uniqueness of the function $g(z)$.

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L'analogue dans $\mathscr{C}^p$ des theorems de convexité de M. Riesz et G. O. Thorin

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Sommaire. Les théorèmes de convexité de M. Riesz et G. O. Thorin dans les espaces de fonctions L^p ont leurs analogues dans les espaces d'opérateurs C^p d'espaces hilbertiens (Espaces de R. Schatten).

La méthode utilisée dans cet article est celle de l'interpolation holomorphe.

§ 1. Les espaces vectoriels considérés sont complexes, sauf mention du contraire. Nous notons $|\ |$ les différentes normes rencontrées.

Soit H un espace de Hilbert; $\mathscr{C}^{\infty+1}(H)$ (resp. $\mathscr{C}^{\infty}(H)$; resp. $\mathfrak{F}(H)$) désignera l'espace vectoriel des opérateurs linéaires et continus de H dans H (resp. l'idéal des opérateurs compacts; resp. l'idéal des opérateurs de rang fini) $\mathscr{C}^{\infty+1}(H)$ et $\mathscr{C}^{\infty}(H)$ seront munis de la norme des opérateurs; ce sont des espaces de Banach. $\mathfrak{F}(H)$ est dense dans $\mathscr{C}^{\infty}(H)$.

Soit p une valeur numérique réelle (finie), $p \geqslant 1$. Nous désignons par $\mathscr{C}^p(H)$ l'ensemble des $T\ \mathscr{C}^{\infty}(H)$ tels que la suite $\{\lambda_n(T)\}_{n\geqslant 1}$ décroissante, tendant vers zéro (avec répétition éventuelle suivant la multiplicité) des valeurs propres de $(T^*T)^{1/2}$, la racine carrée de T^*T, soit dans l^p, espace des suites de puissance p-ième sommable.

Si $T \in \mathscr{C}^p(H)$, notons $|T|_p = \{\sum_1^\infty \lambda_n^p(T)\}^{1/p}$.

On sait que (cf [2] par exemple), pour tout $p \geqslant 1$, $\mathscr{C}^p(H)$ est un espace vectoriel et $|\ |_p$ est une norme sur $\mathscr{C}^p(H)$; muni de cette norme $\mathscr{C}^p(H)$ est un espace de Banach et $\mathfrak{F}(H)$ est dense dans $\mathscr{C}^p(H)$. De plus, pour $\forall p, q$ tels que $1 \leqslant p \leqslant q$

$$\mathscr{C}^p(H) \subset \mathscr{C}^q(H) \subset \mathscr{C}^\infty(H)$$

avec injections continues.

Nous allons prouver dans ce travail que les analogues des théorèmes de convexité bien connues des espaces de fonctions L^p sont vrais pour les espaces d'opérateurs $\mathscr{C}^p(H)$.

§ 2. Soit H un espace de Hilbert.

Si $T \in \mathscr{C}^1(H)$, la suite $\{\mu_j(T)\}_{j\geqslant 1}$ des valeurs propres de T étant rangée par ordre de module décroissant (avec répétition éventuelle suivant la multiplicité), la série $\sum_j \mu_j(T)$ est absolument convergente (cf [2]).