Generalized convolutions II

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Abstract. The purpose of this paper is to prove the uniqueness theorem for
a representation of the characteristic function of infinitely divisible measures in a gen-
eralized convolution algebra. This result is used to investigate stable and self-decom-
posable measures.

1. Introduction. For the terminology and notation used here, see
[3]. In particular, \( \mathfrak{B} \) denotes the class of all probability measures defined
on Borel subsets of the positive half-line. Further, \( E_a (a > 0) \) denotes
the probability measure concentrated at the point \( a \). For any positive
number \( a \) the transformation \( T_a \) of \( \mathfrak{B} \) onto itself is defined by means of
the formula \( (T_a P)(\sigma) = P(a^{-1} \sigma) \) where \( P \in \mathfrak{B} \), \( \sigma \) is a Borel set and
\( a^{-1} \sigma = \{ a^{-1} x : x \in \sigma \} \). The transformation \( T_a \) is defined by assuming
\( T_a P = E_a \) for all \( P \in \mathfrak{B} \).

A commutative and associative \( \mathfrak{B} \)-valued binary operation \( \circ \) defined
on \( P \) is called a generalized convolution if it satisfies the following condi-
tions:

(i) the measure \( E_a \) is a unit element, i.e. \( E_a \circ P = P \) for all \( P \in \mathfrak{B} \);
(ii) \( (aP \circ bQ) \circ R = a(P \circ (bQ)) \) whenever \( P, Q, R \in \mathfrak{B} \)
and \( a > 0 \), \( b > 0 \), \( a + b = 1 \);
(iii) \( (T_a P) \circ (T_b Q) = T_{a+b} (P \circ Q) \) for any \( P, Q \in \mathfrak{B} \) and \( a > 0 \);
(iv) if \( P_n \to P \), then \( P_n \circ Q \to P \circ Q \) for all \( Q \in \mathfrak{B} \) where the convergence
is the weak convergence of probability measures;
(v) there exists a sequence \( c_1, c_2, \ldots \) of positive numbers such that
the sequence \( \{ T_{c_n} P_{n+1} \} \) weakly converges to a measure different from \( E_a \).

The power \( E_a^n \) is taken here in the sense of the operation \( \circ \). The class
\( \mathfrak{B} \) with a generalized convolution \( \circ \) is called a generalized convolu-
tion algebra and denoted by \( (\mathfrak{B}, \circ) \). Algebras admitting a non-trivial
homomorphism into the real field are called regular. We say that an
algebra \( (\mathfrak{B}, \circ) \) admits a characteristic function if there exists one-to-one
correspondence \( P \leftrightarrow \Phi_P \) between probability measures \( P \) from \( \mathfrak{B} \)
and real-valued functions \( \Phi_P \) defined on the positive half-line such that
\( \Phi_{aP+bQ} = a \Phi_P + b \Phi_Q \) (\( a, b > 0 \), \( a + b = 1 \)), \( \Phi_{aP} = a \Phi_P \), \( \Phi_{P(t)} = \Phi_{P(\alpha t)} \)
(\( a, t > 0 \)) and the uniform convergence in every finite interval of
\( \Phi_p \) is equivalent to the weak convergence of \( P_n \). The function \( \Phi_p \) is called the characteristic function of the probability measure \( P \) in the algebra \( (\Psi, \circ) \). It plays the same fundamental role in generalized convolution algebra as in ordinary convolution algebra, i.e. in classical problems concerning the addition of independent random variables.

It is proved in [3] (Theorem 6) that an algebra admits a characteristic function if and only if it is regular. Moreover, each characteristic function is an integral transform

\[ \Phi_p(t) = \int \frac{1 - \Omega(ax)}{1 - \Omega(x)} m(dx), \]

where the kernel \( \Omega \) satisfies the inequality \( \Omega(x) < 1 \) in a neighborhood of the origin and

\[ \lim_{x \to 0} \frac{1 - \Omega(ax)}{1 - \Omega(x)} = \gamma, \]

uniformly in every finite interval. The positive constant \( \gamma \) does not depend upon the choice of a characteristic function and is called a characteristic exponent of the algebra in question. Moreover, there exists a probability measure \( M \) called a characteristic measure of the algebra for which

\[ \Phi_{\psi^*}(t) = \exp(-\gamma x) \]

([3], Theorem 7).

Throughout this paper, we assume that the algebra \( (\Psi, \circ) \) is regular and \( \Phi_p \) is a fixed characteristic function in \( (\Psi, \circ) \).

2. Infinitely divisible measures. A measure \( P \in \Psi \) is said to be infinitely divisible if for every positive integer \( n \) there exists a measure \( P_n \in \Psi \) such that \( P = P_n^\otimes n \). The class of infinitely divisible measures coincides with the class of all limit distributions of sequences \( P_{a_1} \circ P_{a_2} \circ \ldots \circ P_{a_n} \), where \( P_{a_k} (k = 1, 2, \ldots, n; n = 1, 2, \ldots) \) are uniformly asymptotically negligible (see [3], Theorem 12).

Taking an arbitrary number \( a_0 > 0 \) such that \( \Omega(x) < 1 \) whenever \( 0 < x < a_0 \), we put

\[ \omega(x) = \begin{cases} 1 - \Omega(x) & \text{if } 0 < x < a_0, \\ 1 - \Omega(x_0) & \text{if } a_0 > x_0. \end{cases} \]

In [3] (Theorem 13) I proved that the class of characteristic functions of infinitely divisible measures \( P \in \Psi \) coincides with the class of functions

\[ \Phi_{\psi^*}(t) = \exp \int \frac{1 - \Omega(ax)}{1 - \Omega(x)} m(dx), \]

where \( m \) runs over all finite Borel measures on the positive half-line and the integrand is defined as its limiting value \(-\gamma x^* \) when \( x = 0 \).

The aim of the present paper is to prove that the representation (5) is unique, i.e. that the function \( \Phi_p \) determines the measure \( m \). The uniqueness of the representation (5) will lead to some results concerning stable and self-decomposable probability measures in the algebra \( (\Psi, \circ) \).

**Theorem 1.** The representation (5) of the characteristic function of infinitely divisible measures is unique.

**Proof.** Suppose that \( \Phi_p \) is given by formula (3). We introduce an auxiliary finite measure \( m_\delta \) defined on the positive half-line by means of the formula

\[ m_\delta(dx) = \int \frac{1 - \exp(-x^*)}{1 - \Omega(ax)} \frac{1 - \Omega(ax)}{\omega(ax)} m(dx). \]

We note that, by Theorems 1, 5 and 6 in [3], the inequality \( \int \Omega(ax) \, dx < 1 \) is true. Consequently, the density function in (6) is positive for \( x > 0 \). Moreover, by (3) and (4), this density function is bounded which implies the finiteness of \( m_\delta \).

First we shall prove that the function \( \Phi_p \) determines the measure \( m_\delta \) on the open half-line \((0, \infty)\). Of course, to prove this it suffices to prove that \( \Phi_p \) determines the measure \( m_\delta \). Let us introduce the notation

\[ I(t) = \int_0^\infty \frac{1 - \Omega(ax)}{1 - \Omega(x)} \frac{1 - \Omega(ax)}{\omega(ax)} m(dx). \]

Taking into account the formula

\[ \Omega(ax) \Omega(ax) = \Phi_{\psi^*}(ax) \Phi_{\psi^*}(ax) = \Phi_{\psi^*}(ax) = \int_0^\infty \Omega(ax) \circ \circ (dy), \]

by a simple computation we get the equation

\[ I(t) = -\log \Phi_p(t) - \int \frac{1}{2} \log \Phi_p(u) \, du + \int \frac{1}{2} \log \Phi_p(u) \, du \, du. \]

Further, integrating with respect to the characteristic measure \( M \) of the algebra we get, by virtue of (1) and (3), the formula

\[ \int \frac{1}{2} I(t) M(dy) = \int \frac{1 - \exp(-t^* x^*)}{1 - \Omega(ax)} \frac{1 - \Omega(ax)}{\omega(ax)} m(dx). \]

Thus, by (6),

\[ \int \exp(-t^* x^*) m_\delta(dx) = \int \frac{1 - \exp(-t^* x^*)}{1 - \Omega(ax)} \frac{1 - \Omega(ax)}{\omega(ax)} m(dx). \]
Hence and from (7) it follows that the function $\Phi_p$ determines the modified Laplace transform of the measure $m$. This proves that the measure $m$ is uniquely determined by $\Phi_p$ on the open half-line $(0, \infty)$. It remains to prove that $m([0])$ is also determined by $\Phi_p$. But this is a direct consequence of the formula
\[
m([0]) = -\log \Phi_p(0) + \int_{(0,\infty)} \frac{m(dx)}{\omega(x)}
\]
which completes the proof.

3. Stable measures. A measure $P \in \mathcal{P}$ is said to be stable if for any pair $a, b$ of positive numbers there exists a positive number $c$ such that $T_a P \circ T_b P = T_c P$. The class of stable measures coincides with the class of all limit distributions of sequences $T_{a_n}P^n$ where $a_n > 0$ ($n = 1, 2, \ldots$) and $P \in \mathcal{P}$ (see (3), Theorem 15). A description of the characteristic function of stable measures was given by Theorem 16 in [3]. By the uniqueness of the representation (5) we are now in a position to establish a simpler description of these functions. We start with a Lemma.

**Lemma 1.**
\[
\lim_{z \to 0} \frac{\omega(z)}{z^p} < \infty.
\]

**Proof.** Suppose the contrary, i.e.,
\[
\lim_{n \to \infty} \frac{\omega(z_n)}{z_n^p} = \infty
\]
for a sequence $\{z_n\}$ tending to $0$. Since, by (2) and (4),
\[
\lim_{n \to \infty} \frac{1 - \Omega(z_n)}{\omega(z_n)} = z_p
\]
we have, for every positive number $x$, the formula
\[
\lim_{n \to \infty} \frac{1 - \Omega(z_n)}{z_n^p} = \infty.
\]
Obviously, the characteristic function $\mathcal{M}$ of the algebra is not concentrated at the origin. Consequently, the Fubini Lemma yields the equation
\[
\lim_{n \to \infty} \int_{0}^{\infty} \frac{1 - \Omega(z_n)}{z_n^p} \mathcal{M}(dz) = \infty.
\]
On the other hand, by (3),
\[
\int_{0}^{\infty} \frac{1 - \Omega(z_n)}{z_n^p} \mathcal{M}(dz) = \frac{1 - \exp(-z_n^p)}{z_n^p} \to 1
\]
which implies a contradiction. The Lemma is thus proved.

**Theorem 2.** The class of characteristic functions of stable measures in $(\mathcal{P}, \circ)$ coincides with the class of functions
\[
\Phi_p(t) = \exp(-ct^p),
\]
where $c > 0$ and $0 < p < \infty$; $\kappa$ being the characteristic exponent of the algebra in question.

**Proof.** By Theorem 16 in [3] it suffices to prove that the integral
\[
\int_{\mathbb{R}} \frac{\omega(z)}{z^p} \, dz
\]
is finite if and only if $p < \kappa$. The finiteness of this integral for $p < \kappa$ is a direct consequence of Lemma 1. It remains to prove that
\[
\int_{\mathbb{R}} \frac{\omega(z)}{z^p} \, dz = \infty.
\]
Contrary to this let us assume that the last integral is finite. Then the measure $m$ defined by the formula
\[
m_n(X) = b \int_{\mathbb{R}} \frac{\omega(z)}{z^p} \, dz,
\]
where $b^{-1} = \int_{\mathbb{R}} \frac{1}{z^p} \, dz$ is finite. Moreover,
\[
\int_{\mathbb{R}} \frac{\Omega(z)}{\omega(z)} m_n(z) \, dz = -1.
\]
Thus, by (3), $m_n$ is the representing measure in (5) corresponding to the measure $M$. On the other hand, the unit measure $\delta_z$ has the same property which contradicts the Theorem 1. Theorem 2 is thus proved.

4. Self-decomposable measures. A measure $P \in \mathcal{P}$ is said to be self-decomposable if for every number $c$ satisfying the condition $0 < c < 1$ there exists a measure $Q_c \in \mathcal{P}$ such that $P = T_c P \circ Q_c$.

The following Lemmas is used in the sequel. They are a generalization of Lemmas which are well-known for ordinary convolution algebra.

**Lemma 2.** The characteristic function of a self-decomposable measure does not vanish.
Proof. Suppose the contrary and assume that $\Phi_p(a) = 0$ and $\Phi_p(t) \neq 0$ whenever $0 < t < a$. We note that for each number $c$ satisfying the condition $0 < c < 1$ the formula

$$\Phi_p(t) = \Phi_p(ct) \Phi_{t/a}(t)$$

is true. Hence we get the relation

$$\lim_{c \to 1} \Phi_{t/a}(t) = 1$$

in the interval $0 < t < c$. Applying the Compactness Lemma ([3], 230) we can choose then a sequence $\{a_n\}$ $(0 < a_n < 1)$ tending to 1 such that the sequence of measures $\{\Phi_{a_n}\}$ is weakly convergent to a measure $\mathcal{Q}$. Thus

$$\lim_{n \to \infty} \Phi_{a_n}(t) = 1$$

uniformly in the interval $0 < t < a$ and, consequently, $\Phi_{a_n}(a) \neq 0$ for sufficiently large indices $n$. Hence and from (8) it follows that $\Phi_p(a_n) = 0$ for sufficiently large $n$ which yields a contradiction. The Lemma is thus proved.

**Lemma 3.** Let $P$ be a self-decomposable measure. Then for each $c$ $(0 < c < 1)$ the associated measure $Q_c$ is infinitely divisible. Further, setting

$$P_1 = P, \quad P_n = T_{n-1} P_n \quad (n = 2, 3, \ldots)$$

we have

$$P = T_{-1} P_1 \circ T_2 \circ \ldots \circ T_n \quad (n = 1, 2, \ldots).$$

Moreover, the measures $T_{n-1} P_n \quad (k = 1, 2, \ldots, n; n = 1, 2, \ldots)$ are uniformly asymptotically negligible.

Proof. By Lemma 2 the characteristic function $\Phi_p$ does not vanish.

Since

$$\Phi_p(t) = \Phi_p(ct) \Phi_{t/a}(t),$$

we have the equations

$$\Phi_{P_k} = \Phi_{P_1}, \quad \Phi_{P_n}(t) = \frac{\Phi_{P_n}(nt)}{\Phi_{P_1}([n-1]t)} \quad (n = 2, 3, \ldots),$$

which imply

$$\Phi_{P_n}(nt) = \prod_{k=1}^{n} \Phi_{P_1}(t).$$

Formula (10) is a direct consequence of the last equation. Further, by (12), $\Phi_{P_n} \to \mathcal{Q}$ uniformly in $k$ $(k \leq n)$ which shows that the measures $T_{n-1} P_n \quad (k = 1, 2, \ldots, n; n = 1, 2, \ldots)$ are uniformly asymptotically negligible.

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Given a number $c$ $(0 < c < 1)$ we put

$$R_n = T_{n-1} (P_n \circ \ldots \circ P_1) \quad (n = 1, 2, \ldots),$$

where the square brackets denote the integral part of a real number. By (11) and (12)

$$\Phi_{R_n}(t) = \frac{\Phi_{P_n}(t)}{\Phi_{P_1}([nt]/n)} \to \Phi_{P_1}(t)$$

uniformly in every finite interval. Thus $R_n \to \mathcal{Q}_1$. Since the measures $T_{n-1} P_n \quad (k = [n]+1, [n]+2, \ldots, n; n = 1, 2, \ldots)$ are uniformly asymptotically negligible, the limit measure $Q_1$ is infinitely divisible ([3], Theorem 13) which completes the proof.

**Lemma 4.** If the measures $T_{n-1} P_n \quad (k = 1, 2, \ldots; n = 1, 2, \ldots)$ are uniformly asymptotically negligible and the sequence $T_{n-1} (P_1 \circ \ldots \circ P_n)$ converges to a probability measure $P$ different from $E_1$, then

$$\lim_{n \to \infty} a_n = 0$$

and

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1.$$
is fulfilled. First we consider the case \( d < \infty \). Since \( T_{\infty} P_n \to E_0 \), we have
\[
T_{\infty \times 1} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) = T_{\infty} T_{\infty} \left( P_1 \circ P_2 \circ \ldots \circ P_n \circ T_{\infty} \right) \to T_{\infty} P
\]
where \( d \) denotes the quotient \( \sigma_{d \times 1} / \sigma_d \). Thus \( P = T_{\infty} P \) and, consequently, \( P = T_{\infty} P (k = 0, \pm 1, \pm 2, \ldots) \). Since \( 0 < \infty \) is a limit point of the sequence \( d^n (k = 0, \pm 1, \pm 2, \ldots) \) the last equation yields \( P = T_{\infty} P = E_0 \) which contradicts the assumption.

It remains the case \( d = \infty \). Then, denoting \( \sigma_{d \times 1} \) shortly by \( \sigma_d \), we have the relation
\[
T_{\infty} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) = T_{\infty} \left( T_{\infty} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) \right) \to P.
\]
On the other hand this sequence being equal to
\[
T_{\infty} \left( T_{\infty \times 1} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) \right)
\]
tends to \( T_{\infty} P \), i.e. to \( E_0 \). Consequently, \( P = E_0 \) which contradicts the assumption. The Lemma is thus proved.

We are now in a position to give a characterization of self-decomposable measures.

**Theorem 3.** The class of self-decomposable measures in \( (B, c) \) coincides with the class of limit distributions of sequences \( T_{\infty} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) \) where \( T_{\infty} P_n \) \((k = 1, 2, \ldots; \sigma_{d \times 1} )\) are uniformly asymptotically negligible.

Proof. First suppose that \( P \) is a limit distribution of a sequence \( T_{\infty} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) \) where \( T_{\infty} P_n \) \((k = 1, 2, \ldots; \sigma_{d \times 1} )\) form a triangular array of uniformly asymptotically negligible measures. Since the unit measure \( E_0 \) is obviously self-decomposable, we may assume that \( P \neq E_0 \). Then, by Lemma 4, for any number \( c \) \((0 < c < 1)\) we can find sequences \( k_n < k_{n+1} \) such that \( k_n < k \) and \( k_{n+1} > k \), setting \( \lim_{n \to \infty} k_n = c \). Setting \( t_n = T_{k_n} \)
\[
U_n = T_{k_n} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right),
\]
\[
V_n = T_{k_n} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right),
\]
\[
W_n = T_{k_n} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right),
\]
we have the relations
\[
U_n = T_{k_n} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) \quad (n = 1, 2, \ldots),
\]
\[
U_n \to P, \quad T_{\infty} \left( P_1 \circ P_2 \circ \ldots \circ P_n \right) \to T_{\infty} P .
\]

From (18) it follows that \( \Phi P_n (\sigma) \) tends to \( \Phi P (\sigma) \) in a neighborhood of the origin. Applying the Compactness Lemma (33, p. 330) we infer that the sequence \( \{W_n\} \) is compact. Let \( Q \) be its limit point. Then, by (16) and (17),
\[
P = T_{\infty} P = Q,
\]
which shows that \( P \) is a self-decomposable measure.

The converse implication is a direct consequence of Lemma 3, which completes the proof.

We proceed now to a representation problem for characteristic functions of self-decomposable measures. First we establish some properties of measures \( m \) corresponding to self-decomposable probability measures by the representation formula (3).

Let \([0, \infty)\) denote the compactified half-line. A subset of \([0, \infty)\) is said to be separated from the origin iff its closure is contained in \([0, \infty)\).

Let \( m \) be a finite Borel measure on \([0, \infty)\). For any Borel subset \( \Delta \) of \([0, \infty)\) separated from the origin we put
\[
J_m (\Delta) = \int_{\Delta} \frac{m(du)}{au},
\]
where, according to (4), the integrand is assumed to be \((1 - \Omega(ax))^{-1}\) if \( x = \infty \). Denote by \( \mathcal{M} \) the set of all finite Borel measures \( m \) on \([0, \infty)\) satisfying for all numbers \( c \) \((0 < c < 1)\) and all Borel subsets \( \Delta \) separated from the origin the following condition
\[
J_m (\Delta) - J_m (\Delta) \leq 0.
\]
It is clear that the set \( \mathcal{M} \) is convex. Let \( \mathcal{R} \) be the subset of \( \mathcal{M} \) consisting of probability measures on \([0, \infty)\). The set \( \mathcal{R} \) is convex and compact.

Suppose that the measure \( m \) is concentrated on the open half-line \((0, \infty)\) and put
\[
J_m (x) = \int_{0}^{\infty} \frac{m(du)}{au} \quad (a > 0).
\]
Obviously, \( J_m ([a, b]) = J_m (a) - J_m (b) \). It is easy to see that \( m \in \mathcal{R} \) if and only if the inequality (18) holds for all \( c \) \((0 < c < 1)\) and all Borel subsets \( \Delta \) of the form \([a, b] \). Consequently, \( m \in \mathcal{R} \) if and only if for every triplet \( a, b, c \) satisfying the conditions \( 0 < c < 1, 0 < a < b \) the inequality
\[
J_m (a) - J_m (b) - J_m \left( \frac{a}{c} \right) + J_m \left( \frac{b}{c} \right) \geq 0
\]
is true. Introducing the notation
\[
F(a, b) = J_m (\sigma) \quad (\infty < a < \infty)
\]
and substituting \( a = \sigma - h, b = \sigma, c = \sigma - h \) \((\infty < \sigma < \infty, 0 < h < \infty)\) into (20) we get the inequality
\[
F(a, b) \leq \frac{1}{2} \left( F(a - h) + F(a + h) \right).
\]
Thus the function $F$ is convex on the real line. Moreover, by (19), it is also monotone non-increasing with $F(\infty) = 0$. Consequently, it can be represented in the form

$$F(x) = \int_0^x g_m(u) \, du,$$

where $g_m$ is monotone non-increasing and non-negative. Further, by (19) and (21),

$$m(\mathcal{F}) = \int \omega(\sigma) g_m(\log \sigma) \frac{d\sigma}{\sigma}.$$

Conversely, if $q$ is monotone non-increasing and non-negative function and

$$\int_0^\infty \omega(\sigma) q(\log \sigma) \frac{d\sigma}{\sigma} < \infty,$$

then the measure $m$ defined by means of the formula

$$m(\mathcal{F}) = \int \omega(\sigma) g(\log \sigma) \frac{d\sigma}{\sigma}$$

belongs to $\mathcal{R}$. Indeed, then $J_m(x) = \int x \omega(\log x) \frac{dx}{x}$ and the inequality

(20) is evident. Moreover $g_m = q$ at all continuity points.

We may assume that the function $g_m$ is continuous from the right. In this case $g_m$ is uniquely determined by the measure $m$. Thus we have proved the following Lemma.

Lemma 5. Equation (22) establishes a one-to-one correspondence between measures $m$ from $\mathcal{R}$ concentrated on the open half-line $(0, \infty)$ and non-negative monotone non-increasing continuous from the right functions $g_m$ on the real line satisfying the condition

$$\int_0^\infty \omega(\sigma) g_m(\log \sigma) \frac{d\sigma}{\sigma} < \infty.$$

Further, the measures $m$ from $\mathcal{R}$ corresponds to functions $g_m$ satisfying the condition

(23) $$\int_0^\infty \omega(\sigma) g_m(\log \sigma) \frac{d\sigma}{\sigma} = 1.$$

We define a family $m_{\omega}(x) [0, \infty)$ of probability measures on $[0, \infty]$ as follows: $m_0 = B_0$, $m_\omega = E_\omega$ and

$$m_\omega(\mathcal{F}) = \omega(\sigma) \int_{\mathcal{F}} \omega(u) \frac{du}{u} \quad (0 < u < \infty),$$

where $a(x) - 1 = \int_0^x \omega(u) \frac{du}{u}$. We note that, by Lemma 1, $a(\omega)$ is finite for all $x$. It is obvious that $m_\omega$ and $m_{\omega_1}$ belong to $\mathcal{R}$. Since the measures $m_\omega$ ($0 < \omega < \infty$) are concentrated on the open half-line $(0, \infty)$ and

$$g_m(u) = \begin{cases} a(\omega) & \text{if } u < \log \omega \\ 0 & \text{if } u \geq \log \omega \end{cases},$$

we infer, by Lemma 5, that $m_{\omega} \in \mathcal{R}$ too.

Lemma 6. The set $\{m_\omega: \omega \in [0, \infty)\}$ coincides with the set of extreme points of $\mathcal{R}$.

Proof. Let $m_\omega, m_{\omega_1} \in \mathcal{R}$. It is evident that $m = m_{\omega_1} + (1 - \omega)m_\omega$ if and only if $g_m = \omega g_{m_{\omega_1}} + (1 - \omega)g_{m_\omega}$. Thus a measure $m$ from $\mathcal{R}$ is an extreme point of $\mathcal{R}$ if and only if $g_m$ can not be decomposed into a convex combination of two different $q$-functions satisfying condition (23). It is very easy to verify that for $x \in [0, \infty)$ the function $q_{m_\omega}$ is not a convex combination of two different $q$-functions with property (23). Consequently, the measures $m_\omega \{x \in [0, \infty)\}$ are extreme points of the set $\mathcal{R}$. Further, $m_\omega$ and $m_{\omega_1}$ are extreme points too.

On the other hand the only $q$-functions which can not be decomposed into a convex combination of two different $q$-functions satisfying condition (23) are the functions of the form $q_{m_\omega}(u) = b$ whatever $u < y$ and $q_{m_\omega}(u) = 0$ in the remaining case. By (23) we have the relation $b = a(\sigma^\omega)$. Thus $m = m_\omega$. Consequently, each extreme point of $\mathcal{R}$ concentrated on the open half-line coincides with one of the measures $m_\omega \{x \in [0, \infty)\}$. It is clear that a measure belongs to $\mathcal{R}$ if and only all its restriction to $(0, \infty)$, $(0, \infty)$ and $(\infty)$ respectively belong to $\mathcal{R}$. Hence it follows that the extreme points of $\mathcal{R}$ which are not concentrated on the open half-line $(0, \infty)$ are supported by the one-point sets $(0)$ and $(\infty)$ respectively. Consequently, they coincide with one of the measures $m_\omega$ and $m_{\omega_1}$. The Lemma is thus proved.

One can easily prove that the mapping $x \rightarrow m_\omega$ is a homeomorphism between $[0, \infty]$ and the set of extreme points of $\mathcal{R}$. Once the extreme points of $\mathcal{R}$ are found we can apply a Theorem by Choquet (11). Since each element of $\mathcal{R}$ is of the form $m_\omega$, where $\omega > 0$ and $\omega \in \mathcal{R}$ we then get the following Lemma.

Lemma 7. A measure $m$ belongs to $\mathcal{R}$ if and only if there exists a finite Borel measure $\rho$ on $[0, \infty]$ such that

$$\int f(u) m(du) = \int f(u) m_\omega(du) \rho(du)$$

for all continuous functions $f$ on $[0, \infty]$. 

Corollary. A measure \( m \) concentrated on \([0, \infty)\) belongs to \( \mathfrak{M} \) if and only if there exists a finite Borel measure \( p \) on \([0, \infty)\) such that
\[
\int_{0}^{\infty} g(s) m(ds) = \int_{0}^{\infty} g(s) m(s) p(ds)
\]
for all continuous bounded functions \( g \) on \([0, \infty)\).

Now we shall give a description of measures associated by representation formula (5) to self-decomposable probability measures. We note that by Theorem 3 of the present paper and Theorem 12 in [3] self-decomposable measures are infinitely divisible.

**Lemma 8.** A measure \( m \) concentrated on \([0, \infty)\) is a representing measure in (5) of a self-decomposable probability measure if and only if \( m \in \mathfrak{M} \).

Proof. Suppose that the characteristic function of a probability measure \( P \) is given by formula (5). Taking into account Lemma 3 we infer that \( P \) is self-decomposable if and only if the quotient \( \Phi_{p}/\Phi_{\nu,p} \) for every number \( e \) satisfying the condition \( 0 < e < 1 \) is the characteristic function of an infinitely divisible measure. Since
\[
\Phi_{p}(t)/\Phi_{\nu,p}(t) = \Phi_{p}(t)/\Phi_{\nu}(at) = \exp \int_{0}^{\infty} \frac{\Omega(tu) - 1}{\omega(u)} \left( \frac{\omega(u)}{\omega(e^{-u})} - m(dx) \right) dx
\]
we infer, by Theorem 3, that the measure
\[
r_{p}(e) = \int_{0}^{\infty} m(dx) - \int_{0}^{\infty} \frac{\omega(u)}{\omega(e^{-u})} m(e^{-u} dx)
\]
for every \( 0 < e < 1 \) is non-negative. Of course, the last condition is equivalent to the condition
\[
\int_{0}^{\infty} \frac{r_{p}(dx)}{\omega(e^{-u})} > 0
\]
for every \( 0 < e < 1 \) and every Borel set \( \mathcal{E} \) separated from the origin. But the left-hand side of the last inequality is equal to \( I_{\mathcal{E}}(\mathcal{E}) - I_{\mathcal{E}}(e^{-1} \mathcal{E}) \). Consequently, \( P \) is self-decomposable if and only if \( m \in \mathfrak{M} \) which completes the proof of the Lemma.

Lemma 8, Corollary to Lemma 7 and representation formula (5) yield the following Theorem.

**Theorem 4.** The class of characteristic functions of self-decomposable measures in \((\mathfrak{M}, \circ)\) coincides with the class of all functions of the form
\[
\Phi_{p}(t) = \int_{0}^{\infty} \frac{\Omega(tu) - 1}{u} du \int_{0}^{\infty} \frac{\omega(u)}{u} du^{-1} p(ds),
\]
where \( p \) is a finite Borel measure on \([0, \infty)\).

5. An example. As an example of a generalized convolution we quote the \((1, r)\)-convolutions \( 1 \leq r < \infty \) considered by J. F. Kingman in [2] (see also [3], p. 218). The \((1, r)\)-convolution is defined by means of the formula
\[
\int_{0}^{\infty} f(x) P \circ Q(dx) = \frac{1}{r} \int_{0}^{\infty} \int_{0}^{\infty} f(x + y) + f(x - y) P(dx) Q(dy)
\]
where \( f \) runs over all bounded continuous functions on \([0, \infty)\). The \((1, r)\)-convolution for \( r > 1 \) is defined by the formula
\[
\int_{0}^{\infty} f(x) P \circ Q(dx) = \frac{2}{r} \int_{0}^{\infty} \int_{0}^{\infty} f(x + y + 2xy) (1 - x^{2})^{r-2} dx P(dx) Q(dy).\]

All \((1, r)\)-convolution algebras are regular. As a characteristic function in these algebras we can take the integral transformation
\[
\Phi_{p}(t) = \int_{0}^{\infty} J_{\frac{t}{2}}(\mathcal{J}_{x}^{-1}(x)) J_{\frac{t}{2}}^{-1}(x) P(dx)
\]
where \( \mathcal{J}_{x} \) is the Bessel function.

The \((1, r)\)-convolution is closely connected with a random walk problem in Euclidean \( r \)-space. Namely, consider a random walk in \( r \)-space given by
\[
S_{n} = X_{1} + X_{2} + \ldots + X_{n} \quad (n = 1, 2, \ldots)
\]
where \( X_{1}, X_{2}, \ldots \) are independent random \( r \)-vectors having spherical symmetric distribution. The probability distribution of the length \( |S_{n}| \) is determined by that of the length \( |X_{1}|, |X_{2}|, \ldots, |X_{n}| \) (see [2]). More precisely, the probability distribution of \( |S_{n}| \) is the \((1, r)\)-convolution of the probability distributions of \( |X_{1}|, |X_{2}|, \ldots, |X_{n}| \). The asymptotic behaviour of \( |S_{n}| \) \( (n = 1, 2, \ldots) \) can be described in terms of the limit distribution of the sequence \( \alpha_{n} \) \( (n = 1, 2, \ldots) \) where \( \alpha_{n} \) are suitable chosen positive numbers. It is clear that the class of all possible limit distributions coincides with the class of all self-decomposable probability distributions in the \((1, r)\)-convolution algebra. Since
\[
\int_{0}^{\infty} \frac{\alpha(u)}{u} du \sim \log(1 + x^{2})
\]
on the whole positive half-line, we get, by virtue of Theorem 4, the following statement:
The class of all possible limit distributions of sequences \( c_n \{S_n\} \), where \( c_n > 0 \) and \( S_n = X_1 + X_2 + \ldots + X_n \) (\( n = 1, 2, \ldots \)), \( X_n \) being independent random \( r \)-vectors with spherical symmetric distribution coincides with the class of all probability distributions \( F \) on \([0, \infty)\) whose integral transform (36) is of the form

\[
\varphi_F(t) = \exp \left( \int_0^\infty \frac{1}{s} \left( \frac{\Gamma \left( \frac{r}{2} \right)}{\Gamma \left( \frac{r-1}{2} \right)} \right) \int_0^s \frac{J_{r-1}(tu)}{u^{1-r}} \, du \right) m(du)
\]

where \( m \) is a finite Borel measure on \([0, \infty)\).

References


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On generalized variations (II)

by

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Abstract. A \( \phi \)-function is a non-decreasing function, continuous for \( u > 0 \), \( \varphi(u) = 0 \) only for \( u = 0 \) and \( \lim_{u \to \infty} \varphi(u) = \infty \) when \( u \to \infty \). For a function \( \varphi \) with domain \([a, b]\) put

\[
V_\varphi(x) = \sup \sum_{n=1}^\infty \varphi \left( \pi_1 \left( \frac{x-n}{b-a} \right) \right),
\]

supremum is taken over all partitions of \([a, b]\). \( \varphi^{**} \) denotes the class of all functions \( \varphi \) defined on \([a, b]\) for which \( \varphi(x) = 0 \) and \( V_\varphi(x) < \infty \) for certain \( x > 0 \), and \( \varphi^{**} \) denotes the class of all functions continuous on \([a, b]\) belonging to \( \varphi^{**} \). Among all \( \varphi \)-functions the log-convex \( \varphi \)-functions are distinguished i.e. ones satisfying the condition

\[
\varphi(u\varphi(v)) < \varphi(u) + \varphi(v)
\]

for \( u, v > 0 \), and \( \alpha, \beta > 0 \), \( \alpha + \beta = 1 \).

There are presented two proofs of L. C. Young's Theorem that if \( \varphi \) and \( \varphi^{**} \) are log-convex \( \varphi \)-functions satisfying the following L. C. Young's condition

\[
\sum_{n=1}^\infty \varphi^{-1}(n) \varphi^{**}(1/n) < \infty
\]

where \( \varphi^{-1} \) and \( \varphi^{**} \) are the inverse functions to \( \varphi \) and \( \varphi^{**} \) respectively then the integral \( \int_0^\infty \pi(t) dt \) for functions \( \pi \in \varphi^{**} \) and \( \psi \in \varphi^{**} \) exists in the sense of Riemann–Stieltjes.

Estimations of this integral with the use of series in (1) are given. On the same assumptions is proved the theorem on passing to the limit under the sign of Riemann–Stieltjes integral, in particular the analogues of Hölder's theorem. It is shown that if \( \varphi, \psi \) are convex \( \varphi \)-functions satisfying the certain conditions for which L. C. Young's condition (1) does not hold then there are functions \( \alpha \in \varphi^{**} \) and \( \beta \in \varphi^{**} \) such that their Riemann–Stieltjes integral does not exist. These results proved for scalar functions are generalized for functions with values in Banach spaces.

0. Introduction. The present paper can be regarded as a second part of paper [9] which, under the same title, appeared in Studia Math. in 1969 (results of [9] were earlier announced in [8]). In the present paper the notations essentially differ from those employed in [9] i.e. in all places where in [9] and other papers dealing with the theory of Orlicz spaces symbols \( M, N \) etc. were used we now write \( \varphi, \psi, \ldots \) The purpose