So λ is rational and f is not R-linear.

(2.53) *f preserves equality of distance.*

Any additive map f from R into a normed real vector space Y preserves equality of distance. For, define $p(t) = \|f(t)\|$ for t in R_0^+. Then for all x and y in R, $\|fx-fy\| = \|f(x-y)\| = \|\pm f(|x-y|)\| = \|f(|x-y|)\| = p(|x-y|)$.

Properties 2.51, 2.52 and 2.53 of Example 2.5 show that Theorem 2.4 fails if X is permitted to be one-dimensional.

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On the conjugates of some function spaces

by

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Abstract. For $p < 1$ and the underlying measure space non atomic, $L(p, \infty)^* = \{0\}$. Results are also given in the atomic case.

I. Introduction. The function spaces $L(p, q)$ form a two parameter family which incorporates the familiar L^p spaces ($L^p = L(p, p)$) as well as other important function spaces. The family $L(p, q)$ is a convenient setting for interpolation theorems for operators, and so is of interest for problems in harmonic analysis.

The dual spaces $L(p, q)^*$ have been studied, and in many cases characterised. (See [1], [2]). This note considers the previously untreated case when $0 < p < 1$ and $q = \infty$.

Throughout this note (X, Σ, μ) is a σ-finite measure space with $0 \leqslant \mu$.

DEFINITION 1. For each measurable f we define

$$f_*(y) = \mu\{x \mid |f(x)| > y\}.$$

Oconfining ourselves to those f such that $f_*(y) < \infty$ for some $y > 0$ define.

DEFINITION 2.

$$f^*(t) = \inf\{y \mid f_*(y) \leqslant t\}.$$

DEFINITION 3. For $0 < p < \infty$, $0 < q < \infty$

$$\|f\|_{p,q}^* = \left[\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right]^{1/q}$$

and for $0 < p \leqslant \infty$

$$\|f\|_{p,\infty}^* = \sup_{t>0} t^{1/p} f^*(t)$$

Define also $L(p, q) = \{f \mid \|f\|_{p,q}^* < \infty\}$.

A detailed discussion of $L(p, q)$ spaces may be found in [2].

II. $L(p, \infty)^*$.

THEOREM 1. *For* $0 < p < 1$ *and non atomic measure space*

$$L(p, \infty)^* = \{0\}.$$

Proof. For conciseness of presentation, take spaces over the real field. Assume a continuous linear functional φ exists on $L(p, \infty)$. Define the functional φ^+ on all $f \geqslant 0$ in $L(p, \infty)$ by

$$\varphi^+(f) = \sup\{\varphi(g)|\ 0 \leqslant g \leqslant f,\ g \text{ measurable}\}. \tag{2}$$

It follows that

$$\varphi^+(\lambda f) = \lambda\varphi^+(f), \tag{3}$$

$$\varphi^+(f+g) = \varphi^+(f)+\varphi^+(g), \tag{4}$$

$$|\varphi^+(f)| \leqslant C\|f\|^*_{p,\infty} \tag{5}$$

for all non negative f, g, in $L(p, \infty)$, all non negative scalars λ, and some fixed constant C. (3) and (5) are obvious. To prove (4)

$$\begin{aligned}\varphi^+(f)+\varphi^+(g) &= \sup\{\varphi(h_1+h_2)|\ 0 \leqslant h_1 \leqslant f,\ 0 \leqslant h_2 \leqslant g\}\\ &\leqslant \sup\{\varphi(h)|\ 0 \leqslant h \leqslant f+g\}.\end{aligned}$$

For the reverse inequality, taking $\varepsilon > 0$, there exists $h \leqslant f+g$ such that

$$\varphi^+(f+g) \leqslant \varphi(h)-\varepsilon.$$

Put $h_1(x) = \min\{h(x), f(x)\}$, $h_2 = h-h_1$. Then $h_1 \leqslant f$, $h_2 \leqslant g$.

So $\varphi(h) = \varphi(h_1)+\varphi(h_2) \leqslant \varphi^+h+(f)+\varphi^+(g) \leqslant \varphi^+(f)+\varphi^+(g)$.

Since ε is arbitrary, $\varphi^+(f+g) \leqslant \varphi^+(f)+\varphi^+(g)$.

By (3), (4), (5) φ^+ extends linearly to a continuous linear functional on all $L(p, \infty)$ with

$$\varphi^+(f) \geqslant 0 \quad \text{for all } f \geqslant 0.$$

$\varphi^- = \varphi^+ - \varphi$ is also in $L(p, \infty)^*$ with the same positivity property: $\varphi^-(t) \geqslant 0$ for all $f \geqslant 0$.

The set

$$K = \{f \in L(p, \infty)|\ |\varphi^+(f)| \leqslant 1\}$$

is convex, and contains some neighborhood of the origin, $N_\alpha = \{f|\ \|f\|^*_{p,\infty} < \alpha\}$. To complete the proof it will be shown that K is the whole of $L(p, \infty)$ and thus $\varphi^+ = 0$. By identical reasoning $\varphi^- = 0$. This implies $\varphi = 0$ and so $L(p, \infty)^* = \{0\}$.

LEMMA 6. *Let* f *be a step function in* $L(p, \infty)$

$$f = \sum_{n=-\infty}^{\infty} a_n \chi_{A_n}$$

where the sets A_n *are disjoint. Then* $f \in K$.

Proof.

$$0 \leqslant f^*(t) \leqslant t^{-1/p}\|f\|^*_{p,\infty} \leqslant \alpha 2^{(1/p-1)m} t^{-1/p}$$

for all $t > 0$ and some large enough integer m. Divide each set A_n into 2^m disjoint sets, $A_n^1, A_n^2 \ldots A_n^{2^m}$, each of measure $2^{-m}\mu(A_n)$ and define 2^m functions in $L(p, \infty)$, $f_1, f_2 \ldots f_{2^m}$, where

$$f_i = \sum_{n=-\infty}^{\infty} a_n \chi_{A_n^i}$$

$$f_i^*(t) = f^*(2^m t) \leqslant \alpha\, 2^{(1/p-1)m}\, 2^{-m/p} t^{-1/p} \quad \text{and so } 2^m f_i \in N_\alpha \subset K.$$

Since K is convex

$$f = \sum_{i=1}^{2^m} f_i = 2^{-m}\sum_{i=1}^{2^m} 2^m f_i \in K$$

proving the lemma.

LEMMA 7. *For arbitrary* f *in* $L(p, \infty)$ *there exists a step function* $g = \sum_{n=-\infty}^{\infty} a_n \chi_{A_n}$ *also in* $L(p, \infty)$ *such that* $-g \leqslant f \leqslant g$.

Proof. Let Λ be the set of all positive numbers λ such that $|f(x)| = \lambda$ for all x on some set A_λ of positive measure. Λ is a countable set since (X, Σ, μ) is σ-finite. Accordingly if Λ is non empty write $\Lambda = \{\lambda_n\}_1^\infty$ and introduce the function

$$g_1 = \sum_{n=1}^{\infty} \lambda_n \chi_{A_{\lambda_n}}.$$

$$g_1 \in L(p, \infty) \text{ and for all } x \in A = \bigcup_{n=1}^{\infty} A_{\lambda_n}$$

$$g_1(x) = |f(x)|, \quad \text{so } -g_1(x) \leqslant f(x) \leqslant g_1(x).$$

It remains to bound f by a step function on $X - A$. For the function $h = f\cdot\chi_{X-A}$ the equation

$$h_*(h^*(t)) = t$$

holds for all $t > 0$. This is a consequence of the inequalities

$$t - \mu\{x|\ |h(x)| = h^*(t)\} \leqslant h_*(h^*(t)) \leqslant t$$

which are readily deduced for any measurable function h from Definitions I.1 and I.2.

$$h^*(t) \leqslant t^{-1/p}\|f\|^*_{p,\infty} \leqslant 2t^{-1/p}\|f\|^*_{p,\infty}.$$

Therefore $h^*(t) \leqslant s(t)$ where $s(t)$ is a decreasing step function inscribed between the two curves $t^{-1/p}\|f\|^*_{p,\infty}$ and $2t^{-1/p}\|f\|^*_{p,\infty}$. The jumps of $s(t)$ occur at the points $t = a_n$ where $a_n \downarrow 0$ as $n \to -\infty$, $a_n \uparrow \infty$ as $n \to +\infty$ (see Fig. 1.)

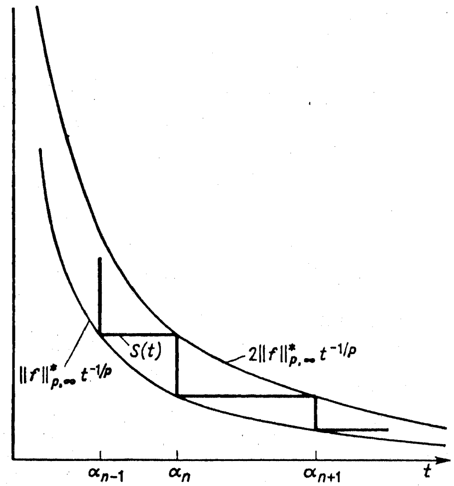

Figure 1.

Let

$$B_n = \{x|\ h^*(a_n) < |h(x)| \leqslant h^*(a_{n-1})\}$$
$$= \{x|\ |h(x)| > h^*(a_n)\} - \{x|\ |h(x)| > h^*(a_{n-1})\}$$

so $\mu(B_n) = h_*\big(h^*(a_n)\big) - h_*\big(h^*(a_{n-1})\big) = a_n - a_{n-1}$.
Introduce the function

$$g_2 = \sum_{n=-\infty}^{\infty} h^*(a_{n-1})\,\chi_{B_n}$$

$g_2 \in L(p, \infty)$ since $g_2^*(t) \leqslant s(t) \leqslant 2t^{-1/p}\|f\|^*_{p,\infty}$ and for $x \in X - A$

$$-g_2(x) \leqslant f(x) = h(x) \leqslant g_2(x).$$

Therefore $g = g_1 + g_2$ is a step function in $L(p, \infty)$ satisfying $-g \leqslant f \leqslant g$, and the lemma is proved.

Furthermore by Lemma 6 $g \in K$ and by the positivity of φ^+, $f \in K$. Therefore $K = L(p, \infty)$ and the theorem is proved.

III. $L(p, \infty)^*$ for a measure space with atoms. For a measure space containing atoms, $L(p, \infty)$ is a direct sum of the two subspaces of functions vanishing respectively on the atomic and non atomic portions of the measure space, and thus to describe $L(p, \infty)^*$ it is sufficient to treat the case where the measure space is purely atomic.

Put $X = \bigcup_{n=-\infty}^{\infty} A_n$, where $\{A_n\}_{-\infty}^{\infty}$ are the atoms arranged in order of increasing measure. (It may thus be in order to index with only the positive or only the negative integers.)

If $f = \sum_{-\infty}^{\infty} f_n \chi_{A_n} \in L(p, \infty)$ then

$$\mu(A_n)^{1/p}|f_n| \leqslant \|f\|^*_{p,\infty} \quad \text{for each } n. \tag{1}$$

Thus every absolutely convergent series $\sum a_n$ defines a continuous linear functional by:

$$\varphi(f) = \sum a_n \mu(A_n)^{1/p} f_n. \tag{2}$$

In general $L(p, \infty)^*$ is strictly larger than the space of functionals of the above form. By (1), the linear transformation T,

$$T\Big(\sum f_n \chi_{A_n}\Big) = \big(\ldots\ \mu(A_{-1})^{1/p} f_{-1},\ \mu(A_0)^{1/p} f_0,\ \mu(A_1)^{1/p} f_1, \ldots\big)$$

maps $L(p, \infty)$ continuously into the sequence space l^∞. If

$$\sum_{k=-\infty}^{n} \mu(A_k)/\mu(A_n) \leqslant C < \infty \quad \text{for all } n \tag{3}$$

T is readily seen to be a homeomorphism, and $L(p, \infty)^* \equiv (l^\infty)^*$, the space of all bounded finitely additive set functions on the integers.

If (3) does not hold, the characterisation of $L(p, \infty)^*$ is rather more elusive. However, given some restrictions on the sequence $\{\mu(A_n)\}$, $L(p, \infty)^*$ consists solely of functionals of form (2) with the sequences $\{a_n\}$ ranging over a class larger than l^1.

THEOREM 4. *If (X, Σ, μ) is atomic and the atoms $\{A_n\}$ satisfy*

$$a \leqslant \mu(A_n) \leqslant b \quad \textit{for each } n, \tag{5}$$

where a and b fixed positive constants, then $L(p, \infty)^* \equiv l^\infty$ the correspondence being in the sense of equation (2).

Proof. Assume that

$$\mu(A_n) = 1 \quad \text{for all } n. \tag{5'}$$

The same proof with a few technical elaborations works for (5) in place of (5′).

The main step is to show that $L(p,\infty)^*$ is purely a sequence space in the sense of equation (2). This is equivalent to proving that any two functionals φ, ψ in $L(p,\infty)^*$ such that $\varphi(f)=\psi(f)$ for every simple function f must be identical.

For such a φ and ψ, $\theta=\varphi-\psi$ vanishes on all simple functions. Any function dominated by a simple function must also be simple, so θ^+ (Definition II.2) and also $\theta^-=\theta^+-\theta$, both vanish on simple functions.

As in Theorem II.1, we prove $\theta=0$ by showing

$$K=\{f\in L(p,\infty)\,\big|\,|\theta^+(f)|\leqslant 1\}\quad \text{to be all of } L(p,\infty).$$

K contains $N_a=\{f\mid \|f\|^*_{p,\infty}\leqslant a\}$ for some $a>0$. For arbitrary $f=\Sigma\beta_n\chi_{A_n}$ in $L(p,\infty)$ re-index the atoms using the positive integers so that the sequence $\{|\beta_n|\}$ is non-increasing.

$$0\leqslant f^*(t)\leqslant t^{-1/p}\|f\|^*_{p,\infty}\leqslant \tfrac{1}{2}a2^{(1/p-1)m}t^{-1/p}$$

for all $t>0$ and some large enough integer m. Choose an integer r satisfying

$$r>2^m(2^p-1)^{-1}\tag{6}$$

and define the function g by

$$g=\sum_{n=1}^{r}|\beta_n|\chi_{A_n}+\sum_{k=0}^{\infty}|\beta_{r+k\cdot 2^m+1}|\Big(\sum_{j=1}^{2^m}\chi_{A_{r+k2^m+j}}\Big).$$

For any integer k, observe that on the 2^m sets $\{A_{r+k2^m+j}\}_{j=1}^{2^m}$ g takes the constant value which is the maximum value taken by f on these 2^m sets. So on the interval $\big(r+k2^m,\ r+(k+1)2^m\big)$ $g^*(t)$ takes a constant value which in view of (6) lies beneath the curve $2\|f\|^*_{p,\infty}t^{-1/p}$. Therefore $\|g\|^*_{p,\infty}\leqslant 2\|f\|^*_{p,\infty}$ and $g^*(t)\leqslant a2^{(1/p-1)m}t^{-1/p}$.

But $g=s+\sum_{j=1}^{2^m}g_j$ where s is the simple function $\sum_{n=1}^{r}|\beta_n|\chi_{A_n}$ and $g_j=\sum_{k=0}^{\infty}|\beta_{r+k2^m+1}|\chi_{A_{r+k2^m+j}}$.

$$g_j^*(t)\leqslant g^*(2^mt)\leqslant a2^{-m}t^{-1/p}$$

so $2^mg_j\in N_a\subset K$ for each $j=1,2\ldots 2^m$, $\theta^+(s)=0$ and by convexity of K,

$$g=2^{-m}\sum 2^mg_j+s\in K.$$

$f\in K$ since $|f|\leqslant g$. This proves $\theta^+=0$ and similarly $\theta^-=0$. Consequently $\theta=\varphi-\psi=0$ and $L(p,\infty)^*$ is a sequence space.

To show finally that $L(p,\infty)^*=l^\infty$ is a simple deduction from the observation that if $f=\sum a_n\chi_{A_n}\in L(p,\infty)$,

$$\sum|a_n|\leqslant\|f\|^*_{p,\infty}\sum_{n=1}^{\infty}n^{-1/p}.$$

The details are left to the reader.

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