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On generalized right modular complemented algebras

by

T. HUSAIN* and PAK-KEN-WONG** (Hamilton, Canada)

Abstract. In this paper, we introduce a class of algebras called generalized right modular complemented algebras (g.r.m.c.). Some fundamental properties of these algebras are obtained. For example, it is shown that a generalized right modular complemented algebra which contains no proper two-sided ideal with zero annihilator is the direct topological sum of its minimal closed two-sided ideals each of which is a simple g.r.m.c. algebra.

1. Introduction. In [5], Yood introduced a class of algebras called modular complemented algebras. A structure and ideal theory is developed for these algebras. The present work is an attempt to generalize these algebras.

In § 2, which is introductory in nature, we give the concept of generalized right modular complemented (g.r.m.c.) algebras. A structure theorem for these algebras is obtained in § 3. In § 4, we study induced complementors by given complementors. Finally in § 5, we investigate the relationship between g.r.m.c. algebras and modular annihilator algebras.

2. Notation and preliminaries. Let A be a topological algebra, $\mathfrak{R}$ the set of all non-zero closed right ideals of A and $\mathfrak{M}$ the set of all closed modular maximal right ideals of A. We call A a generalized right modular complemented algebra if it satisfies the following properties:

(2.1) *there exists a mapping p from $\mathfrak{M}$ into $\mathfrak{R}$ such that $M \cap M^p = (0)$ for all $M \in \mathfrak{M}$.*

(2.2) $$\cap\{M\colon M \in \mathfrak{M}\} = (0).$$

The mapping p is called a *complementor on* A. A generalized right modular complemented algebra will be abbreviated to g.r.m.c. algebra. It is clear that a g.r.m.c. algebra is semi-simple.

Analogously we define a generalized left modular complemented algebra. We shall restrict our attention to g.r.m.c. algebras with the remark that similar properties hold for generalized left modular complemented algebras.

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For any subset E in an algebra A, let $L(E)$ ($R(E)$) denote the left (right) annihilator of E in A. Suppose A is endowed with a topology. Then we call A *an annihilator algebra* provided that, for every closed left ideal M and for every closed right ideal N, we have $R(M) = (0)$ if and only if $M = A$ and $L(N) = (0)$ if and only if $N = A$. If $M = L(R(M))$ and $N = R(L(N))$, then A is called a *dual algebra*.

Let X be a topological space and E a subset of X. Then $\mathrm{cl}(E)$ will denote the closure of E in X.

In this paper, all algebras and linear spaces under consideration are over the complex field C.

3. A Structure theorem. Various forms of the following lemma are probably well known.

LEMMA 3.1. *Let A be a g.r.m.c. algebra with a complementor p and M a closed modular maximal right ideal of A. Then there exists a unique minimal idempotent e of A such that $M = (1-e)A$ and $M^p = eA$.*

Proof. Let f be a left identity for A modulo M. Since $M^p \neq \{0\}$ and since M is a maximal ideal and $M+M^p$ is an ideal, $M+M^p = A$. Therefore we can write $f = m+e$, where $m \in M$ and $e \in M^p$. It is easy to see that e is also a left identity for A modulo M. Let $x \in M^p$. Since $x-ex \in M \cap M^p$, we have $x = ex$. Hence $M^p = eA$ and $e^2 = e$. It follows now that $(1-e)A = M$.

Suppose there is an idempotent e' such that $e'A = M^p$ and $(1-e')A = M$. Then $e' \in L(M) = Ae$. Therefore $e' = e'e$. Since $e-e'e \in M \cap M^p$, we have $e = e'e$. Therefore $e = e'e = e'$ and so e is unique. This completes the proof.

We now establish the following structure theorem for a g.r.m.c. algebra.

THEOREM 3.2. *Let A be a g.r.m.c. algebra which contains no proper two-sided ideal with zero annihilator. Then A is the direct topological sum of its minimal closed two-sided ideals each of which is a simple g.r.m.c. algebra.*

Proof. Let N be a closed maximal modular right ideal of A. By Lemma 3.1, $N = (1-e)A$ for some minimal idempotent e. Hence the socle S of A is defined. Let $x \in R(S)$. Since $e \in S$, we have $x = x-ex \in N$. Since N is arbitrary, it follows from (2.2) that $x = 0$ and so by assumption, S is dense in A. Therefore by ([4], p. 41, Lemma 3.11) A is the topological direct sum of its minimal closed two-sided ideals.

Let I be a minimal closed two-sided ideal of A. We show that I is a g.r.m.c. algebra. Let S_I be the socle of I. By ([4], p. 41, Lemma 3.10), $S_I = S \cap I = SI = IS$. Since S is dense in A, it follows that $S_I \neq (0)$ and so $\mathrm{cl}(S_I) = I$. Let M be a closed maximal modular right ideal of I.

Since $M \not\supset S_I$, by ([4], p. 38, Lemma 3.3), we can write $M = (1-e)I$, where e is a minimal idempotent of I. By the proof of ([4], p. 41, Lemma 3.10), e is a minimal idempotent of A. Hence $N = (1-e)A$ is a closed modular maximal right ideal of A; clearly $M = N \cap I$. We show that $N^p \cap I \neq (0)$, where p is the given complementor on A. Suppose this is not true. By Lemma 3.1 we can write $N^p = fA$ and $N = (1-f)A$ for some minimal idempotent $f \in A$. Since $N^pI \subset N^p \cap I = (0)$, it follows that $fI = (0)$. Hence $I = (1-f)I \subset N$. Therefore $M = N \cap I = I$, a contradiction. This shows that $N^p \cap I \neq (0)$. Since N^p is a minimal right ideal, we have $N^p \subset I$. Now define $M^{p'} = N^p$. It is clear that $M^{p'} \cap M = (0)$. Therefore condition (2.1) is satisfied. Let N be a closed modular maximal right ideal of A and write $N = (1-e)A$ for some minimal idempotent e in A. Suppose $N \cap I \neq (0)$. If $e \notin I$, then $eA \cap I = (0)$ and so $eI = (0)$. Therefore $I = (1-e)I \subset N$. If $e \in I$, then $(1-e)I = N \cap I$. Hence we have either $N \supset I$ or $N \cap I$ is a closed modular maximal right ideal of I. We have shown before that each closed maximal modular right ideal M of I is of the form $M = N \cap I$. It follows now that condition (2.2) holds for I. Hence I is a g.r.m.c. algebra. This completes the proof.

4. Annihilator Banach algebras. In this section, A will be a semi-simple Banach algebra with norm $\|\cdot\|$ which is a dense subalgebra of a semi-simple Banach algebra B with norm $|\cdot|$. Further A and B have the following properties:

(i) *There exists a constant k such that $k\|x\| \geqslant |x|$ for all $x \in A$, i.e., $\|\cdot\|$ majorizes $|\cdot|$.*

(ii) *Every proper closed left (right) ideal in B is the intersection of maximal modular left (right) ideals in B.*

If A is an A^*-algebra and B is the completion of A in an auxiliary norm, then the above conditions automatically hold (see [3]).

Notation. For any subset E of A, $\mathrm{cl}_A(E)$ (resp. $\mathrm{cl}(E)$) will denote the closure of E in A (resp. B).

LEMMA 4.1. *Suppose for each minimal idempotent e of A, we have $Ae = Be$ and $eB = eA$. If A has the dense socle, then A and B have the same socle.*

Proof. Let S be the socle of A. It is clear that S is contained in the socle of B. By assumption, S is a two-sided ideal of B. Let f be a minimal idempotent in B. Since S is dense in A, it follows that S is dense in B. Therefore $fB \cap S \neq (0)$ and so $f \in fB \subset S \subset A$. Thus the socle of B is contained in S. Hence A and B have the same socle.

LEMMA 4.2. *If A has dense socle, then B is a dual algebra.*

Proof. This is proved in [3].

THEOREM 4.3. *Let A and B be given as before. Then the following statements are equivalent*:

(i) *A is an annihilator algebra.*

(ii) *The socle S of A is dense in A and $eA = eB$, $Ae = Be$ for all minimal idempotent e of A.*

Proof. (i) $\Rightarrow$ (ii). This follows immediately from ([2], p. 100, Corollary (2.8.16)) and the proof of Lemma 3.2 in [3].

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Let us assume first that A is simple. Then B is simple. In fact let M be a non-zero closed two-sided ideal of B and let S be the socle of A. By Lemma 4.1, S is the socle of B. Hence M contains a minimal idempotent e of A. Therefore $M \supset AeA$. Since A is simple, $\mathrm{cl}_A(AeA) = A$ and so $M \supset \mathrm{cl}(A) = B$. Hence $M = B$ and therefore B is simple.

Let e be a minimal idempotent of A and let $I = Ae$. Since $Ae = Be$, I is a minimal left ideal of A and B. It is clear that $\|\cdot\|$ and $|\cdot|$ are equivalent on I. By ([2], p. 61, Theorem (2.4.12)), A and B can be considered as operator algebras on I. Since by Lemma 4.2, B is a dual algebra, it follows from ([2], p. 101, Lemma (2.8.20)) that B contains all operators with finite rank on I. By ([2], p. 104, Theorem (2.8.23)), I is reflexive. Since A and B have the same socle S and since S coincides with the set of all operators with finite rank on I, it follows that A contains all operators with finite rank on I. Since $\mathrm{cl}_A(S) = A$ and since I is reflexive, A is an annihilator algebra by ([2], p. 104, Theorem (2.8.23)).

Now suppose that A is not simple. Let e be a minimal idempotent in A and let $J = \mathrm{cl}_A(AeA)$. Then J is a minimal closed two-sided ideal of A and J is simple. Let $K = \mathrm{cl}(J)$. Since B is a dual algebra so is K by ([2], p. 100, Theorem (2.8.14)). Hence J is an annihilator algebra. Since A has dense socle, A is equal to the topological sum of its minimal closed two-sided ideals. Therefore by ([2], p. 106, Theorem (2.8.29)), A is an annihilator algebra. This completes the proof.

COROLLARY 4.4. *A is annihilator algebra if and only if A and B have the same socle S and $\mathrm{cl}_A(S) = A$.*

Proof. This follows from Lemma 4.1 and Theorem 4.3.

THEOREM 4.5. *Let A be an annihilator algebra and p a complementor on A. Then p induces a complementor p' on B.*

Proof. By Corollary 4.4, A and B have the same socle. Let M be a maximal modular right ideal of B. Then we can write $M = (1-e)B$, where e is a minimal idempotent in A. Let $N = (1-e)A$ and so N is a maximal modular right ideal of A. Hence we can write $N = (1-f)A$ and $N^p = fA$, where f is a minimal idempotent of A. Define $M^{p'} = fB$. It is easy to show that p' is a complementor on B and the proof is complete.

We now establish the converse of Theorem 4.5.

THEOREM 4.6. *Let A be an annihilator algebra and let p' be a complementor on B. Then it induces a complementor p on A.*

Proof. Let N be a maximal modular right ideal of A and write $N = (1-e)A$ with e a minimal idempotent in A. Then $M = \mathrm{cl}(N) = (1-e)A$ is a maximal modular right ideal of A. It is clear that $N = M \cap A$. We can write $M = (1-f)B$ and $M^{p'} = fB$, where f is a minimal idempotent in B. By Corollary 4.4, $f \in A$. Let $N^p = fA$. It is easy to see that p is a complementor on A. This completes the proof.

5. Miscellaneous properties of g. m. r. c. algebras. An algebra A is called a *modular annihilator algebra* if, for every maximal modular left ideal I and for every maximal modular right ideal J, we have $R(I) = (0)$ if and only if $I = A$ and $L(J) = (0)$ if and only if $J = A$.

THEOREM 5.1. *Let A be a semi-simple topological $*$-algebra with continuous involution $*$ such that $x*x = 0$ implies $x = 0$. Suppose every maximal modular right ideal of A is closed. Then A is a modular annihilator algebra if and only if it is a g.r.m.c. algebra.*

Proof. Suppose that A is a modular annihilator algebra. Let M be a maximal modular right ideal of A. Define $M^p = L(M)^*$. It is clear that M^p is a closed right ideal of A. If $x \in M^p \cap M$ then $x^* \in L(N)$ and so $x*x = 0$. Hence $x = 0$. Therefore $M^p \cap M = (0)$. Hence A is a g.r.m.c. algebra. The converse of the theorem follows easily from Lemma 3.1.

COROLLARY 5.2. *Let A be a $B*$-algebra. Then A is a dual algebra if and only if it is g.r.m.c.*

Proof. This follows immediately from the above theorem and ([4], p. 42, Theorem 4.1).

THEOREM 5.3. *Let A be a semi-simple commutative topological algebra. Then we have*

(i) *If A is g.r.m.c., then A has a unique complementor.*

(ii) *Suppose every modular maximal ideal M of A is closed.*

Then A is a modular annihilator algebra if and only if it is g.r.m.c.

Proof. (i). Suppose A is g.r.m.c. and p is a given complementor on A. Let M be a closed modular maximal right ideal of A. Since $M^p M \subset M \cap M^p = (0)$, we have $M^p \subset L(M)$. If $x \in L(M) \cap M$, then $xM = (0)$ and $xM^p = (0)$. Hence $xA = (0)$ and so $x = 0$. Therefore $L(M) \cap M = (0)$. Since $L(M) \supset M^p$ and $A = M + M^p$, it follows that $L(M) = M^p$. Therefore p is uniquely determined.

(ii). Suppose A is a modular annihilator algebra. For each maximal modular ideal M of A, let $M^p = L(M)$. It is clear that $M \cap M^p = (0)$. Hence A is g.r.m.c. The converse follows from Lemma 3.1 and this completes the proof.

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MCMASTER UNIVERSITY,
HAMILTON, CANADA.

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Maps which preserve equality of distance

by

ANDREW VOGT (Corvallis, Oreg.)

Abstract. If $f\colon X \to Y$ is a continuous surjection, $f(0) = 0$, and $\|fx - fy\|$ depends functionally on $\|x - y\|$, then f is linear.

A theorem due to Mazur and Ulam ([2], p. 166 and [4]) states that every isometry of a normed real vector space onto a normed real vector space is linear up to translation. Charzyński [3] and Rolewicz [6] have shown, respectively, that surjective isometries of finite-dimensional F-spaces and of locally bounded spaces with concave norm are also linear.

The present paper extends the result of Mazur and Ulam in a different direction. The spaces remain normed real vector spaces, but we replace isometries by the more general notion of equality of distance preserving maps, maps with the property that the distance between image points depends functionally on the distance between domain points.

We prove in Section 1 that every continuous equality of distance preserving map from a normed real vector space onto a normed real vector space is affine-linear. This result generalizes the Mazur–Ulam theorem and yields a characterization of the similarity group of a space which does not presuppose linearity. In Section 2 the continuity hypothesis of Section 1 is shown to be a consequence of surjectivity when the domain has dimension $\geqslant 2$.

Schoenberg [7] and von Neumann and Schoenberg [5] investigated and classified all continuous equality of distance preserving maps from one separable or finite-dimensional Hilbert space into another. Corollary 2.3 in Section 2 shows that their continuity assumption is also redundant when the domain has dimension $\geqslant 2$.

1. Let R_0^+ denote the set of non-negative real numbers. Let X and Y be normed real vector spaces of dimension $\geqslant 1$, the norms in each space being denoted by the symbol $\|\ \|$.

DEFINITION 1.1. A map $f\colon X \to Y$ *preserves equality of distance* iff there exists a function $p\colon R_0^+ \to R_0^+$ such that for each x and y in X $\|fx - fy\| = p(\|x - y\|)$. The function p is called the *gauge function* for f.