STUDIA MATHEMATICA, T. XLV. (1973)

On the individual ergodic theorem for subsequences

by

RYOTARO SATO (Sakado)

Abstract. In this paper it is shown that the Dunford–Schwartz ergodic theorem holds for uniform sequences. The result, obtained below, generalizes and extends a result of Brunel and Keane Z. Wahrscheinlichkeitstheorie verw. Geb. 12 (1969), pp. 231–240.

Let $(\Omega, \mathscr{B}, m)$ be a σ-finite measure space and let T be a positive linear operator from $L^1(\Omega)$ to $L^1(\Omega)$ with $\|T\|_1 \leqslant 1$. We shall say that *the individual ergodic theorem holds for* T if for any uniform sequence $k_1, k_2, \ldots$ (for the definition, see [2]) and for any $f \in L^1(\Omega)$, the limit

$$f^*(\omega) = \lim_n \frac{1}{n} \sum_{i=1}^{n} T^{k_i} f(\omega)$$

exists and is finite almost everywhere. In [2], Brunel and Keane showed that the individual ergodic theorem holds for every measure preserving transformation on a finite measure space. In the present paper we shall generalize and extend this result to one at the operator theoretic level.

THEOREM 1. *If T maps, in addition, $L^p(\Omega)$ into $L^p(\Omega)$ and $\|T\|_p \leqslant 1$ for some p with $1 < p < \infty$ then the individual ergodic theorem holds for T.*

Proof. Let $k_1, k_2, \ldots$ be a uniform sequence, and let $(X, \mathscr{X}, \mu, \varphi)$ and y, Y be the apparatus connected with this sequence. Φ will denote the operator in $L^1(X)$ induced by φ. Taking $(\Omega', \mathscr{B}', m')$ to be the direct product of $(\Omega, \mathscr{B}, m)$ and $(X, \mathscr{X}, \mu)$ and T' the direct product of T and Φ, it follows that T' is a positive linear operator from $L^1(\Omega')$ to $L^1(\Omega')$ and $\|T'\|_1 \leqslant 1$. Since $\|T\|_p \leqslant 1$ by hypothesis, it also follows that T' maps $L^p(\Omega')$ into $L^p(\Omega')$ and $\|T'\|_p \leqslant 1$. Suppose first that $f \in L^1(\Omega) \cap L^p(\Omega)$ and $f \geqslant 0$. As in [2], for any fixed $\varepsilon > 0$, choose open subsets Y', Y'' and W of X such that $Y' \subset Y \subset Y''$, $\mu(Y'' - Y') < \varepsilon$, $y \in W$ and for any $x \in W$ and any $n \geqslant 0$,

$$1_{Y'}(\varphi^n x) \leqslant 1_Y(\varphi^n y) \leqslant 1_{Y''}(\varphi^n x).$$

Define

$$g(\omega, x) = f(\omega)1_Y(x),$$
$$g'(\omega, x) = f(\omega)1_{Y'}(x),$$
$$g''(\omega, x) = f(\omega)1_{Y''}(x).$$

It follows from [1] that

$$\bar{g}'(\omega, x) = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} T'^k g'(\omega, x)$$

and

$$\bar{g}''(\omega, x) = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} T'^k g''(\omega, x)$$

exist and are finite almost everywhere. Clearly $\bar{g}'$ and $\bar{g}''$ belong to $L^p(\Omega')$, and the mean ergodic theorem implies that

$$\lim_n \left\| \frac{1}{n}\sum_{k=0}^{n-1} T'^k g' - \bar{g}'\right\|_p = 0$$

and

$$\lim_n \left\| \frac{1}{n}\sum_{k=0}^{n-1} T'^k g'' - \bar{g}''\right\|_p = 0.$$

Put

$$S(\omega) = \limsup_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f(\omega) 1_Y(\varphi^k y).$$

$$s(\omega) = \liminf_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f(\omega) 1_Y(\varphi^k y).$$

It is clear that

$$\bar{g}'(\omega, x) \leqslant s(\omega) \leqslant S(\omega) \leqslant \bar{g}''(\omega, x)$$

almost everywhere on $\Omega \times W$. Thus for any $\Omega_1 \epsilon \mathscr{B}$ with $m(\Omega)_1 < \infty$ we have

$$\int_{\Omega_1} [S(\omega) - s(\omega)]\, dm = \frac{1}{\mu(W)} \int_{\Omega_1 \times W} [S(\omega) - s(\omega)]\, dm'$$
$$\leqslant \frac{1}{\mu(W)} \int_{\Omega_1 \times W} [\bar{g}'' - \bar{g}']\, dm'$$
$$= \frac{1}{\mu(W)} \lim_n \int_{\Omega_1 \times W} \frac{1}{n}\sum_{k=0}^{n-1} T^k f(\omega) 1_{Y''-Y'}(\varphi^k x)\, dm'$$
$$\leqslant \frac{1}{\mu(W)} \|f\|_1 \int_W \lim_n \frac{1}{n}\sum_{k=0}^{n-1} 1_{Y''-Y'}(\varphi^k x)\, d\mu \leqslant \varepsilon \|f\|_1.$$

Since ε was arbitrary, this demonstrates that $S(\omega) = s(\omega)$ almost everywhere on Ω_1. Since $(\Omega, \mathscr{B}, m)$ is a σ-finite measure space, we conclude that

$$\bar{S}(\omega) = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} T^k f(\omega) 1_Y(\varphi^k y)$$

exists and is finite almost everywhere. Hence we can apply the argument of [2] to infer that

$$f^*(\omega) = \lim_n \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f(\omega)$$

exists and is finite almost everywhere. Next we suppose that $f \epsilon L^1(\Omega)$ and $f \geqslant 0$. It is easily checked by an analogous argument as above that

$$\sup_n \frac{1}{n}\sum_{i=1}^{n} T^{k_i} f(\omega) < \infty$$

almost everywhere. Since $L^1(\Omega) \cap L^p(\Omega)$ is dense in $L^1(\Omega)$, it follows from Banach's theorem (for example, see [3], p. 332) that for any $f \epsilon L^1(\Omega)$,

$$\frac{1}{n}\sum_{i=1}^{n} T^{k_i} f(\omega)$$

converges almost everywhere. The proof is complete.

Corollary. *Let φ be a point transformation from Ω into Ω such that $\varphi^{-1}A \epsilon \mathscr{B}$ if $A \epsilon \mathscr{B}$ and $m(\varphi^{-1}A) = 0$ if $m(A) = 0$. Suppose there exists a constant K such that*

$$0 < \limsup_n \frac{1}{n}\sum_{k=0}^{n-1} m(\varphi^{-k}A) \geqslant Km(A)$$

for every measurable set A of positive measure. Then for any uniform sequence $k_1, k_2, \ldots$ and for any $f \epsilon L^1(\Omega)$, the limit

$$f^*(\omega) = \lim_n \frac{1}{n}\sum_{i=1}^{n} f(\varphi^{k_i}\omega) \tag{1}$$

exists and is finite almost everywhere.

Proof. It follows [5], [7] that there exists a σ-finite measure ν on $(\Omega, \mathscr{B})$ such that

(a) $\nu(A) \leqslant K^2 m(A)$ for all $A \epsilon \mathscr{B}$;
(b) ν is invariant under φ;
(c) $\nu(A) = 0$ if and only if $m(A) = 0$.

Thus Theorem 1 implies that (1) exists and is finite almost everywhere with respect to ν. This together with (c) completes the proof of the corollary.

Remark. Using the above corollary, it may be readily shown that if φ is as in the corollary then for any uniform sequence $k_1, k_2, \ldots$ and for any $f \in L^p(\Omega)$, where $1 \leqslant p < \infty$, (1) exists and is finite almost everywhere. In particular if φ is a measure preserving transformation and if $1 < p < \infty$, then it can also be shown [6] that

$$\lim_n \left\| \frac{1}{n} \sum_{i=1}^{n} f(\varphi^{k_i}\omega) - f^*(\omega) \right\|_p = 0. \tag{2}$$

In case $m(\Omega) < \infty$, (2) is true for $p = 1$ (see [2]).

THEOREM 2. *If there exists a strictly positive function $h \in L^1(\Omega)$ such that the set*

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T^k h \right\}$$

is weakly sequentially compact then the individual ergodic theorem holds for T.

Proof. If we define an integrable function $h'(\omega, x)$ on $\Omega' = \Omega \times X$ by $h'(\omega, x) = h(\omega)$, then the set

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} T'^k h' \right\}$$

is weakly sequentially compact in $L^1(\Omega')$. Therefore a slightly modified argument of [4] shows that for any $f' \in L^1(\Omega')$,

$$\frac{1}{n} \sum_{k=0}^{n-1} T'^k f'(\omega, x)$$

converges almost everywhere and in the norm of $L^1(\Omega')$ to a function in $L^1(\Omega')$. So an analogous argument as in the proof of Theorem 1 is sufficient for the proof, and we omit the details.

References

[1] M. A. Akcoglu and R. V. Chacon, *A convexity theorem for positive operators*, Z. Wahrscheinlichkeitstheorie verw. Geb. 3 (1965), pp. 328–332.

[2] A. Brunel and M. Keane, *Ergodic theorems for operator sequences*, Z. Wahrscheinlichkeitstheorie verw. Geb. 12 (1969), pp. 231–240.

[3] N. Dunford and J. T. Schwartz, *Linear operators* I, New York, 1958.

[4] Y. Ito, *Uniform integrability and the pointwise ergodic theorem*, Proc. Amer. Math. Soc. 16 (1965), pp. 222–227.

[5] C. Ryll-Nardzewski, *On the ergodic theorems. I. (Generalized ergodic theorems)*, Studia Math. 12 (1951), pp. 65–73.

[6] R. Sato, *On a decomposition of transformations in infinite measure spaces*, Pacific J. Math. (to appear).

[7] S. Tsurumi, *On the ergodic theorem*, Tôhoku Math. J. (2) 6 (1954), pp. 53–68.

DEPARTMENT OF MATHEMATICS, JOSAI UNIVERSITY
SAKADO, JAPAN

Received July 28, 1971 (367)