PROBLEM 1. *If E is a Fréchet space with a basis in which all block basic sequences have block extensions, must E be either a Banach space or isomorphic to ω?*

One property of ω which seems to be of importance for the existence of block extensions is the fact that ω has no continuous norms.

A more general question, which in the case of nuclear Fréchet spaces is closely related to the existence of complements for subspaces with bases, is the following:

PROBLEM 2. *If (y^n) is a block basic sequence (or any basic sequence) in a Fréchet space E with a basis, is there any basis for E containing (y^n) as a subsequence?*

Thus, for the example given in Section 3, there may be some less restrictive method for obtaining an extension.

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On the differentiability of Lipschitz mappings in Fréchet spaces

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Abstract. The probem of differentiability of mappings from a subset of a Fréchet space into another Fréchet space satisfying the first order Lipschitz condition is studied. Some extensions of the classical theorem of Rademacher are obtained. Applications of the result to the problem of the topological classification of Fréchet spaces are given.

1. Introduction. A classical theorem of Rademacher [11], [6] states that for every mapping F from the cube C_n in $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ satisfying the first order Lipschitz condition, the differential $(DF)_p$ exists for almost all p in C_n. The aim of this note is to give an extension of this theorem for the case of a mapping F satisfying the first order Lipschitz condition from a subset of a Fréchet space into another Fréchet space.

Some difficulties arise with the definition of the first order Lipschitz condition. (A simple example of a Fréchet space X and two metrics ϱ_1 and ϱ_2 on X can be given such that the identity mapping I from (X, ϱ_1) onto (X, ϱ_2) does not satisfy the first order Lipschitz condition with respect to the metrics ϱ_1 and ϱ_2). This leads us to apply the definition introduced in [9] which states that the mapping F from X into Y satisfies the first order Lipschitz condition if and only if for every continuous pseudonorm on Y there exists a continuous pseudonorm on X such that F induces a mapping between suitable quotient spaces satisfying the first order Lipschitz condition (with respect to the norms).

The other difficulty is the following. There are known examples in which a mapping satisfying the first order Lipschitz condition from the interval $[0,1]$ into a Fréchet space Y does not possess a differential at any point of the interval $[0,1]$. For example the mapping F from the interval $[0,1]$ into $L_1([0,1])$ defined by the formula

$$F(t) = \chi_{[0,t]} \quad \text{for } t \in [0,1],$$

where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0,t]$. (A similar example can be given for the space c_0).

On the other hand Gelfand proved in [7] (see also [6]) that for every mapping F from the interval $[0,1]$ into a separable conjugate Banach space X satisfying the first order Lipschitz condition, the derivative $F'(t)$ exists for almost all t in $[0,1]$ with respect to one-dimensional Lebesgue

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measure. It seems to us to be reasonable to restrict our considerations to the case of mappings satisfying the first order Lipschitz condition into Fréchet spaces which satisfy the following property:

(GF) *For every mapping F from the interval $[0,1]$ into X satisfying the first order Lipschitz condition the derivative $F'(t)$ exists for almost all t in $[0,1]$.*

A Fréchet space satisfying the property (GF) is said to be a *Gelfand–Fréchet space.*

In Section 2 we study some immediate consequences of the definition of Gelfand–Fréchet space. We prove that the class of Gelfand–Fréchet spaces is rather rich (it contains some important well known spaces), however it does not contain any space with a subspace isomorphic to L_1 or c_0. Moreover we prove that the property "being the Gelfand–Fréchet space" is in fact a "separable property".

In Section 3 we consider the existence of the derivatives of mappings from a Hilbert cube into a Gelfand–Fréchet space satisfying the first order Lipschitz condition. We obtain some extensions of the theorem of Gelfand which are useful in Section 4. We prove that in suitable sense, for every such mapping and every direction in the cube the derivative in this direction exists for almost all points in the cube.

Section 4 contains the main results of this note. In this section we study the differentiability of mappings satisfying the first order Lipschitz condition from a Hilbert cube into a Gelfand–Fréchet space in order to derive an extension of the theorem of Rademacher (Theorems 4.4 and 4.5). We prove that if F is a mapping from the Hilbert cube Q into a Gelfand–Fréchet space satisfying the first order Lipschitz condition then the differential $(DF)_p$ exists for almost all p in Q. The corresponding result for mappings from a separable Fréchet space into a Gelfand–Fréchet space is proved too.

In the last section we give some applications of Theorem 4.5 to the problem of the topological classification of Fréchet spaces.

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2. Gelfand–Fréchet spaces. Unless otherwise specified, all vector spaces will be assumed to be vector spaces over the field $\mathbf{R}$ of the reals.

Let X and Y be locally convex spaces and let $A\subset X$.

DEFINITION 2.1. The mapping F from A into Y is said to satisfy the *first order Lipschitz condition* if and only if for every continuous pseudonorm $Q(\cdot)$ on Y there exists a continuous pseudonorm $P(\cdot)$ on X such that for every pair $x_1, x_2\in A$ the following inequality holds

$$Q\big(F(x_1)-F(x_2)\big)\leqslant P(x_1-x_2).$$

Observe that if X and Y are Banach spaces then the definition above coincides with the standard definition (with respect to the metrics). In the following we shall often say "F is a *Lipschitz mapping*" instead of "F satisfies the first order Lipschitz condition".

Let F be a Lipschitz mapping from a subset $A\subset X$ into Y and let $x\in A$, $a\in X$ be such that $x+\lambda a\in A$ for sufficiently small $\lambda\in\mathbf{R}$.

DEFINITION 2.2. The mapping F is said to possess a derivative at the point x and in the direction a iff

$$\lim_{\lambda\to 0}\frac{F(x+\lambda a)-F(x)}{\lambda}$$

exists (in the topology of Y). If it exists we denote this limit by $F'_a(x)$.

Let I denote the interval $[0,1]$. In the following we shall consider I as a subset of a one-dimensional Banach space $(\mathbf{R}, |\cdot|)$.

If F is a Lipschitz mapping from I into Y then it is easy to see that for every $x\in I$ and every $0\neq a\in\mathbf{R}$, the existence of $F'_a(x)$ is equivalent to the existence of $F'_{\bar a}(x)$ for every $0\neq\bar a\in\mathbf{R}$. In this case we say that the mapping F possesses the derivative at the point x and we write $F'(x)$ instead of $F'_1(x)$.

THEOREM 2.3. (Gelfand) *Let F be a Lipschitz mapping from I into a Banach space Y. Then F possesses a derivative $F'(x)$ for almost all x in I (with respect to the one-dimensional Lebesgue measure μ^1) provided that Y is isomorphicaly embeddable in a separable conjugate*(1) *Banach space.*

DEFINITION 2.4. A Fréchet space X is said to be a *Gelfand–Fréchet space* (abbreviation "*a GF-space*") iff for every Lipschitz mapping F from I into X, F possesses a derivative $F'(x)$ for almost all x in I (with respect to the one-dimensional Lebesgue measure).

The following theorem is an easy consequence of Definition 2.4 and Theorem 2.3.

THEOREM 2.5. (i) *every reflexive Banach space is a GF-space,*

(ii) *every closed subspace of a separable conjugate Banach space is a GF-space.*

THEOREM 2.6. (i) *every closed subspace of a GF-space is a GF-space,*

(ii) *a finite or countable Cartesian product of GF-spaces is a GF-space,*

(iii) *a space isomorphic to a GF-space is a GF-space.*

Proof. (i) and (iii) are trivial. In order to prove (ii) it is enough to observe that a mapping F from I into a Cartesian product of Fréchet spaces satisfies the first order Lipschitz condition if and only if for every

(1) A Banach space Y is said to be a conjugate space iff there exists a Banach space X such that $X^*=Y$.

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$n \in N$ the mapping $P_n \circ F$ satisfies the first order Lipschitz condition, where P_n denotes the canonical projection onto the nth component of the product.

COROLLARY 2.7. *Let X be a Fréchet space. If the topology on X can be defined by a sequence of pseudonorms $\{P_n\}_{n \in N}$ such that the completion $\tilde{X}_n$ of a quotient space $X_n = X/P_n$ is a GF-space for $n = 1, 2, \ldots$ then X is a GF-space.*

Indeed, in this case we have that X is isomorphic to a closed subspace of the Cartesian product $Y = \prod_{n=1}^{\infty} \tilde{X}_n$ and Corollary 2.7 becomes an easy consequence of Theorem 2.6.

The following theorem shows that in fact the property "being a GF-space" is a "separable" property.

THEOREM 2.8. *A Fréchet space X is a GF-space if and only if every closed separable subspace of X is a GF-space.*

Proof. Since by Theorem 2.6 every closed subspace of a GF-space is a GF-space it is sufficient to prove that if every separable closed subspace of a Fréchet space X is a GF-space then X is a GF-space. Let F be a Lipschitz mapping from I into X. The space $X_0 = \overline{\operatorname{span} F(I)}$ is a separable closed subspace of X, hence it is a GF-space. It follows from the definition of GF-spaces that the derivative $F'(x)$ exists for almost all x in I, if we consider F as a Lipschitz mapping from I into X_0. This implies that the derivative $F'(x)$ exists for almost all x in I, if we consider F as a Lipschitz mapping from I into X, which concludes the proof of the theorem.

THEOREM 2.9. *Every Montel-Fréchet space is a GF-space.*

Proof. Let X be a Montel-Fréchet space and let $\{\xi_n\}_{n \in N}$ be a fundamental sequence of continuous linear functionals on X. Let F be a Lipschitz mapping from I into X. Consider the mappings $f_n = \xi_n \circ F$ for $n = 1, 2, \ldots$ from I into R. Obviously for every $n \in N$ the mapping f_n satisfies the first order Lipschitz condition. Hence by a well known theorem from the theory of real functions we have that the measure of the set

$$M = \{x \in I : f_n'(x) \text{ exists for } n = 1, 2, \ldots\}$$

is equal to 1. This implies that for every $x \in M$ the differential quotient

$$g_\lambda(x) = \frac{F(x+\lambda) - F(x)}{\lambda}$$

is weakly convergent as $\lambda \to 0$ with respect to the fundamental system of functionals. Since for the Montel–Fréchet space such a convergence of a bounded sequence is equivalent to the strong convergence we conclude that the derivative $F'(x)$ exists for every $x \in M$ and the theorem is proved.

3. The existence of the derivatives. We start with two facts which are immediate consequences of the Fubini Theorem.

LEMMA 3.1. *Let (X_1, μ_1) and (X_2, μ_2) be two measure spaces with $\mu_1(X_1) < \infty$ and $\mu_2(X_2) < \infty$ and let S be a measurable subset of the product measure space $(X_1 \times X_2, \mu_1 \times \mu_2)$ such that for almost all $x_2 \in X_2$ we have*

$$\mu_1(\{x_1 \in X_1 : (x_1, x_2) \in S\}) = 0.$$

Then $(\mu_1 \times \mu_2)(S) = 0$.

For every $n \in N$ denote by μ^n the n-dimensional Lebesgue measure on $\boldsymbol{R}^n$. Then we have the following corollary.

COROLLARY 3.2. *Let $0 \neq a \in \boldsymbol{R}^n$ and S be a measurable subset of $\boldsymbol{R}^n$ with the property that for every $x \in S$ the one-dimensional measure of the set*

$$S \cap \{p \in \boldsymbol{R}^n : p = x + \lambda a,\ \lambda \in \boldsymbol{R}\}$$

is equal to 0. Then $\mu^n(S) = 0$.

LEMMA 3.3. *Let A be an open subset of $\boldsymbol{R}^n$ and let F be a Lipschitz mapping from A into a GF-space X. Then for every $a \in \boldsymbol{R}^n$ the measure of the set*

$$S = \{x \in A : F_a'(x) \text{ does not exist}\}$$

is equal to 0.

Proof. Let $\{P_n\}_{n \in N}$ be a fundamental system of pseudonorms on X and let

$$\|x\| = \sum_{i=1}^{\infty} 2^{-i} P_i(x) / (1 + P_i(x))$$

for $x \in X$. For $\lambda, \lambda' \in \boldsymbol{R} \setminus \{0\}$, $p \in A$ put

$$f_{\lambda,\lambda'}(p) = \begin{cases} \left\| \dfrac{F(p+\lambda a) - F(p)}{\lambda} - \dfrac{F(p+\lambda' a) - F(p)}{\lambda'} \right\| & \text{if it is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for every pair $\lambda, \lambda' \in \boldsymbol{R} \setminus \{0\}$ the function $f_{\lambda,\lambda'}$ is lower semicontinuous in A. Hence the functions defined by the identity

$$g_n(p) = \sup \left\{ f_{\lambda,\lambda'}(p) : 0 < |\lambda|, |\lambda'| < \frac{1}{n} \right\}$$

are semicontinuous as well for $n = 1, 2, \ldots$ Since for every $p \in A$ the sequence $\{g_n(p)\}_{n \in N}$ is monotone decreasing we infer that the function

$$g(p) = \lim_{n \to \infty} g_n(p)$$

is measurable. Observe that

$$S = \{p \in A : g(p) > 0\}.$$

This implies that the set S is measurable. For every $x \in S$ consider the mapping F restricted to the set

$$I_x = \{p \in \boldsymbol{R}^n\colon\ p = x + \lambda a,\ \lambda \in \boldsymbol{R}\} \cap A.$$

It is easily seen that for every $x \in S$ the mapping F/I_x satisfies the first order Lipschitz conditions as a mapping from the set I_x into X. On the other hand observe that for every $x \in S$, I_x is a union of an at most countable family of open intervals. It follows from the definition of GF-space that the assumptions of the Corollary 3.2 are satisfied. Hence $\mu^n(S) = 0$ and the lemma is proved.

Let Q be a Hilbert cube, $Q = \prod_{i=1}^{\infty} I_i$, where $I_i = [-2^{-i}, 2^{-i}]$ for $i \in N$, with standard metric ϱ_Q, defined for $p_1, p_2 \in Q$, $p_1 = (p_{1,1}, p_{2,1}, \ldots)$, $p_2 = (p_{1,2}, p_{2,2}, \ldots)$ by the formula

$$\varrho_Q(p_1, p_2) = \sum_{i=1}^{\infty} |p_{i,1} - p_{i,2}|.$$

Therefore we can consider Q as a subset of l_1. Let us define the product measure $\bar{\mu} = \bigtimes_{i=1}^{\infty} \bar{\mu}_i$ on Q, where for every $i \in N$ the measure $\bar{\mu}_i$ is normalized Lebesgue measure on I_i. From the definition of $\bar{\mu}_i$ we have $\bar{\mu}_i(I_i) = 1$ for $i \in N$, so $\bar{\mu}(Q) = 1$. It is easy to see that $\bar{\mu}$ is a Radon measure on Q. Let μ be a completion of the measure $\bar{\mu}$.

Theorem 3.4. *Let F be a Lipschitz mapping from the Hilbert cube Q into a GF-space X, and let $a = (a_1, a_2, \ldots, a_k, 0, 0, \ldots) \in Q$. Then the derivative $F'_a(p)$ of F in the direction a exists for μ-almost all p in Q.*

Proof. Let $D = \{p \in Q\colon F'_a(p) \text{ exists}\}$. It is obvious that

$$D = \{p \in Q\colon\ g(p) = 0\},$$

where $g(p)$ for $p \in Q$ is a measurable real valued function defined in the same manner as in the proof of the previous lemma. Hence we conclude that the set D is measurable.

Now we consider Q as the Cartesian product of a k-dimensional cube $C_k = \prod_{i=1}^{k} I_i$ and a Hilbert cube $Q_k = \prod_{i=k+1}^{\infty} I_i$. In the following we shall consider C_k as a subset of $\boldsymbol{R}^k$. In this interpretation the measure μ becomes the product of the measures $\mu_k = \bigtimes_{i=1}^{k} \bar{\mu}_i$ and $\underline{\mu_k} = \bigtimes_{i=k+1}^{\infty} \bar{\mu}_i$. Moreover we have that μ_k is absolutely continuous with respect to the k-dimensional Lebesgue measure μ^k. Therefore we can consider F as a mapping from the Cartesian product $C_k \times Q_k$ into X putting for $p \in Q$ $p = (p', p'')$, where $p' \in C_k$ and $p'' \in Q_k$. Since, according to the assumption, $a = (a', 0)$ where $a' \in C_k$ and $0 = (0, 0, \ldots) \in Q_k$, we infer that the derivative $F'_a(p)$ exists at the point $p = (p', p'')$ if and only if the mapping $F_{p''}(p') = F((p', p''))$ from C_k into X possesses the derivative in the direction $a' \in C_k$ and at the point $p' \in C_k$. It is easy to see that for every fixed $p'' \in Q_k$ the mapping $F_{p''}$ defined above considered as a mapping from the k-dimensional cube $C_k \subset \boldsymbol{R}^k$ into a GF-space X satisfies the first order Lipschitz condition. For every $p'' \in Q_k$ put

$$D_{p''} = \{p' \in C_k\colon\ (p', p'') \in D\}.$$

According to the previous lemma we have that for every fixed $p'' \in Q_k$

$$\mu_k(D_{p''}) = \mu_k(\{p' \in C_k\colon\ (F_{p''})'_{a'}(p') \text{ exists}\}) = 1.$$

Let $S = Q \setminus D$. The last equality means that the assumptions of Lemma 3.1 are satisfied and we infer that the derivative $F'_a(p)$ exists for μ-almost all p in Q and therefore the theorem is proved.

4. The differentiability of Lipschitz mappings. For every $\varepsilon > 0$ denote by Q_ε the Hilbert cube, $Q_\varepsilon = (1+\varepsilon)Q \subset l_1$ and put $\tilde{Q} = \operatorname{span} Q$. Observe that for every $p \in Q$ and every direction $a \in \tilde{Q}$ there exists $\delta > 0$ such that $p + \lambda a \in Q_\varepsilon$ for $|\lambda| < \delta$.

Definition 4.1. A Lipschitz mapping F from the Hilbert cube Q_ε into a locally convex space X is differentiable at the point $p \in Q$ if and only if for every direction $a \in \tilde{Q}$ the derivative $F'_a(p)$ exists and the mapping $(DF)_p$ from $\tilde{Q}$ into X defined by the formula

$$(DF)_p(a) = F'_a(p) \quad \text{for } a \in \tilde{Q}$$

is linear. The mapping $(DF)_p$ is said to be the *differential* of F at the point $p \in Q$.

It is easily seen that if the differential $(DF)_p$ exists for some $p \in Q$ then it is a continuous linear mapping.

Lemma 4.2. *Let f be a real valued function from the Hilbert cube Q_ε satisfying the first order Lipschitz condition. Then the differential $(Df)_p$ exists for μ-almost all $p \in Q$.*

Proof. Since the space of real numbers considered as a one-dimensional Banach space $(\boldsymbol{R}, |\cdot|)$ is a GF-space we have that for every $a = (a_1, a_2, \ldots, a_k, 0, 0, \ldots) \in \tilde{Q}$ the set of p in Q such that $f'_a(p)$ exists has measure equal to 1 (Theorem 3.4). Denote by $\boldsymbol{Q}$ the set of rational numbers and put

$$W = \{a = (a_1, a_2, \ldots) \in \tilde{Q}\colon\ a_i \in \boldsymbol{Q} \text{ for } i \in N \text{ and only a finite number of } a_i \text{ is different from } 0\}.$$

Let

$$D = \bigcap_{a \in W} \{p \in Q\colon\ f'_a(p) \text{ exists}\}.$$

Obviously D is measurable and has measure equal to 1. Observe that for every $a \in W, p \in D$ we have

$$f'_a(p) = \lim_{n\to\infty} g_n(p), \quad \text{where } g_n(p) = n\left(f\left(p+\frac{a}{n}\right)-f(p)\right)$$

for sufficiently large n (such that $\left(Q+\frac{a}{n}\right) \subset Q_s$). Hence for every $a \in W$ the function f'_a is measurable. For every $a, b \in W$ let $M(a, b)$ denote the set

$$\{p \in D\colon f'_a(p)+f'_b(p) = f'_{a+b}(p)\}.$$

We shall prove that for every $a, b \in W$ such that $a = (a_1, a_2, \ldots)$ and $b = (\beta_1, \beta_2, \ldots)$ we have $\mu(M(a, b)) = 1$. Indeed, let k be a positive integer such that $a_i = \beta_i = 0$ for $i > k$. Now (in the notation of the proof of Theorem 3.4) for every $p'' \in Q_k$ we consider the function $f_{p''}(p') = f((p', p''))$ for $p' \in C_k$. Since for every $p'' \in Q_k$ the function $f_{p''}$ satisfies the first order Lipschitz condition as a function from $C_k \subset \boldsymbol{R}^k$ into $\boldsymbol{R}$ then by a well known theorem of Rademacher [11] we obtain that the measure μ_k of the set $p' \in C_k$ such that $(Df_{p''})_{p'}$ exists is equal to 1. Denote this set by $M(p'')$. Obviously we have

$$M(p'') \subset \{p' \in C_k\colon (p', p'') \in M(a, b)\}.$$

Put $S = Q \setminus M(a, b)$. It is easy to see that the assumptions of Lemma 3.1 are satisfied (for the measure spaces (C_k, μ_k) and (Q_k, μ_k)). Hence $M(a, b)$ has the measure equal to 1 for every $a, b \in W$. Since the set W is countable, the measure μ of the set

$$M = \bigcap_{a,b \in W} M(a, b)$$

is equal to 1. We shall prove that for every $p \in M$ the differential $(Df)_p$ exists. Let $a = (r_1, r_2, \ldots) \in \tilde{Q}$, where $r_i \in \boldsymbol{R}$ for $i = 1, 2, \ldots$, be an arbitrary direction in $\tilde{Q}$. It is enough to prove that for every $p \in M$ the derivative $f'_a(p)$ exists and that

$$f'_a(p) = \sum_{i=1}^{\infty} r_i f'_{e_i}(p),$$

where $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ (the unit on the ith place) for $i \in N$. Observe that if K is a Lipschitz constant(²) for the function f, then

$$|f'_{e_i}(p)| \leqslant K \quad \text{for } p \in M,\ i = 1, 2, \ldots$$

(²) Which means that $|f(p)-f(q)| \leqslant K\|p-q\|$ for every $p, q \in Q_s$.

Fix $p \in M$ and $a \in \tilde{Q}$. Given $\varepsilon' > 0$. Let $b \in W$, $b = (w_1, w_2, \ldots)$ where $w_i \in \boldsymbol{Q}$ and $w_i = 0$ for $i > k$ be a direction in $\tilde{Q}$ such that $\|a-b\| < \varepsilon'$. It follows from the first part of this proof that there exists $\delta > 0$ such that for every $0 \neq |\lambda| < \delta$

$$\left|\frac{f(p+\lambda b)-f(p)}{\lambda} - \sum_{i=1}^{k} w_i f'_{e_i}(p)\right| < \varepsilon'.$$

Hence for $0 \neq |\lambda| < \delta$ we have

$$\left|\frac{f(p+\lambda a)-f(p)}{\lambda} - \sum_{i=1}^{\infty} r_i f'_{e_i}(p)\right| \leqslant \left|\frac{f(p+\lambda a)-f(p+\lambda b)}{\lambda}\right| + \left|\frac{f(p+\lambda b)-f(p)}{\lambda} - \sum_{i=1}^{k} w_i f'_{e_i}(p)\right| + \left|\sum_{i=1}^{\infty}(w_i - r_i) f'_{e_i}(p)\right|$$
$$\leqslant \frac{K\|p+\lambda a-(p+\lambda b)\|}{|\lambda|} + \varepsilon' + \sum_{i=1}^{\infty}|w_i - r_i| \sup\{|f'_{e_i}(p)|\colon i \in N\} \leqslant K\varepsilon' + \varepsilon' + K\varepsilon' = (2K+1)\varepsilon',$$

where K is a Lipschitz constant for the function f, which concludes the proof of the lemma.

Lemma 4.3. *Let X be a separable Fréchet space. Then there exists a sequence of continuous linear functionals $\{\xi_n\}_{n \in N}$ such that for every $x, y \in X$ the equality $\xi_n(x) = \xi_n(y)$ holds for $n = 1, 2, \ldots$ if and only if $x = y$.*

Proof. Since every separable Fréchet space can be isomorphically embedded in the space $C(R)$ of all continuous real valued functions defined on R, it is enough to observe that such a sequence of continuous linear functionals on $C(R)$ exists (for example the evaluations at the rational numbers).

Theorem 4.4. *Let F be a Lipschitz mapping from the Hilbert cube Q_s into a GF-space X. Then the differential $(DF)_p$ exists for μ-almost all p in the Hilbert cube Q.*

Proof. Since the Hilbert cube Q_s is separable we can assume that X is a separable GF-space. If it is not the case we restrict our consideration to $\overline{\operatorname{span} F(Q_s)}$ which is separable and by Theorem 2.6 (i) is a GF-space. Let $\{\xi_n\}_{n \in N}$ be a sequence of continuous linear functionals on X, existence of which was stated in the lemma above. Consider a sequence of real valued functions $\{f_n\}_{n \in N}$ on Q_s defined by the formula $f_n = \xi_n \circ F$, for $n = 1, 2, \ldots$ It is trivial that for every $n \in N$ the function f_n satisfies the first order Lipschitz condition. Let

$$M = \{p \in Q\colon (Df_n)_p \text{ exists for every } n \in N\}.$$

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By Lemma 4.2 we have that $\mu(M) = 1$. Consider the set

$$\tilde{M} = \{p \in Q\colon F'_a(p) \text{ exists for every } a \in W\}.$$

It follows from Theorem 3.4 that $\mu(\tilde{M}) = 1$. Hence $\mu(M \cap \tilde{M}) = 1$. We shall prove that for every $p \in M_0 = M \cap \tilde{M}$ and for every $a, b \in W$

$$F'_a(p) + F'_b(p) = F'_{a+b}(p).$$

Indeed, it suffices to prove that for every $n \in N$, $a, b \in W$ and $p \in M_0$

$$\xi_n(F'_a(p)) + \xi_n(F'_b(p)) = \xi_n(F'_{a+b}(p)).$$

But this is an immediate consequence of the following easily verifiable identity

$$\xi_n(F'_c(p)) = \lim_{\lambda \to 0} \frac{\xi_n(F(p+\lambda c)) - \xi_n(F(p))}{\lambda} = (Df_n)_p(c)$$

for $n = 1, 2, \ldots, p \in M_0$ and $c \in W$.

Using a similar argument as in the second part of the proof of Lemma 4.2 we can show the existence of the differential $(DF)_p$ for every $p \in M_0$, which concludes the proof of the theorem.

THEOREM 4.5. *Let F be a Lipschitz mapping from a separable Fréchet space X into a GF-space Y. Then the set M of $x \in X$ such that the differential $(DF)_x$ exists is dense in X.*

Proof. Without any loss of generality it is sufficient to prove that the origin belongs to the closure of M. The general case when x_0 is an arbitrary point in X can be reduced to the previous one by the substitution $\tilde{F}(x) = F(x - x_0)$.

Let $\{a_n\}_{n \in N}$ be an arbitrary bounded, linearly independent and linearly dense sequence in X. Fix an $\varepsilon > 0$ and define the mapping F_0 from Q_ε into X by the following formula

$$F_0(p) = F_0((\alpha_1, \alpha_2, \ldots)) = \sum_{i=1}^{\infty} \alpha_i a_i \quad \text{for } p = (\alpha_1, \alpha_2, \ldots) \in Q_\varepsilon.$$

It is easy to see that the mapping F_0 satisfies the first order Lipschitz condition. Indeed, let $P(\cdot)$ be an arbitrary continuous pseudonorm on X and let $p_1, p_2 \in Q_\varepsilon$, $p_1 = (\alpha_1, \alpha_2, \ldots)$, $p_2 = (\beta_1, \beta_2, \ldots)$. Then

$$P(F_0(p_1) - F_0(p_2)) = P\Big(\sum_{i=1}^{\infty} (\alpha_i - \beta_i) a_i\Big) \leqslant \sum_{i=1}^{\infty} |\alpha_i - \beta_i| P(a_i)$$
$$\leqslant \sup\{P(a_i)\colon i \in N\}\, \varrho(p_1, p_2) = K\|p_1 - p_2\|.$$

Since the composition of Lipschitz mappings satisfies the first order Lipschitz condition then $\tilde{F} = F_0 \circ F$ is a Lipschitz mapping from the Hilbert cube Q_ε into a GF-space Y. Hence the differential $(D\tilde{F})_p$ exists

for μ-almost all p in Q (Theorem 4.4). Assume that for some $p_0 \in Q$ the differential $(D\tilde{F})_{p_0}$ exists. We shall show that this implies that the differential $(DF)_{F_0(p_0)}$ exists. Denote $F_0(p_0)$ by x_0. It follows from the existence of the differential $(D\tilde{F})_{p_0}$ that for every $a \in \tilde{Q}$, $a = (\alpha_1, \alpha_2, \ldots)$ the limit

$$\lim_{\lambda \to 0} \frac{\tilde{F}(p_0 + \lambda a) - \tilde{F}(p_0)}{\lambda} = \tilde{F}'_a(p_0)$$

exists and moreover

$$(*) \qquad \tilde{F}'_a(p_0) = \sum_{i=1}^{\infty} \alpha_i \tilde{F}'_{e_i}(p_0).$$

Observe that by the definition of the mapping $\tilde{F}$ we have that for $a \in Q$, $a = (\alpha_1, \alpha_2, \ldots)$

$$\frac{\tilde{F}(p_0 + \lambda a) - \tilde{F}(p_0)}{\lambda} = \frac{F(F_0(p_0 + \lambda a)) - F(F_0(p_0))}{\lambda} = $$
$$= \frac{F\big(x_0 + \sum_{i=1}^{\infty} \alpha_i a_i\big) - F(x_0)}{\lambda}.$$

Hence we conclude that the existence of the derivative $\tilde{F}'_a(p_0)$ implies the existence of the derivative $F'_{F_0(a)}(x_0)$ and $\tilde{F}'_a(p_0) = F'_{F_0(a)}(x_0)$. It follows from the differentiability of $\tilde{F}$ at the point p_0 that for every $a \in \operatorname{span} Q$ the derivative $F'_{F_0(a)}(x_0)$ exists. The linearity of F_0 and $(*)$ imply that for every $a_1, a_2 \in \operatorname{span} Q$ we have

$$F'_{F_0(a_1)}(x_0) + F'_{F_0(a_2)}(x_0) = F'_{F_0(a_1) + F_0(a_2)}(x_0).$$

Since for every $a \in Q$, $\lambda \in R$ we have $F'_{\lambda F_0(a)}(x_0) = \lambda F'_{F_0(a)}(x_0)$ then the mapping $\tilde{F}_{x_0}$ from $X_0 = \operatorname{span}\{a_i\}_{i \in N}$ into Y defined by the formula

$$\tilde{F}_{x_0}(x) = F'_x(x_0) \quad \text{for } x \in X_0$$

is defined and linear. It can easily be proved ([9], Section 3) that the mapping $\tilde{F}_{x_0}$ satisfies the first order Lipschitz condition with the same set of constants as F and therefore is continuous. Since X_0 is a dense subspace of X the mapping $\tilde{F}_{x_0}$ can be uniquely extended by continuity to the continuous linear mapping F_{x_0} from X into Y. We shall show that $(DF)_{x_0}$ exists and moreover $(DF)_{x_0} = F_{x_0}$. It is sufficient to show that for every $a \in X$, $F'_a(x_0) = F_{x_0}(a)$.

Fix $\varepsilon' > 0$ and $a \in X$. Let $P(\cdot)$ be an arbitrary continuous pseudonorm on Y. Since F satisfies the first order Lipschitz condition there exists a continuous pseudonorm $Q(\cdot)$ on X such that

$$P(F(x_1) - F(x_2)) \leqslant Q(x_1 - x_2) \quad \text{for every } x_1, x_2 \in X.$$

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Since X_0 is dense in X there exists $a_1 \in X_0$ such that $Q(a_1 - a) < \varepsilon'$. Finally, since $F'_{a_1}(x_0)$ exists we infer that there exists $\delta > 0$ such that for $0 \neq |\lambda| < \delta$

$$P\left(\frac{F(x_0) + \lambda a_1 - F(x_0)}{\lambda} - F'_{a1}(x_0)\right) < \varepsilon'.$$

Then we have

$$P\left(\frac{F(x_0+\lambda a) - F(x_0)}{\lambda} - F_{x_0}(a)\right)$$
$$\leqslant P\left(\frac{F(x_0+\lambda a) - F(x_0+\lambda a_1)}{\lambda}\right) + P\left(\frac{F(x_0+\lambda a_1) - F(x_0)}{\lambda} - F'_{a_1}(x_0)\right) +$$
$$+ P\big(F'_{a_1}(x_0) - F_{x_0}(a)\big) \leqslant \frac{Q\big(\lambda(a-a_1)\big)}{|\lambda|} + \varepsilon' + P\big(F_{x_0}(a_1) - F_{x_0}(a)\big)$$
$$\leqslant \varepsilon' + \varepsilon' + \varepsilon' = 3\varepsilon'$$

for $|\lambda| < \delta$ which concludes the proof of the existence of the differential $(DF)_{x_0}$.

In order to prove that the closure of M contains the origin it is sufficient to observe that the set M_0 of p in Q such that the differential $(D\tilde{F})_p$ exists has measure μ equal to 1 and therefore is dense in Q. Let $\{p_n\}_{n\in N}$ be a sequence of points in M_0 tending to $\mathbf{0} = (0, 0, \ldots)$. It was proved above that if this is the case then $(DF)_{F_0(p_n)}$ exists for $n = 1, 2, \ldots$ Hence $F_0(p_n) \in M$ for $n \in N$. It follows from the definition of the mapping F_0 that the sequence $\{F_0(p_n)\}_{n\in N}$ is convergent to the origin which completes the proof of the theorem.

Observe that in the same manner we can prove the following

THEOREM 4.5′. *Let F be a Lipschitz mapping from an open nonempty subset U of a separable Fréchet space X into a GF-space Y. Then the set M of $x \in U$ such that the differential $(DF)_x$ exists is dense in U.*

Let us note that Theorems 3.4 and 4.4 remain valid if we replace the measures $\bar{\mu}_k$ by arbitrary normalized measures absolutely continuous with respect to Lebesgue measure.

Remark. Let F be a Lipschitz mapping from Q into a GF-space X. Since there exists a Lipschitz retraction of l_1 onto the Hilbert cube Q there always exist extensions of F to a Lipschitz mapping from l_1 into X. However, it follows from the proof of Theorem 4.4 that the differentiability of the extended mapping at the point $p \in Q$ does not depend on the extension (in particular it does not depend on the way we extend F to the Lipschitz mapping defined on Q_ε). In fact we have proved that for $p \in Q$, $(DF)_p$ exists if and only if for every a in W the derivative $(F/Q)'_a(p)$ exists and moreover the eqality

$$(F/Q)'_a(p) + (F/Q)'_b(p) = (F/Q)'_{a+b}(p)$$

holds for every $a, b \in W$.

5. Applications. In this section we give some applications of Theorem 4.5 to the problem of the topological classification of Fréchet spaces [1]. Let us remark that the similar arguments to those we use in this section were presented by Lindenstrauss in [8] and Enflo [5].

DEFINITION 5.1. A mapping F from a subset A of a locally convex space X into another locally convex space Y is said to be a *Lipschitz embedding of A into Y* iff F is a one-to-one mapping and both F and F^{-1} satify the first order Lipschitz condition. A locally convex space X is said to be *Lipschitz embeddable in a locally convex space Y* iff there exists a Lipschitz embedding of X into Y.

It is obvious that if a complete locally convex space X is Lipschitz embeddable in a Fréchet space then X is a Fréchet space.

THEOREM 5.2. *If a Fréchet space X is Lipschitz embeddable in a GF-space Y then X is a GF-space.*

Proof. By Theorem 2.8 it is enough to prove that every separable closed subspace of X is a GF-space. Let X_0 be a closed separable subspace of X and let F be a Lipschitz embedding of X into Y. Consider F restricted to X_0. By Theorem 4.5 there exists $x_0 \in X_0$ such that the mapping F restricted to X_0 is differentiable at the point x_0. It can be easily shown that the mapping $(DF/X_0)_{x_0}$ from X_0 into Y is a Lipschitz embedding of a Fréchet space X_0 into Y (see e.g. [9], Section 3). Since the mapping $(DF/X_0)_{x_0}$ is linear we infer that $(DF/X_0)_{x_0}$ is an isomorphic embedding of X_0 into Y. Hence by Theorem 2.6 (i) and (iii) we have that X_0 is a GF-space which concludes the proof of the theorem.

It follows from Theorem 2.5 (i) and Corollary 2.7 that every pre-Hilbertian Fréchet space is a GF-space. Using the same argument as before we can prove the following theorem.

THEOREM 5.3. *If a separable Fréchet space X is Lipschitz embeddable in a pre-Hilbertian Fréchet space Y then X is a pre-Hilbertian Fréchet space.*

Observe that a space X Lipschitz embeddable into a separable Fréchet space is a separable Fréchet space. Since every nuclear Fréchet space is a separable pre-Hilbertian Fréchet space [10] and a space X isomorphically embeddable in a nuclear space is nuclear we obtain in a similar manner as the previous theorem the following result.

THEOREM 5.4. *If a Fréchet space X is Lipschitz embeddable in a nuclear Fréchet space then X is a nuclear Fréchet space.*

In [9] it is shown (Lemma 4) that if F is a Lipschitz embedding of a Fréchet space X onto a Montel–Fréchet space Y (such an embedding is said to be a Lipschitz homeomorphism) and if the differential $(DF)_x$ exists for some $x \in X$ then the differential $(DF)_x$ is a (linear) isomorphism of X onto Y. This and Theorem 2.9 imply

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THEOREM 5.5. *If a Fréchet space X is Lipschitz homeomorphic to a Montel–Fréchet space Y then X is isomorphic to Y.*

It can be proved ([9], Section 5) that if a Fréchet space X is uniformly homeomorphic to a Montel–Fréchet space Y then X is Lipschitz embeddable in Y. Combining this result with Theorem 5.4 we obtain

THEOREM 5.6. (i) *If a Fréchet space X is uniformly homeomorphic to a nuclear space then X is a nuclear space,*

(ii) *if a Fréchet space X is uniformly homeomorphic to a Montel–Fréchet space then X is a Montel–Fréchet space.*

We conclude with the following result.

THEOREM 5.7. *If a Fréchet space X is uniformly homeomorphic to the space s of all sequences then X is isomorphic to s.*

Proof. s is a Montel–Fréchet space. Hence X is Lipschitz embeddable in s. Let F be a Lipschitz embedding of X into s. Since s is a separable space by Theorem 4.5 there exists $x \in X$ such that F is differentiable at the point x. This implies that $(DF)_x$ is an isomorphic embedding of X into s (linear). It was shown in [2] that every closed infinite dimensional subspace of s is isomorphic to s which concludes the proof of the theorem.

For more details and other applications of Theorem 4.5 the reader is refered to [9].

Remark. Let X be a Fréchet space over the field $\boldsymbol{C}$ of the complex numbers. It is easily seen that X is a pre-Hilbertian (nuclear, Montel) Fréchet space if and only if X considered as a vector space over the field $\boldsymbol{R}$ of the real numbers is a pre-Hilbertian (resp. nuclear, Montel) space. Observe that this implies that Theorems 5.2, 5.3, 5.4, 5.6 and 5.7 remain valid if X and Y (or only one of them) is a complex vector space.

QUESTION. *Does Theorem 5.5 hold provided that X and Y are vector spaces over the field $\boldsymbol{C}$ of the complex numbers?*

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