STUDIA MATHEMATICA, T. XLV. (1973)

Extensions of basic sequences in Fréchet spaces*

by

WILLIAM B. ROBINSON (Potsdam, N. Y.)

Abstract. Criteria are established for extending a block basic sequence to a basis in a Fréchet space. An example is constructed in a nuclear Fréchet space E of a block basic sequence which has no block extension to a basis for E.

In [15], M. Zippin proved that every block basic sequence relative to a basis for a Banach space has a block extension, and applied this result to reflexivity of Banach spaces with bases. The purpose of the present paper is to show that this result cannot be extended to the case of arbitrary nuclear Fréchet spaces. One expects this situation since in nuclear spaces every basis is unconditional ([9]). On the other hand, in a Banach space with an unconditional basis it is possible that a block basic sequence has no extension to an unconditional basis [13]. Another reason explaining the difference between the case of Banach spaces and nuclear spaces is provided by theorems of Dragilev type (cf. [10] and [2]) on the equivalence of bases in some nuclear spaces. Roughly speaking, these results show that there are not many bases in such spaces.

The key to the example is Theorem 3.3, which implies that if certain block basic sequences have extensions they must have trivial extensions. The example is obtained in an echelon space of order 1. On the other hand, it is shown in Theorem 4.1 that every block basic sequence in the space ω has a block extension.

For general terminology we shall follow Köthe [8]. If $\langle E, F\rangle$ is a dual system, then we shall denote the weak topology induced by F on E by $\mathscr{I}_s(F, E)$, and the strong topology induced by F on E by $\mathscr{I}_b(F, E)$. By a *Fréchet space* we mean a complete, metrizable, locally convex space.

We shall denote by N the collection of positive integers, and the phrase "for all (each) n" will mean "for all (each) $n \in N$". Sequences indexed by N will be written as (x^n), and any other indexing set will be explicitly mentioned. By a *total sequence* in a locally convex space E we mean a sequence (x^n) whose linear span is dense in E.

---

* This paper is taken from the author's dissertation at Tulane University, written under the direction of Professor Ed Dubinsky.

The author wishes to acknowledge several helpful suggestions from the referee.

1. Basic definitions and theorems. Let E be a Fréchet space with basis (x^n) and let $(p_n)_{n=0}^{\infty}$ be an increasing sequence of integers such that $p_0 = 0$. A sequence (y^n) of the form

$$y^n = \sum_{i=p_{n-1}+1}^{p_n} c_i x^i;$$

where $y^n \neq 0$ for all n, is called a *block basic sequence* (relative to (x^n)). The finite dimensional space $E^n = [x^{p_{n-1}+1}, \ldots, x^{p_n}]$ is called the *n-th block space* determined by $(p_n)_{n=0}^{\infty}$. Given a block basic sequence (y^n) in E, we define a *block extension* of (y^n) to be a basis (z^n) for E such that

(i) $$z^{p_n} = y^n \quad \text{for all } n,$$

and

(ii) $$z^i \in E^n \quad \text{for all } n, \text{ for all } i \text{ with } p_{n-1}+1 \leqslant i \leqslant p_n.$$

We now state a theorem which establishes a useful criterion for collecting algebraic bases for the nth block spaces into a basis for E.

THEOREM 1.1. *Let E be a Fréchet space with the sequence of semi-norms $(\|_n)_{n=1}^{\infty}$ defining the topology, and let (x^n) be a basis for E. Let $(p_n)_{n=0}^{\infty}$ be an increasing sequence of integers with $p_0 = 0$, and suppose that (z^n) is a sequence in E such that*

$$E^n = [z^{p_{n-1}+1}, \ldots, z^{p_n}], \quad \text{for all } n.$$

Then (z^n) is a basis for E if and only if for each $\|_k$ there exists $\|_m$ and $M(k) > 0$ such that

$$\Big|\sum_{i=p_{n-1}+1}^{r} t_i z^i\Big|_k \leqslant M(k) \Big|\sum_{i=p_{n-1}+1}^{p_n} t_i z^i\Big|_m$$

for all n, for all integers r with $p_{n-1}+1 \leqslant r \leqslant p_n$, and for all scalars $t_{p_{n-1}+1}, \ldots, t_{p_n}$.

The proof will be omitted. One should observe that by making use of the basis criterion of Retherford and McArthur in [14], Theorem 1.1 can be extended to the class of complete, barreled spaces.

2. Fréchet spaces and echelon spaces. We now restrict our attention to Fréchet spaces which have a continuous norm, or equivalently to Fréchet spaces whose topology is generated by a sequence Γ of norms $(\| \, \|_k)_{k=1}^{\infty}$, where it may be assumed that $\|x\|_k \leqslant \|x\|_{k+1}$ for all k and for all $x \in E$. In this setting, a basis is always a Schauder basis ([12]).

If $A: F_1 \to F_2$ denotes a linear map between two subspaces of a Fréchet space $(E, (\| \, \|_k))$, then we shall refer to the norm of A as an operator from the normed space $(F_1, \| \, \|_m)$ to be normed space $(F_2, \| \, \|_k)$ as the *(k, m)-norm* of A, for arbitrary positive integers k and m. We shall denote this by $\|A\|_{k,m}$, and from the standard definition of the operator norm we have

$$\|A\|_{k,m} = \inf\{M > 1 \colon \|Ax\|_k \leqslant M\|x\|_m \text{ for all } x \in F_1\}.$$

Let (x^n) be a basis for E, and let $(p_n)_{n=0}^{\infty}$ be an increasing sequence of integers such that $p_0 = 0$, and let $E^n = [x^{p_{n-1}+1}, \ldots, x^{p_n}]$, for all n. Let $q_n = p_n - p_{n-1}$. Given integers $n \geqslant 1$ and r such that $1 \leqslant r \leqslant q_n$, define the projection $\pi_r^n \colon E^n \to E^n$ by

$$\pi_r^n\Big(\sum_{i=p_{n-1}+1}^{p_n} t_i x^i\Big) = \sum_{i=p_{n-1}+1}^{p_{n-1}+r} t_i x^i.$$

The following theorem reduces the problem of finding block extensions from an infinite dimensional problem to a sequence of problems involving the existence of finite dimensional operators.

THEOREM 2.1. *Let $(E, (\| \, \|_k))$ be a Fréchet space with a continuous norm, and let (x^n) be a basis for E. Let (y^n) be a block basic sequence in E of the form*

$$y^n = \sum_{i=p_{n-1}+1}^{p_n} c_i x^i$$

with block spaces $E^n = [x^{p_{n-1}+1}, \ldots, x^{p_n}]$. Then (y^n) has a block extension if and only if for each n there exists an isomorphism $A_n \colon E^n \to E^n$ such that $A_n x^{p_n} = y^n$ for all n, and such that for each k there exists $m = m(k) \geqslant k$ and $M = M(k) \geqslant 1$ such that for all n, and for each positive integer $r < q_n$,

$$\|A_n \pi_r^n A_n^{-1}\|_{k,m} \leqslant M.$$

Proof. Let (z^n) be a block extension of (y^n). For each n, $E^n = [z^{p_{n-1}+1}, \ldots, z^{p_n}]$, so the linear maps A_n on E^n defined by

$$A_n\Big(\sum_{i=p_{n-1}+1}^{p_n} t_i x^i\Big) = \sum_{i=p_{n-1}+1}^{p_n} t_i z^i$$

are isomorphisms. Now by Theorem 1.1 applied to (z^n), it is true that for each k there exists $m \geqslant k$ and $M \geqslant 1$ such that for all n and for any integer r such that $1 \leqslant r \leqslant q_n$, and for any $x = \sum_{i=p_{n-1}+1}^{p_n} t_i x^i \in E^n$,

$$\|A_n \pi_r^n x\|_k \leqslant M \|A_n x\|_m.$$

But since each A_n is an isomorphism, we may make the substitution $y = A_n x$, so that for each n the inequality becomes

$$\|A_n \pi_r^n A_n^{-1} y\|_k \leqslant M \|y\|_m, \quad \text{for all } y \in E^n,$$

which is equivalent to the inequality

$$\|A_n \pi_r^n A_n^{-1}\|_{k,m} \leqslant M.$$

It is easily seen that $A_n x^{p_n} = y^n$ for all n.

Conversely, if we assume the existence of the isomorphisms A_n, we define (z^n) by $z^i = A_n x^i$ if $p_{n-1}+1 \leqslant i \leqslant p_n$. Then $z^{p_n} = y^n$ for all n and a straightforward application of 1.1 shows that (z^n) is a basis, completing the proof.

2.2. An important class of Fréchet spaces with a continuous norm is the class of echelon spaces. Echelon spaces of order p, $1 \leqslant p < \infty$, were introduced by Köthe ([8], § 30), and echelon spaces of a more general type were defined by Dubinsky in [4] and [5].

If λ is a sequence space, its *Köthe dual* is defined to be the sequence space

$$\lambda^x = \Big\{(u_i): \sum_{i=1}^{\infty} |x_i u_i| < +\infty \text{ for all } x \in \lambda\Big\}.$$

Then the pair $\langle \lambda, \lambda^x \rangle$ forms a dual system via the duality

$$\langle x, u \rangle = \sum_{i=1}^{\infty} x_i u_i, \quad \text{for } x \in \lambda,\ u \in \lambda^x.$$

λ is said to be *perfect* if $\lambda = \lambda^{xx}$. λ is called a *step* if λ is perfect, $\lambda[\mathscr{I}_b(\lambda^x)]$ is a Banach space, and $l^1 \subset \lambda \subset l^\infty$. Note that each of the spaces l^p, $1 \leqslant p \leqslant \infty$, is a step. Also, λ is a step if and only if λ^x is. ([5], p. 188).

Let λ_k be a sequence of steps, (a^k) a sequence of sequences such that

(i) $$0 < a_n^k < a_n^{k+1} \quad \text{for all } n \text{ and } k,$$

and

(ii) $$\frac{1}{a^{k+1}} \lambda_{k+1} \subset \frac{1}{a^k} \lambda_k, \quad \text{for all } k.$$

Then (a_k, λ_k) is called an *echelon system*, and the sequence spaces $\lambda = \bigcap_{k=1}^{\infty} \frac{1}{a^k} \lambda_k$ and $\mu = \bigcup_{k=1}^{\infty} a^k \lambda_k^x$ are called the corresponding *echelon* and *co-echelon spaces*, respectively. If $\lambda_k = l^p$, $1 \leqslant p \leqslant \infty$, λ is called an *echelon space of order* p.

THEOREM 2.3. *A co-echelon space is the Köthe dual of the corresponding echelon space, and both are perfect. Moreover, every echelon space is a Fréchet space in its strong topology.*

The proof of Theorem 2.3 is given in [5], p. 189. The following result is well known ([7]).

THEOREM 2.4. *An echelon space* $E = \bigcap_{k=1}^{\infty} \frac{1}{a^k} \lambda_k$ *is nuclear if and only if* $E = \bigcap_{k=1}^{\infty} \frac{1}{b^k} l^1$, *where* (b^k) *is such that for each* k *there exists* $\eta(k) > k$ *with* $\frac{a^k}{a^{\eta(k)}} \in l^1$.

Thus for nuclear echelon spaces we may restrict our attention to echelon spaces of order 1, and if $E = \bigcap \frac{1}{a^k} l^1$, the topology on E is generated by the norms $\| \ \|_k$, where $\|x\|_k = \sum_{n=1}^{\infty} |x_n| a_n^k = \|x \cdot a^k\|_{l^1}$, for all $x \in E$. Moreover, if e^n denotes the sequence in E which has 1 in the nth-coordinate and 0 elsewhere, it is clear that (e^n) is a basis for E. Finally, if (z^n) is a basis for E and $b_n^k = \|z^n\|_k$ for all n, k, then (z^n) is similar to the basis (e^n) of $\bigcap_{k=1}^{\infty} \frac{1}{b^k} l^1$, so that by (1.5) it suffices to consider only block basic sequences relative to (e^n).

In order to apply Theorem 2.1 we need to compute the (k, m)-norm of operators on finite-dimensional subspaces of E. Let $F = [e^r, \ldots, e^s]$ be such a subspace, and let $q = s-r+1$. For each k, let D_k: $E \to l^1$ be the diagonal map defined by $D_k x = a^k \cdot x$ for all $x \in E$. D_k is continuous by nature of the norms on E, and is one-to-one since $a_n^k > 0$ for all n.

If A: $F \to F$ is a linear map, then A has a unique $q \times q$ matrix representation relative to the basis $\{e^r, \ldots, e^s\}$ of F; that is, there exists a unique $q \times q$ matrix $(\beta_{ij})_{ij=r}^s$ such that if $x = \sum_{i=r}^{s} x_i e^i$, then

$$Ax = \sum_{i=r}^{s} \Big(\sum_{j=r}^{s} \beta_{ij} x_j\Big) e^i.$$

Let $D_k A D_m^{-1}|_F(x) = a^k \cdot x \cdot \frac{1}{a^m}$ for $x \in F$. Then $D_k A D_m^{-1}|_F$ is a linear map of F onto itself and has matrix $\left(\beta_{ij} \frac{a_i^k}{a_j^m}\right)_{i,j=r}^{s}$. In the following lemma we record the results of the elementary computations needed to apply (2.1) in echelon spaces of order 1.

LEMMA 2.4. *Let* $F = [e^r, \ldots, e^s]$ *be a finite dimensional subspace of* $E = \bigcap_{k=1}^{\infty} \frac{1}{a^k} l^1$, A: $F \to F$ *a linear map with matrix* $(\beta_{ij})_{ij=r}^s$.

(i) $\|A\|_{l^1} = \sup_{r\leqslant j\leqslant s}\big(\sum_{i=r}^{s}|\beta_{ij}|\big)$.

(ii) *For arbitrary* k *and* m, $\|A\|_{k,m} = \|D_k A D_m^{-1}|_F\|_{l^1}$.

(iii) $\|A\|_{k,m} = \sup_{r\leqslant j\leqslant s}\Big(\sum_{i=r}^{s}|\beta_{ij}|\,\dfrac{a_i^k}{a_j^m}\Big)$.

Proof. (i) and (ii) are straightforward computations and (iii) follows from (i) and (ii).

3. The example. In this section we construct a block basic sequence in a nuclear echelon space which has no block extension. The key to the example is Theorem 3.3, which implies that if all block spaces E^n have dimension 2, then a block extension will exist if and only if a trivial extension exists.

Let $p_n = 2n$, so that $q_n = 2$ and $E^n = [e^{2n-1}, e^{2n}]$ for $n\geqslant 1$. Let (y^n) be a block basic sequence in E of the form

$$y^n = \sum_{i=2n-1}^{2n} c_i e^i,$$

so that for each n, either $c_{2n-1}\neq 0$ or $c_{2n}\neq 0$.

By Theorem 2.1 it suffices to consider all sequences (A_n) of isomorphisms on E^n such that $A_n e^{2n} = y^n$ for all n. Each A_n has a 2×2 matrix representation relative to the basis $\{e^{2n-1}, e^{2n}\}$ of E^n, which we shall write as

$$A_n = \begin{pmatrix} b_{2n-1} & c_{2n-1}\\ b_{2n} & c_{2n}\end{pmatrix}.$$

Note that $\det A_n \neq 0$. We then obtain directly from Lemma 2.4 (iii), that for each n and for arbitrary k and m,

$$\|A_n\pi_1^n A_n^{-1}\|_{k,m} = \frac{1}{\det A_n}\max\Big\{|c_{2n}|\,|b_{2n-1}|\frac{a_{2n-1}^k}{a_{2n-1}^m} + |c_{2n}|\,|b_{2n}|\frac{a_{2n}^k}{a_{2n-1}^m},\quad |c_{2n-1}|\,|b_{2n-1}|\frac{a_{2n-1}^k}{a_{2n}^m} + |c_{2n-1}|\,|b_{2n}|\frac{a_{2n}^k}{a_{2n}^m}\Big\}. \tag{1}$$

Remark 3.1. We shall say that a block basic sequence (y^n) is in *standard form* if $y^n = \sum_{i=2n-1}^{2n} c_i e^i$, and if for each n either $c_{2n} = 1$ or $c_{2n} = 0$ and $c_{2n-1} = 1$. If (y^n) is an arbitrary sequence of the form $(y^n) = \sum_{i=2n-1}^{2n} c_i e^i$, and if

$$\gamma_n = \begin{cases} \dfrac{1}{c_{2n}} & \text{if } c_{2n}\neq 0,\\[2mm] \dfrac{1}{c_{2n-1}} & \text{if } c_{2n} = 0,\end{cases}$$

then $(\gamma_n y^n)$ is in standard form, and (y^n) will have a block extension if and only if $(\gamma_n y^n)$ does. Moreover, it suffices to consider isomorphisms A_n such that $\det A_n = 1$, so that for each n, $b_{2n-1}c_{2n} - b_{2n}c_{2n-1} = 1$.

Finally the choice of b_{2n-1} and b_{2n} for those n for which either c_{2n-1} or c_{2n} is 0 can be made easily. If $c_{2n-1} = 0$, so that $c_{2n} = 1$, choose $b_{2n} = 0$ and $b_{2n-1} = 1$. If $c_{2n} = 0$, so that $c_{2n-1} = 1$, let $b_{2n-1} = 0$ and $b_{2n} = -1$. In both cases we obtain from (1) that for each k and $m\geqslant k$, $\|A_n\pi_1 A_n^{-1}\|_{k,m}\leqslant 1$, since (a^k) is increasing.

Thus, if (y^n) is in standard form, it suffices to consider $J = \{n\colon c_{2n-1}\neq 0$ and $c_{2n} = 1\}$. For $n\in J$, $\det A_n = 1$ implies that $b_{2n} = \dfrac{b_{2n-1}-1}{c_{2n-1}}$.

Lemma 3.2. *Let E be an echelon space of order 1, and let (y^n) be a block basic sequence in E in standard form, and let $J = \{n\colon c_{2n-1}\neq 0,\ c_{2n} = 1\}$. Then (y^n) has a block extension if and only if there exists a sequence $(b_{2n-1})_{n\in J}$ such that for each k there exist $m\geqslant k$ and $M\geqslant 1$ such that for all $n\in J$*

(a) $$|b_{2n-1}|\frac{a_{2n-1}^k}{a_{2n-1}^m}\leqslant M,$$

(b) $$|c_{2n-1}|\,|b_{2n-1}|\frac{a_{2n-1}^k}{a_{2n}^m}\leqslant M,$$

(c) $$\Big|\frac{b_{2n-1}-1}{c_{2n-1}}\Big|\frac{a_{2n}^k}{a_{2n-1}^m}\leqslant M,$$

(d) $$|1-b_{2n-1}|\frac{a_{2n}^k}{a_{2n}^m}\leqslant M.$$

Proof. If conditions (a) through (d) hold for a sequence $(b_{2n-1})_{n\in J}$, then we let

$$A_n = \begin{pmatrix} b_{2n-1} & c_{2n-1}\\ b_{2n} & 1\end{pmatrix},\quad\text{where } b_{2n} = \frac{b_{2n-1}-1}{c_{2n-1}},\ \text{for } n\in J.$$

Each A_n, $n\in J$, determines an isomorphism on E^n such that $A_n e^{2n} = y^n$ and $\det A_n = 1$. Then the inequalities (a)–(d) inserted in (1) imply that for $n\in J$,

$$\|A_n\pi_1^n A_n^{-1}\|_{k,m}\leqslant 2M,$$

so that Theorem 2.1 is satisfied and we have an extension.

Conversely if (y^n) has a block extension then by the Remark 3.1 and Theorem 2.1 we obtain isomorphisms A_n on E^n for $n\in J$, with matrices

$$A_n = \begin{pmatrix} b_{2n-1} & c_{2n-1}\\ b_{2n} & c_{2n}\end{pmatrix},$$

such that $\det A_n = 1$. Then the inequalities (a) through (d) follow from Theorem 2.1 and (1). This completes the proof.

In the remainder of this section we shall say that a sequence $(b_{2n-1})_{n\epsilon J}$ *determines a block extension* of a block basic sequence (y^n) in standard form if (b_{2n-1}) satisfies the statement of Lemma 3.2.

THEOREM 3.3. *Let* $E = \bigcap_{k=1}^{\infty} \frac{1}{a^k} l^1$ *be an echelon space of order* 1. *Let* (y^n) *be a block basic sequence in* E *in standard form,* $J = \{n\colon c_{2n-1} \neq 0, c_{2n} = 1\}$. *Then* (y^n) *has a block extension if and only if there exist disjoint sequences* I_1 *and* I_2 *in* J, *whose union is* J, *such that if* $b_{2n-1} = 0$ *for* $n \epsilon I$, *and if* $b_{2n-1} = 1$ *for* $n \epsilon I_2$, *then* $(b_{2n-1})_{n\epsilon J}$ *determines a block extension of* (y^n).

Proof. Given $(b_{2n-1})_{n\epsilon J}$ as in the hypothesis, then it follows from Lemma 3.2 that (y^n) has a block extension. For the converse, Lemma 3.2 guarantees the existence of a sequence $(b'_{2n-1})_{n\epsilon J}$ determining an extension, and satisfying (a) through (d).

Then let $I_1 = \{n\colon |b'_{2n-1}| \leqslant \frac{1}{2}\}$ and $I_2 = \{n \epsilon J\colon |b'_{2n-1}| > \frac{1}{2}\}$. Clearly $I_1 \cap I_2 = \emptyset$, and $I_1 \cup I_2 = J$. For $(b_{2n-1})_{n\epsilon J}$ as defined in the theorem, the inequalities of Lemma 3.2 are equivalent to

$$(\bar{\mathrm{b}}) \qquad |c_{2n-1}| \frac{a_{2n-1}^k}{a_{2n}^m} \leqslant M, \quad \text{for } n \epsilon I_2,$$

and

$$(\bar{\mathrm{c}}) \qquad \frac{1}{|c_{2n-1}|} \frac{a_{2n}^k}{a_{2n-1}^m} \leqslant M, \quad \text{for } n \epsilon I_1.$$

But (b'_{2n-1}) does satisfy (a)–(d) with some $M' \geqslant 1$, and $n \epsilon I$, implies that $\frac{1}{|b'_{2n-1}|} \leqslant 2$, and $n \epsilon I_2$ implies that $|1 - b'_{2n-1}| > \frac{1}{2}$. Hence for $n \epsilon I_2$, (b) reduces to $(\bar{\mathrm{b}})$ with $M = 2M'$, and for $n \epsilon I_1$, (c) reduces to $(\bar{\mathrm{c}})$ with $M = 2M'$, completing the proof.

COROLLARY 3.4. *Let* E *be an echelon space of order* 1, *let* (y^n) *be a block basic sequence of the form*

$$y^n = \sum_{i=2n-1}^{2n} c_i e^i, \quad \textit{and} \quad J = \{n \epsilon c_{2n-1} \neq 0,\ c_{2n} \neq 0\}.$$

Then (y^n) *has a block extension* (z^n) *if and only if* z^n *may be chosen to be either* e^{2n-1} *or* e^{2n}.

Proof. Let $\gamma_n = \frac{1}{c_{2n}}$ if $c_{2n} \neq 0$, $\gamma_n = \frac{1}{c_{2n-1}}$ if $c_{2n} = 0$. Then $(\gamma_n y^n)$ is in standard form and has an extension if and only if (y^n) does. By Theorem 3.3 and the Remark 3.1 this is equivalent to the existence of an extension (z^n) of $(\lambda_n y^n)$ with $z^{2n} = \lambda_n y^n$ for all n, and

$$z^{2n-1} = \begin{cases} -e^{2n}, & \text{if } c_{2n} = 0, \\ e^{2n-1}, & \text{if } c_{2n-1} = 0, \text{ or if } n \epsilon I_2, \\ \dfrac{-c_{2n}}{c_{2n-1}} e^{2n}, & \text{if } n \epsilon I_1, \end{cases}$$

where I_1 and I_2 are disjoint sequences in J whose union is J. Define (λ_n) by

$$\lambda_n = \begin{cases} -1, & \text{if } c_{2n} = 0, \\ 1, & \text{if } c_{2n-1} = 0, \text{ or if } n \epsilon I_2, \\ \dfrac{-c_{2n-1}}{c_{2n}}, & \text{if } n \epsilon I_1. \end{cases}$$

Then if $u^{2n-1} = \lambda_n z^{2n-1}$ and $u^{2n} = y^n$ for all n, (u^n) is the desired block extension. Since the converse is clear, the proof is complete.

We are now in a position to give the example. Let (a^k) be the sequence of sequences defined by $a_{2n-1}^k = k^n$ and $a_{2n}^k = n^k$ for all k and n. Then for each k,

$$\sum_{i=1}^{\infty} \frac{a_i^k}{a_i^{k+2}} = \sum_{i=1}^{\infty} \left(\frac{k}{k+2}\right)^i + \sum_{i=1}^{\infty} \frac{1}{i^2} < +\infty,$$

so the echelon space $E = \bigcap_{k=1}^{\infty} \frac{1}{a^k} l^1$ is nuclear, by 2.4.

As an immediate consequence of Lemma 3.2 and Theorem 3.3 we have the following:

COROLLARY 3.5. *Let* (y^n) *be a block basic sequence in* E *in standard form. Then* (y^n) *has a block extension if and only if there exist disjoint sequences* I_1 *and* I_2 *such that* $I_1 \cup I_2 = J = \{n\colon c_{2n-1} \neq 0,\ c_{2n} = 1\}$, *and such that for each* k *there exists* $m \geqslant k$ *and* $M \geqslant 1$ *satisfying*

$$(\mathrm{B}_1) \qquad \frac{1}{|c_{2n-1}|} \leqslant M \frac{m^n}{n_k}, \quad \text{for } n \epsilon I_1,$$

$$(\mathrm{B}_2) \qquad |c_{2n-1}| \leqslant M \frac{n^m}{k^n}, \quad \text{for } n \epsilon I_2.$$

LEMMA 3.6. (i) (B_1) *holds on a subsequence* I_1 of J *only if*

$$(2) \qquad \varlimsup_{\substack{n\to\infty \\ n\epsilon I_1}} |c_{2n-1}|^{1/n} > 0.$$

On the other hand, if (14) *holds for a subsequence* I_1 of J, *there exists a subsequence* I'_1 *of* I_1 *such that* (B_1) *holds on* I'_1.

(ii) (B_2) *holds on a subsequence* I_2 *of* J *if and only if*

$$\overline{\lim_{\substack{n\to\infty\\ n\in I_2}}}\ |c_{2n-1}|^{1/n} = 0. \tag{3}$$

Proof. (i) (B_1) implies that for each k there exists $m \geqslant k$ and $M \geqslant 1$ such that for all $n \in I_1$, $|c_{2n-1}| \geqslant \frac{1}{M}\frac{n^k}{m^n}$, so that

$$\overline{\lim_{\substack{n\to\infty\\ n\in I_1}}}\ |c_{2n-1}|^{1/n} \geqslant \lim_{\substack{n\to\infty\\ n\in I_1}} M^{-1/n}\frac{n^{k/m}}{m} = \frac{1}{m} > 0.$$

On the other hand, if $\overline{\lim\limits_{\substack{n\to\infty\\ n\in I_1}}}\ |c_{2n-1}|^{1/n} > 0$, there exists a subsequence I_1' of I_1 and $\varepsilon > 0$ such that for all $n \in I_1'$, $|c_{2n-1}|^{1/n} > \varepsilon$. Given k, choose $m \geqslant \max\{k, 2/\varepsilon\}$, so that if $n \in I_1'$, $|c_{2n-1}|^{1/n} \geqslant 2/m$. Now $\inf\limits_{n\in I_1'} n^{m/n} = 1$, so given k and m as above, there exists n_k such that $n \geqslant n_k$ and $n \in I_1'$ imply that $2 \geqslant n^{m/n} \geqslant 1$, and hence $|c_{2n-1}|^{1/n} \geqslant \frac{n^{m/n}}{m}$. Thus given k we have $m \geqslant k$ and n_k such that for $n \in I_1'$, $n \geqslant m_k$, $\frac{1}{|c_{2n-1}|} \leqslant \frac{m^n}{n^m}$. Let $M = \max\left\{1, |c_{2n-1}|\frac{n^m}{m^n}, n \geqslant n_k\right\}$, so that $\frac{1}{|c_{2n-1}|} \leqslant M\frac{m^n}{n^m}$ for all $n \in I_1'$.

(ii) Similarly, (B_2) holds on a subsequence I_2 of J if and only if for each k there exists $m \geqslant k$ and $M \geqslant 1$ such that

$$\overline{\lim_{\substack{n\to\infty\\ n\in I_2}}}\ |c_{2n-1}|^{1/n} \leqslant \lim_{\substack{n\to\infty\\ n\in I_2}} M^{1/n}\frac{n^{m/n}}{k} = 1/k,$$

which is equivalent to (3).

COROLLARY 3.7. (i) (B_1) *holds on a subsequence* I_1 *of* J *if and only if* (B_2) *fails on every subsequence of* I_1.

(ii) (B_2) *holds on a subsequence* I_2 *of* J *if and only if* (B_1) *fails on every subsequence of* I_2.

The proof is immediate from Lemma 3.6 (i) and (ii).

$(*)$ *To construct a block basic sequence which has no block extension in* E *we will define a sequence* $(c_{2n-1})_{n=1}^{\infty}$ *of non-zero scalars such that neither* (B_1) *nor* (B_2) *hold on* N, *and such that whenever* (B_1) *holds on a subsequence* I_1 *of* N, *then* (B_2) *fails on* $N\setminus I_1$. *Then by Corollary* 3.5, *the block basic sequence* (y^n) *given by* $y^n = c_{2n-1}e^{2n-1} + e^{2n}$, *which is in standard form with* $J = N$, *will have no block extension.*

First define the infinite matrix $(u_{ij})_{i,j=1}^{\infty}$ by

$$u_{ij} = \begin{cases} (i-j+1), & \text{if } i \geqslant j,\\ \left(\dfrac{1}{(j-i+1)!}\right)^{j-i+1}, & \text{if } i < j.\end{cases}$$

Then

$$(u_{ij})_{i,j=1}^{\infty} = \begin{pmatrix} 1 & \left(\frac{1}{2!}\right)^2 & \left(\frac{1}{3!}\right)^3 & \dots\\ 2 & 1 & \left(\frac{1}{2!}\right)^2 & \dots\\ 3 & 2 & 1 & \dots\\ \cdot & \cdot & \cdot & \dots\\ \cdot & \cdot & \cdot & \dots\\ \cdot & \cdot & \cdot & \dots \end{pmatrix}.$$

We may write $(u_{ij})_{i,j=1}^{\infty}$ as a sequence $(u_n)_{n=1}^{\infty}$ via the bijection of $N\times N$ onto N given by

$$n(i,j) = \frac{(i+j-1)(i+j-2)}{2} + i.$$

This bijection may be indicated schematically as

$$\begin{pmatrix} 1 & 2 & 4 & 7\\ 3 & 5 & 8 & \\ 6 & 9 & & \\ 10 & & & \end{pmatrix}.$$

We then define $(c_{2n-1})_{n=1}^{\infty}$ by $c_{2n-1} = u_n$ for all n. The proof of $(*)$ may be broken into three parts.

(a) Let $I_1 = (n_i)_{i=1}^{\infty}$ be the subsequence of N defined by $n_i = n(i,1)$ for all i, so that $(u_{n_i})_{i=1}^{\infty}$ corresponds to the first column of the array (u_{ij}). Then

$$\overline{\lim_{i\to\infty}}\ |u_{n_i}|^{1/n_i} = \overline{\lim_{i\to\infty}}\ (i)^{\frac{2}{i(i-1)+2i}} = 1 > 0,$$

so (B_1) holds on I_1, and hence (B_2) fails on N.

(b) For fixed i, let $I_2^i = (n_j)_{j\geqslant i+1}$, where $n_j = n(i,j)$ for $j \geqslant i+1$. Then $u_{n_j} = \left(\frac{1}{(j-i+1)!}\right)^{j-i+1}$, so that

$$\overline{\lim_{j\to\infty}}\,|u_{n_j}|^{1/n_j} = \overline{\lim_{j\to\infty}}\left(\frac{1}{(j-i+1)!}\right)^{\frac{2(j-i+1)}{(i+j-1)(i+j-2)+2i}} \leqslant \overline{\lim_{j\to\infty}}\left(\frac{1}{(j-i+1)!}\right)^{\frac{2(j-i+1)}{4_j^2}}. \tag{4}$$

Replacing m by $(j-i+1)$, we obtain,

$$\overline{\lim_{j\to\infty}}|u n_j|^{1/n_j} \leqslant \overline{\lim_{m\to\infty}}\left(\frac{1}{m!}\right)^{1/3m} = 0.$$

Hence (B_2) holds on each I_2^i, so that (B_1) fails on N.

(c) Finally, we show that if I_1 is a subsequence of N on which (B_1) holds, then (B_2) fails on $N\setminus I_1 = I_2$. Given I_1, then from step (b) and Corollary 3.7 (ii), it follows that I_1 may not contain any subsequence of any I_2^i. Thus for each i there exists a largest index j_i such that $n(i, j_i)\in I_1$.

Suppose that $j_i \leqslant i$ infinitely often. Then $I_2 = N\setminus I_1$ contains infinitely many integers of the form $n(i, i+1)$, say a sequence $\big(n(i_\nu, i_\nu+1)\big)_{\nu=1}^{\infty}$. Then $u_n(i_\nu, i_\nu+1) = \dfrac{1}{2!}$ for all ν, and on this sequence the required upper limit is positive, so (B_2) fails on I_2 by Lemma 3.6 (ii).

Thus we may suppose there exists an integer i_0 such that $j_i > i$ for all $i \geqslant i_0$. If (B_2) holds on $I_2 = N\setminus I_1$, then in particular (B_2) must hold on the subsequence $(m_i)_{i=i_0+1}^{\infty}$ of I_2 defined by $m_i = n(i, j_i+1)$ for $i > i_0$. From (3) we have

$$(5)\qquad 0 = \overline{\lim_{i\to\infty}}|u_{m_i}|^{1/m_i} = \overline{\lim_{i\to\infty}}\left(\frac{1}{(j_i-i+2)!}\right)^{\frac{2(j_i-i+1)}{(i+j_i)(i+j_i-1)+2i}}.$$

But now we show that (5) implies that (B_1) does not hold in I_1, which is a contradiction.

Consider the subsequence $(s_i)_{i=i_0+1}^{\infty}$ of I_1 given by $s_i = n(i, j_i)$. Then

$$(6)\qquad \overline{\lim_{i\to\infty}}|u_{s_i}|^{1/s_i} = \lim_{i\to\infty}\left(\frac{1}{(j_i-i+1)!}\right)^{\frac{2(j_i-i+1)}{(i+j_i-1)(i+j_i-2)+2i}} \leqslant \overline{\lim_{i\to\infty}}(j_i-i+2)^{\frac{2(j_i-i+1)}{(i+j_i-1)(i+j_i-2)+2i}} \overline{\lim_{i\to\infty}}\left(\frac{1}{(j_i-i+2)!}\right)^{\frac{2(j_i-i+1)}{(i+j_i-1)(i+j_i-2)+2i}}.$$

Now

$$(7)\qquad \overline{\lim_{i\to\infty}}(j_i-i+2)^{\frac{2(j_i-i+1)}{(i+j_i-1)(i+j_i-2)+2i}} \leqslant \overline{\lim_{i\to\infty}}(j_i)^{\frac{2j_i}{j_i^2}} = 1,$$

and

$$(8)\qquad \overline{\lim_{i\to\infty}}\left(\frac{1}{(j_i-i+2)!}\right)^{\frac{2(j_i-i+1)}{(i+j_i-1)(i+j_i-2)+2i}} \leqslant \overline{\lim_{i\to\infty}}\left(\frac{1}{(j_i-i+2)!}\right)^{\frac{2(j_i-i+1)}{(i+j_i)(i+j_i-1)+2i}} \leqslant \overline{\lim_{i\to\infty}}|u\quad|^{1/m_i}\cdot\overline{\lim_{i\to\infty}}\big((j_i-i+2)!\big)^{\frac{2}{(i+j_i)(i+j_i-1)+2i}}.$$

But $\overline{\lim_{i\to\infty}}\big((j_i-i+2)!\big)^{\frac{2}{(i+j_i)(i+j_i-1)+2i}} \leqslant \overline{\lim_{i\to\infty}}(j_i!)^{\frac{2}{j_i^2}} < +\infty$ by Stirling's Formula. Then applying (7) and (8) to (6) we obtain

$$(9)\qquad \overline{\lim_{i\to\infty}}|u_{s_i}|^{1/s_i} = 0.$$

But this is equivalent to the statement that (B_2) holds on $(s_i)_{i=i_0+1}^{\infty}$, by Lemma 3.6(ii), which is contrary to the fact that $(s_i)\subset I_1$. Hence the claim in (*) is true, and the proof is complete.

4. Extensions in the space ω. As one final result, we consider the nuclear space ω, which is a Fréchet space with the topology $\mathscr{T}_b(\varphi, \omega) = \mathscr{T}_s(\varphi, \omega)$ ([8], p. 408).

THEOREM 4.1. *Let (y^n) be a block basic sequence in ω relative to a basis (x^n) for ω. Then (y^n) has a block extension.*

Proof. Since every basis for ω is similar to the unconditional basis (e^n), as was shown in [3], Theorem 7, then we may assume that $x^n = e^n$ for all n. Suppose $y^n = \sum_{i=p_{n-1}+1}^{p_n} c_i e^i$, for all n, and $E^n = [e^{p_{n-1}+1}, \ldots, e^{p_n}]$ is the nth block space. If for each n, $\{z^{p_{n-1}+1}, \ldots, z^{p_n}\}$ is an algebraic basis for E^n, then $\bigcup_{n=1}^{\infty}\{z^{p_{n-1}+1}, \ldots, z^{p_n}\} = (z^n)$ is a basis for ω. For if $x\in\omega$,

$$x = \sum_{n=1}^{\infty} x_n e^n = \sum_{n=1}^{\infty}\Big(\sum_{i=p_{n-1}+1}^{p_n} x_i e^i\Big).$$

But for each n, there exist unique scalars $\{t_i\}_{i=p_{n-1}+1}^{p_n}$ such that

$$\sum_{i=p_{n-1}+1}^{p_n} x_i e^i = \sum_{i=p_{n-1}+1}^{p_n} t_i z^i,$$

and hence for each $x\in\omega$ there exists a unique sequence

$$(t_n)_{n=1}^{\infty} = \bigcup_{n=1}^{\infty}\{t_i\}_{i=p_{n-1}+1}^{p_n} \quad \text{such that } x = \sum_{i=1}^{\infty} t_i z^i.$$

Thus for each n we may select any algebraic basis $\{z^i\}_{i=p_{n-1}+1}^{p_n}$ for E^n such that $z^{p_n} = y^n$, and then $\bigcup_{n=1}^{\infty}\{z^i\}_{i=p_{n-1}+1}$ is the desired block extension.

4.2. Remarks and problems. There are two directions for further investigation which are suggested by the results given here. The first involves characterizing the Fréchet spaces in which block extensions always exist, and the second involves finding more general types of extensions. These can be formulated as follows:

PROBLEM 1. *If E is a Fréchet space with a basis in which all block basic sequences have block extensions, must E be either a Banach space or isomorphic to ω?*

One property of ω which seems to be of importance for the existence of block extensions is the fact that ω has no continuous norms.

A more general question, which in the case of nuclear Fréchet spaces is closely related to the existence of complements for subspaces with bases, is the following:

PROBLEM 2. *If (y^n) is a block basic sequence (or any basic sequence) in a Fréchet space E with a basis, is there any basis for E containing (y^n) as a subsequence?*

Thus, for the example given in Section 3, there may be some less restrictive method for obtaining an extension.

References

[1] S. Banach, *Théories des Operations Linearies*, Warszawa 1932.
[2] C. Bessaga, *Some remarks on Dragilev's theorem*, Studia Math. 31 (1968), pp. 307–318.
[3] — and A. Pełczyński, *Properties of bases in B_0-spaces*, Prace Mat. 3 (1959), pp. 123–142.
[4] E. Dubinsky, *Echelon spaces of order ∞*, Proc. Amer. Math. Soc. 16 (1965), pp. 1178–1183.
[5] — *Perfect Fréchet spaces*, Math. Ann. 174 (1967), pp. 186–194.
[6] — and J. R. Retherford, *Schauder bases and Köthe sequence spaces*, Trans. Amer. Math. Soc. 130 (1968), pp. 265–280.
[7] A. Grothendieck, *Produits tensorial topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955).
[8] G. Köthe, *Topological Vector Spaces I*, New York 1969.
[9] B. S. Mitiagin, *Nuclear spaces and bases in nuclear spaces*, (Russian) Uspchi Mat, Nauk, Vol. 16, No. 4.
[10] — *Fréchet spaces with a unique unconditional basis*, Studia Math. 38 (1970), pp. 23–34.
[11] A. Pietsch, *Nukleare Lokalkonvexe Räume*, Berlin (1965).
[12] A. Pełczyński, *Some problems on bases in Banach and Fréchet spaces*, Israel J. Math. 2, (1964), pp. 132–138.
[13] — *Universal bases*, Studia Math. 32, (1969), pp. 247–268.
[14] J. R. Retherford and C. W. McArthur, *Some remarks on bases in linear topological spaces*, Math. Annal. 64 (1966), pp. 38–41.
[15] M. Zippin, *A remark on bases and reflexivity in Banach spaces*, Israel J. Math. 6 (1968), pp. 74–79.

CLARKSON COLLEGE OF TECHNOLOGY
POTSDAM, NEW YORK

Received December 28, 1970 (406)

STUDIA MATHEMATICA, T. XLV. (1973)

On the differentiability of Lipschitz mappings in Fréchet spaces

by

P. MANKIEWICZ (Warszawa)

Abstract. The probem of differentiability of mappings from a subset of a Fréchet space into another Fréchet space satisfying the first order Lipschitz condition is studied. Some extensions of the classical theorem of Rademacher are obtained. Applications of the result to the problem of the topological classification of Fréchet spaces are given.

1. Introduction. A classical theorem of Rademacher [11], [6] states that for every mapping F from the cube C_n in $\boldsymbol{R}^n$ into $\boldsymbol{R}^m$ satisfying the first order Lipschitz condition, the differential $(DF)_p$ exists for almost all p in C_n. The aim of this note is to give an extension of this theorem for the case of a mapping F satisfying the first order Lipschitz condition from a subset of a Fréchet space into another Fréchet space.

Some difficulties arise with the definition of the first order Lipschitz condition. (A simple example of a Fréchet space X and two metrics ϱ_1 and ϱ_2 on X can be given such that the identity mapping I from (X, ϱ_1) onto (X, ϱ_2) does not satisfy the first order Lipschitz condition with respect to the metrics ϱ_1 and ϱ_2). This leads us to apply the definition introduced in [9] which states that the mapping F from X into Y satisfies the first order Lipschitz condition if and only if for every continuous pseudonorm on Y there exists a continuous pseudonorm on X such that F induces a mapping between suitable quotient spaces satisfying the first order Lipschitz condition (with respect to the norms).

The other difficulty is the following. There are known examples in which a mapping satisfying the first order Lipschitz condition from the interval $[0,1]$ into a Fréchet space Y does not possess a differential at any point of the interval $[0,1]$. For example the mapping F from the interval $[0,1]$ into $L_1([0,1])$ defined by the formula

$$F(t) = \chi_{[0,t]} \quad \text{for } t \in [0,1],$$

where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0,t]$. (A similar example can be given for the space c_0).

On the other hand Gelfand proved in [7] (see also [6]) that for every mapping F from the interval $[0,1]$ into a separable conjugate Banach space X satisfying the first order Lipschitz condition, the derivative $F'(t)$ exists for almost all t in $[0,1]$ with respect to one-dimensional Lebesgue