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On a certain class of non-removable ideals in Banach algebras

by

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Abstract. Let A be a commutative complex Banach algebra with unit element. A subset $S \subset A$ is said to consist of joint topological divisors of zero if there is a net $(z_a) \subset A$, $\|z_a\| = 1$, such that $\lim_a z_a x = 0$ for all $x \in S$. In the paper there are studied ideals consisting of joint topological divisors of zero. The main result states that every maximal ideal belonging to the Šilov boundary of A consists of joint topological divisors of zero.

All algebras in this paper are assumed to be commutative complex Banach algebras with unit elements. If A and B are such algebras then B is called an *extension* or a *superalgebra* of A if there exists a map φ of A into B, sending the unit element of A onto the unit element of B, the map being an algebraic isomorphism and a homeomorphism. We write in this case $A \subset B$ and call the map φ an imbedding of A into B. Two extensions consisting of the same algebra B and two different imbeddings are considered to be different. When the imbedding φ is an isometry we speak about an isometric extension.

A proper ideal $I \subset A$ is called a *non-removable ideal* if for every extension B of A there is a proper ideal J in B such that $I \subset J$ meaning $\varphi(I) \subset J$. The concept of a non-removable ideal was introduced and studied by Arens in papers [1] and [2]. Actually Arens considered only isometric extensions, so his class of non-removable ideals is formally less restrictive than that of this paper. It seems that both classes coincide, this would follow from a positive answer to the following question: Let B be an extension of A under an imbedding φ. Does there exists an equivalent Banach algebra norm on B under which φ becomes an isometry? [1]

In this paper we introduce and study a class of non-removable ideals that is the class consisting of *joint topological divisors of zero* (shortly j.t.d.z., cf. definition below). We do not know whether the introduced class coincides with the class of all non-removable ideals and, in fact, we expect that this is the case. Basing upon our earlier result [5] we obtain the main result of this paper stating that every ideal in $\Gamma(A)$ (the Šilov

[1] A positive answer to this question is given in the paper [7] (added in proof).

boundary of A) consists of j.t.d.z. As corollaries we prove that the radical of A consists of j.t.d.z. and that any functional in $\Gamma(A)$ can be extended to a member of $\Gamma(B)$ if B is an extension of A (we identify maximal ideals with corresponding multiplicative-linear functionals).

DEFINITION. We say that a non-void subset S of a Banach algebra A *consists of joint topological divisors of zero* if for any finite subset $\{x_1, \ldots, x_n\} \subset A$ it is

$$(1) \qquad \delta(x_1, \ldots, x_n) = \inf\Big\{\sum_{i=1}^{n} \|zx_i\| : \|z\| = 1\Big\} = 0.$$

This is equivalent to the fact that there is a net $(z_a) \subset A$, $\|z_a\| = 1$, such that $\lim_a z_a x = 0$ for every $x \in S$ (cf. [5]). We say in this case that the net (z_a) annihilates S.

We denote by $\mathfrak{l}^{\#}(A)$ the family of all ideals of A consisting of j.t.d.z., by $\mathfrak{l}(A)$ — the family of all closed ideals in $\mathfrak{l}^{\#}(A)$, and by $\mathcal{L}(A)$ the intersection $\mathfrak{M}(A) \cap \mathfrak{l}(A)$, i.e. the class of all maximal ideals in $\mathfrak{l}(A)$.

LEMMA 1. *If S is a non-void subset of an algebra A and S consists of j.t.d.z. then S is contained in an ideal I belonging to $\mathfrak{l}^{\#}(A)$.*

Proof. Taking as I the smallest ideal containing S, i.e. the collection of all finite sums of the form $\sum_{i=1}^{n} a_i s_i$, where $a_i \in A$, $s_i \in S$, we see that any net (z_a) annihilating S also annihilates I and thus $I \in \mathfrak{l}^{\#}(A)$.

LEMMA 2. *The closure $\bar{I}$ of any ideal $I \in \mathfrak{l}^{\#}(A)$ belongs to $\mathfrak{l}(A)$.*

Proof. Let $x_1, \ldots, x_n \in \bar{I}$ and for a given $\varepsilon > 0$ choose $y_1, \ldots, y_n \in I$ in such a way that $\|x_i - y_i\| \leqslant \varepsilon/n$. For any $z \in A$ with $\|z\| = 1$ we have

$$\sum_{i=1}^{n} \|x_i z\| \leqslant \sum_{i=1}^{n} \|(x_i - y_i) z\| + \sum_{i=1}^{n} \|y_i z\| \leqslant \varepsilon + \sum_{i=1}^{n} \|y_i z\|$$

and the infimum on the right-hand side with respect to $z \in A$, $\|z\| = 1$, equals ε. This means $\delta(x_1, \ldots, x_n) \leqslant \varepsilon$ for every $\varepsilon > 0$, where $\delta(x_1, \ldots, x_n)$ is given by the formula (1), thus $I \in \mathfrak{l}(A)$.

In the sequel we consider only ideals from $\mathfrak{l}(A)$ and such ideals we call sometimes $\mathfrak{l}$-ideals. Corresponding facts on $\mathfrak{l}^{\#}$-ideals follow immediately from the above lemma.

PROPOSITION 1. *If $I \in \mathfrak{l}(A)$ and B is an extension of A then there is an ideal $J \in \mathfrak{l}(B)$ such that $I \subset J$.*

The proof follows immediately from Lemmas 1 and 2 since the ideal I, treated as a subset of B, consists of j.t.d.z.

COROLLARY 1. *Any ideal in $\mathfrak{l}(A)$ is a non-removable ideal.*

LEMMA 3. *Every ideal $I \in \mathfrak{l}(A)$ is contained in an $\mathfrak{l}$-maximal ideal, i.e. in an ideal $J \in \mathfrak{l}(A)$ such that if $J_1 \supset J$ and $J_1 \in \mathfrak{l}(A)$, then $J = J_1$.*

Proof. Consider $\mathfrak{l}(A)$ as a partially ordered set with inclusion as order relation. If $\{I_a\} \subset \mathfrak{l}(A)$ is a linearly ordered subset of $\mathfrak{l}(A)$ then $\bigcup_a I_a$ consists of j.t.d.z. and so, by Lemmas 1 and 2 it is contained in a certain ideal $I_1 \in \mathfrak{l}(A)$ which contains all ideals I_a. The conclusion follows now from the Kuratowski–Zorn lemma and from Lemma 2.

PROPOSITION 2. *Every $\mathfrak{l}$-maximal ideal is a prime ideal.*

Proof. Let I be an $\mathfrak{l}$-maximal ideal in A and let $xy \in I$ with $x \notin I$. We have to show that $y \in I$. Let (z_a) be any annihilating net for I. There is an index a_0 and a positive real δ such that $\|z_a x\| \geqslant \delta$ for all $a > a_0$. Otherwise there would exist a subnet $(z_\beta) \subset (z_a)$, cofinal with (z_a), which annihilates $S = I \cup \{x\}$; this, by the Lemma 1 contradicts to the $\mathfrak{l}$-maximality of I. Setting now $z_a' = z_a x / \|z_a x\|$ we obtain a net annihilating $I \cup \{y\}$ since it annihilates I and $\|z_a' y\| = \|z_a xy\| / \|z_a x\| \leqslant \delta^{-1} \|z_a xy\| \to 0$. This, as before, implies $y \in I$.

Actually we do not know whether any $\mathfrak{l}$-ideal of A is contained in an ideal belonging to $\mathcal{L}(A)$, i.e. whether any $\mathfrak{l}$-maximal ideal is is a maximal ideal in A. It would be interesting to know whether a corresponding fact is true for non-removable ideals, i.e. whether every non-removable ideal is contained in a maximal ideal which is also non-removable. This problem is connected with some questions posed by Arens in [1], where he asked whether a family of removable ideals of A can be a non-removable family in the sense that there is no single extension B of A which "removes" all ideals in this family(²). If I is a non-removable ideal in A such that every maximal ideal containing I is a removable ideal then the family of all these removable ideals is a non-removable family. On the other hand it can be shown that if every non-removable ideal is contained in a maximal ideal which is a removable ideal, then every finite family of removable ideals is a removable family.

PROPOSITION 3. *The set $\overline{\mathcal{L}(A)}$ is a closed subset of the maximal ideal space $\mathfrak{M}(A)$.*

Proof. If $M_0 \in \overline{\mathcal{L}(A)}$, then for a given neighbourhood U of M_0 in $\mathfrak{M}(A)$ there is an element M' in $\mathcal{L}(A)$ belonging to the neighbourhood U. So for a given $\varepsilon > 0$ and elements $x_1, \ldots, x_n \in M_0$ there is an $M' \in \mathcal{L}(A)$ such that $|\hat{x}_i(M')| < \varepsilon/2n$ for $i = 1, 2, \ldots, n$. If we put $y_i = x_i - \hat{x}_i(M') e$ we have $y_i \in M'$ and $\|y_i - x_i\| < \varepsilon/2n$, $i = 1, 2, \ldots, n$. Since $M' \in \mathcal{L}(A)$, there is an element $z \in A$ such that $\|z\| = 1$ and $\sum_{i=1}^{n} \|zy_i\| < \varepsilon/2$. This implies

$$\sum_{i=1}^{n} \|zx_i\| \leqslant \sum_{i=1}^{n} \|zy_i\| + \sum_{i=1}^{n} \|z(y_i - x_i)\| < \varepsilon/2 + \sum_{i=1}^{n} \|y_i - x_i\| < \varepsilon.$$

(²) It follows from a construction given in [6], that the answer to this question is negative (added in proof).

Since ε was choosen arbitrarily it means that $\delta(x_1, \ldots, x_n) = 0$ and so $M_0 \in \mathcal{L}(A)$.

We pass now to our main result stating that there are "sufficiently many" ideals in $\mathcal{L}(A)$, i.e. that if two elements of an algebra A can be separated by functionals from $\mathfrak{M}(A)$, then they can be also separated by functionals from $\mathcal{L}(A)$. This result is contained in the following

THEOREM. *If $\Gamma(A)$ designates the Šilov boundary of A, then $\Gamma(A) \subset \mathcal{L}(A)$.*

Proof. It is sufficient to show that if $\delta(x_1, \ldots, x_n) > 0$, then the elements $x_1, \ldots, x_n$ cannot belong to the same maximal ideal $M \in \Gamma(A)$. Without loss of generality we can assume $\delta(x_1, \ldots, x_n) \geqslant 1$ which is equivalent to

$$\sum_{i=1}^{n} \|x_i z\| \geqslant \|z\| \tag{2}$$

for all $z \in A$. Consider extension B of A consisting of all formal power series $s = \sum_{i_1 \ldots i_n \geqslant 0} a_{i_1 \ldots i_n} t_1^{i_1} \ldots t_n^{i_n}$, $a_{i_1 \ldots i_n} \in A$, with $\|s\| = \sum_{i_1 \ldots i_n \geqslant 0} \|a_{i_1 \ldots i_n}\| < \infty$. It can be easily verified that B is a Banach algebra under Cauchy multiplication of formal power series, which contains A under an isometric imbedding given by

$$\varphi(x) = x + \sum_{i_1 + \ldots + i_n > 0} 0 t_1^{i_1} \ldots t_n^{i_n}.$$

If we put $w = x_1 t_1 + \ldots + x_n t_n$ we obtain an element of B and from (2) we see that $\|wz\| \geqslant \|z\|$ for all $z \in A$. We want to estimate from below the expression $\|w^k z\|$, $z \in A$. Writing

$$w^{k-1} = \sum_{i_1 + \ldots + i_n = k-1} C_{i_1 \ldots i_n}^{(k-1)} x_1^{i_1} \ldots x_n^{i_n} t_1^{i_1} \ldots t_n^{i_n}$$

we have $C_{i_1 \ldots i_n}^{(k-1)} > 0$ and

$$\|w^{k-1} z\| = \sum_{i_1 + \ldots + i_n = k-1} C_{i_1 \ldots i_n}^{(k-1)} \|x_1^{i_1} \ldots x_n^{i_n} z\|. \tag{3}$$

By use of (2) and (3) we have

$$\begin{aligned} \|w^k z\| &= \Big\| w \sum_{i_1 + \ldots + i_n = k-1} C_{i_1 \ldots i_n}^{(k-1)} x_1^{i_1} \ldots x_n^{i_n} z t_1^{i_1} \ldots t_n^{i_n} \Big\| \\ &= \Big\| \sum_{s=1}^{n} \sum_{i_1 + \ldots + i_n = k-1} C_{i_1 \ldots i_n}^{(k-1)} x_1^{i_1} \ldots x_s^{i_s+1} \ldots x_n^{i_n} z t_1^{i_1} \ldots t_s^{i_s+1} \ldots t_n^{i_n} \Big\| \\ &= \sum_{i_1 + \ldots + i_n = k-1} C_{i_1 \ldots i_n}^{(k-1)} \sum_{s=1}^{n} \|x_s x_1^{i_1} \ldots x_n^{i_n} z\| \\ &\geqslant \sum_{i_1 + \ldots + i_n = k-1} C_{i_1 \ldots i_n}^{(k-1)} \|x_1^{i_1} \ldots x_n^{i_n} z\| = \|w^{k-1} z\|. \end{aligned}$$

Thus, by induction we obtain

$$\|w^k z\| \geqslant \|z\| \tag{4}$$

for all $z \in A$ and all $k = 1, 2, \ldots$ and thus also $\|w^k z^k\| \geqslant \|z^k\|$.

By passing to the spectral norm $\|u\|_s = \lim_n \|u^n\|^{1/n}$ we get

$$\|wz\|_s \geqslant \|z\|_s \tag{5}$$

for all $z \in A$. Since $\|t_i\|_s = 1$ we obtain

$$\|z\|_s \leqslant \|wz\|_s = \Big\| \sum_{i=1}^{n} z x_i t_i \Big\|_s \leqslant \sum_{i=1}^{n} \|x_i z\|_s \tag{6}$$

and so the relation (2) holds also for spectral norm in A.

Designate by $\bar{A}$ the completion of the algebra $\hat{A}$ of Gelfand transforms of elements of A in the sup-norm on the maximal ideal space $\mathfrak{M}(A)$. It is known that every multiplicative linear functional $f \in \mathfrak{M}(A)$ "extends" to a member $\bar{f}$ of $\mathfrak{M}(\bar{A})$, so we can identify $\mathfrak{M}(A)$ with $\mathfrak{M}(\bar{A})$. Under this identification we have $\Gamma(A) = \Gamma(\bar{A})$, i.e. the elements $x_1, \ldots, x_n$ are in an ideal M belonging to the Šilov boundary $\Gamma(A)$ if and only if their Gelfand transforms $\hat{x}_1, \ldots, \hat{x}_n$ are in a Šilov boundary ideal of $\bar{A}$. We apply now the main result of [5] which states that for every function algebra A it is $\Gamma(A) = \mathcal{L}(A)$. The formula (6) proves now that the elements $\hat{x}_1, \ldots, \hat{x}_n$ cannot be in a maximal ideal $\bar{M} \in \Gamma(\bar{A})$, and so the elements $x_1, \ldots, x_n$ cannot belong to a maximal ideal $M \in \Gamma(A)$.

Denoting by $\operatorname{cor} A$ the cortex of A i.e. the set of all non-removable ideals in $\mathfrak{M}(A)$ (cf. [1]) we have the following relation

$$\Gamma(A) \subset \mathcal{L}(A) \subset \operatorname{cor}(A).$$

An example due to Šilov [4], reproduced also in [1] shows that it may be $\operatorname{cor} A \neq \Gamma(A)$ and in this case $\operatorname{cor} A = \mathcal{L}(A)$, so there be $\mathcal{L}(A) \neq \Gamma(A)$. On the other hand we do not know whether it can be $\operatorname{cor} A \neq \mathcal{L}(A)$, and, as mentioned above, we expect that both sets coincide.

COROLLARY 2. *If $\operatorname{rad} A$ denotes the radical of A, then $\operatorname{rad} A \in l(A)$.*

COROLLARY 3. *If f is a functional in $\Gamma(A)$ and B is an extension of A then f extends to a member $\bar{f}$ of $\Gamma(B)$.*

Proof. As in the proof of the theorem the problem can be reduced to the case when A and B are function algebras. The zero set of f is contained in an l-ideal of B (Proposition 1), so in a non-removable ideal (Corollary 1). But $B \subset C(\Gamma(B))$ and this proves thus also that any non-removable ideal of B is contained in an $M \in \Gamma(B)$.

Remark. We do not know whether Corollary 3 holds true if we replace there $\Gamma(A)$ by $\mathcal{L}(A)$ or $\operatorname{cor} A$. It is also possible to have a functional

$f \in \Gamma(A)$ which has an extension $\bar{f}$ not belonging to $\Gamma(B)$, or even $\operatorname{cor} B$ for some superalgebra $B \supset A$. To see this take as B the sup-norm disc algebra of all continuous functions on the unit disc of the complex plane, holomorphic in its interior and let $A = \{x \in B \colon x(0) = x(1)\}$. The maximal ideal space of A is the closed unit disc with identified 0 and 1 and the Šilov boundary of A is the unit circle (with 1 identified with 0). So the functional $f(x) = x(0) = x(1)$ is in $\Gamma(A)$ and it has two extensions onto B: $f_1(x) = x(1)$ and $f_0(x) = x(0)$ such that $f_1 \in \Gamma(B)$ but $f_0 \notin \operatorname{cor} B$.

The following purely algebraic result can support the conjecture that $\mathfrak{l}(A)$ coincides with the family of all non-removable closed ideals of A. Let R and P be arbitrary rings with unit elements. P is an extension of R if there is an isomorphic imbedding of R into P sending the unit of R into unit of P. Call an ideal I of R non-removable if in any extension P of R the ideal I is contained in a proper ideal of P. A subset S of R consists of joint divisors of zero if for any finite subset $\{x_1, \ldots, x_n\} \subset R$ there is a non zero element $y \in R$ such that $x_i y = 0$ for $i = 1, 2, \ldots, n$.

PROPOSITION 4. *An ideal I of a commutative ring R is a non-removable ideal if and only if it consists of joint divisors of zero.*

The proof can be obtained from a reasoning in [2].

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On cosine operator functions and one-parameter groups of operators

by

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Dedicated to Professor Antoni Zygmund

Abstract. If A is a complex number then

$$(*)\qquad \exp\left(t\begin{pmatrix}0 & 1\\ A & 0\end{pmatrix}\right) = \begin{pmatrix}\cos(-A)^{\frac{1}{2}}t & \int_0^t \cos(-A)^{\frac{1}{2}}\tau\, d\tau \\ \frac{d}{dt}\cos(-A)^{\frac{1}{2}}t & \cos(-A)^{\frac{1}{2}}t\end{pmatrix}, \qquad -\infty < t < \infty.$$

The paper gives a generalization of this formula to the case, when A is an unbounded linear operator in a Banach space.

1. Preliminaries.

1.1. If E and F are Banach spaces over the same, real or complex, field of scalars then $\mathscr{L}(E;F)$ denotes the space of all linear bounded operators from E to F. Let $\mathscr{L}_s(E;F)$ denote $\mathscr{L}(E;F)$ equipped with the topology of pointwise convergence (called also the strong topology). An $\mathscr{L}(E;F)$-valued function of a real variable is called *strongly continuous*, or *strongly continuously differentiable*, if it is continuous or continuously differentiable, when regarded as a mapping from $(-\infty, \infty)$ to $\mathscr{L}_s(E;F)$. For instance, by an application of the Banach–Steinhaus theorem, it follows that a function $K\colon (-\infty, \infty) \to \mathscr{L}(E;F)$ is strongly continuously differentiable on $(-\infty, \infty)$ if and only if for any fixed $x \in E$ the F-valued function $t \to K(t)x$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in F.

1.2. Let E be a Banach space. A strongly continuous mapping $G\colon (-\infty, \infty) \to \mathscr{L}(E;F)$ is called a *one-parameter strongly continuous group of operators* if $G(0) = 1$ and

$$G(t)G(s) = G(t+s) \quad \text{for every } s, t \in (-\infty, \infty).$$

The infinitesimal generator of the one parameter group G is the