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The renewal theorem for a random walk in two-dimensional time*

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Dedicated to Prof. A. Zygmund in honor of the 50th anniversary of his first published paper

Abstract. An analog of the classical renewal theorem is proved for a random walk in two dimensional time. The renewal sequence u_n is shown to be of the form $u_n = \mu^{-1}\log n + r_n$, where μ is the mean of the underlying random variables, and r_n is a remainder sequence whose behavior depends on the restrictions imposed on these variables. This contrasts with the result $u_n \to \mu^{-1}$ in the one dimensional case.

1. Introduction. The purpose of this paper is to examine an analog of the renewal theorem for two dimensional time. The classical renewal theorem of Erdös, Feller, and Pollard [2], considers the partial sums $S_n = \sum_{i=1}^{n} X_i$ of independent, identically distributed, integer valued, aperiodic(¹) random variables $X_1, X_2, \ldots$, with finite mean $\mu > 0$; and asserts that the expected number of n's for which $S_n = k$ converges to $1/\mu$ as $k \to +\infty$.

For the two dimensional setting we consider a family $\{X_{i,j};\, i \geqslant 1,\ j \geqslant 1\}$ of random variables, and let

$$S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j}, \qquad EX_{i,j} = \mu > 0 \tag{1}$$

and

$$N_k = \text{the number of pairs } (m, n) \text{ for which } S_{m,n} = k. \tag{2}$$

Geometrically, one may think of $\{S_{m,n}\}$ as defining a discrete (random) surface on the positive quadrant of lattice points in the plane $E^{(2)}$, and N_k as being the number of points of this surface which lie in a plane $E_k^{(2)}$ parallel to $E^{(2)}$, and a distance k above it.

We call $u_k = EN_k$, $k \geqslant 1$, *the renewal sequence associated with* $\{X_{i,j}\}$, and want to study u_k as $k \to \infty$. *We assume throughout this paper that*

(¹) A random variable X is called *aperiodic* if $Ee^{i\theta X} \neq 1$ for $\theta \neq 0$, it is *strongly aperiodic* if $|Ee^{i\theta X}| \neq 1$ for $\theta \neq 0$.

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the $X_{i,j}$ are independent, identically distributed, integer-valued, and strongly aperiodic.

In the one-dimensional case one can very effectively make use of a difference equation satisfied by u_k. (It is a linear integral equation, called the renewal equation, in the general non-lattice case.) There does not appear to be a natural analog of this equation in dimension two, mainly because the lattice points of the plane are not linearly ordered under the natural order. Fortunately, however, a direct Fourier analysis yields some results.

Using standard Tauberian methods one can easily prove the weak (global) result (Theorem 1) that

$$\sum_{k=1}^{n} u_k \sim \frac{n\log n}{\mu} \quad \text{as } n \to \infty. \tag{3}$$

Our main concern in this paper is to examine when the strong (local) result

$$u_n \sim \frac{\log n}{\mu} \tag{4}$$

is valid. With sufficient moment assumptions we can prove (4) by using a sharp local central limit theorem (see Theorem 2). The problem becomes much more difficult if one limits oneself to the existence of a first moment; and we do not yet have the complete solution for this case. The main result so far is a somewhat weaker statement than (4), namely that

$$u_n = \frac{\log n}{\mu} + o(\log n) + \beta_n, \tag{5}$$

where an average of $|\beta_n|^2$ goes to zero. This is done in Theorem 3, where we actually do a little better than (5), in that we express u_n as a function of the ordinary renewal sequence, plus a remainder like β_n.

We have only considered non-negative random variables in this paper; except for Theorem 2, where the methods apply equally to the two-sided case.

2. The global theorem.

THEOREM 1. *Let $\{u_k; k \geqslant 1\}$ be the renewal sequence associated with $\{X_{i,j} \geqslant 0; i \geqslant 0, j \geqslant 0\}$, and assume that $\mu = EX_{i,j} < \infty$. Then*

$$\sum_{k=1}^{n} u_k \sim \mu^{-1} n\log n, \quad \text{as } n \to \infty. \tag{1}$$

Proof. Define $S_{m,n}$ and N_k as in (1.1) and (1.2) and define $\Delta_{m,n}(k) = 1$ if $S_{m,n} = k$, $\Delta_{m,n}(k) = 0$ if $S_{m,n} \neq k$. Then $N_k = \sum_m \sum_n \Delta_{m,n}(k)$, and

$$u_k = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P\{S_{m,n} = k\}. \tag{2}$$

Let $P(k) = P\{X_{i,j} = k\}$ and $P_n(\cdot) =$ the n-fold convolution of $P_1(\cdot)$ with itself. Then from (2) we see that

$$u_k = \sum_{n=1}^{\infty} d_n P_n(k), \tag{3}$$

where $d_k =$ the number of divisors of k.

Let $D_n = \sum_{k=1}^{n} d_k$. Then (see Hardy and Wright [7])

$$D_k \sim k\log k \quad \text{as } k \to \infty. \tag{4}$$

Since $\sum_{n=1}^{\infty} \gamma^k P_n(k) < \infty$ for any $\gamma < (P_1(0))^{-1}$ whenever $\{P_1(k)\}$ has its support on the non-negative integers, the series (3) always converges.

Let

$$f(s) = \sum_{k=0}^{\infty} P_1(k) s^k; \quad U(s) = \sum_{k=1}^{\infty} u_k s^k; \quad D(s) = \sum_{k=1}^{\infty} d_k s^k; \tag{5}$$

where $|s| \leqslant 1$. By (4) and the Hardy–Littlewood Tauberian theorem (see e.g. Theorem 5, p. 423 of Feller, Vol. II [4])

$$D(s) \sim \frac{1}{1-s}\log\frac{1}{1-s} \quad \text{as } s \nearrow 1. \tag{6}$$

But from (3) and (5)

$$U(s) = D[f(s)], \tag{7}$$

and hence, since $f(s) \nearrow 1$ as $s \nearrow 1$

$$U(s) \sim \frac{1}{1-f(s)}\log\frac{1}{1-f(s)}, \quad s \nearrow 1. \tag{8}$$

But $1-f(s) = \mu(1-s)+o(1-s)$, and thus

$$U(s) \sim \frac{1}{\mu(1-s)}\log\frac{1}{1-s}, \quad s \nearrow 1. \tag{9}$$

The Hardy–Littlewood theorem applied in the converse direction implies (1).

3. The local theorem. If $f(s) = s$ then $u_k = d_k$, which is known to oscillate wildly. Clearly

$$\liminf d_n = 2, \tag{1}$$

and on the other hand (see [7]) the statement

$$d_n = O([\log n]^\delta) \tag{2}$$

is false for every $\delta > 0$. The key question is thus whether in replacing s by a generating function $f(s) \neq s$, we sufficiently smooth out the renewal sequence.

In the next theorem we show rather easily that under a fourth moment assumption, the answer is affirmative. (Here the $X_{i,j}$ are not restricted to be non-negative random variables.) Let $\sigma^2 = \text{variance } (X_{i,j})$, and recall that $\mu = EX_{i,j}$.

THEOREM 2. *Let $\{u_k; k \geq 1\}$ be the renewal sequence associated with $\{X_{i,j}\}$, and assume that $EX_{i,j}^4 < \infty$, $\mu > 0$, and $\sigma^2 > 0$. Then*

(3) $$u_n \sim \mu^{-1}\log n \quad \text{as } n \to \infty.$$

Proof. Recall that

(4) $$u_k = \sum_{n=1}^{\infty} d_n P_n(k),$$

where $P_1(\cdot)$ and $P_n(\cdot)$ are as in Section 2. The fourth moment assumption assures us that the series in (4) converges. (To see this just apply the Čebyšeff inequality to $P_n(\cdot)$, and the bound in (7) below to d_k.) We will use the local central limit estimate

(5) $$P_n(k) = \varphi_n(k;\mu,\sigma^2) + O\left\{\frac{1}{\sqrt{n}}\varphi_n(k;\mu,\beta^2)\right\} + O\left(\frac{1}{n^{3/2}}\right),$$

where $\varphi_n(k;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi n}\sigma}\exp\left\{-\frac{(k-n\mu)^2}{2\sigma^2 n}\right\}$, μ and σ^2 are the mean and variance of X_{ij}, and β is some positive number. The remainder terms in (5) are uniform in k. One can deduce (5) from Theorem 1 of § 51 of Gnedenko and Kolmogorov [5]. We also use two more facts about d_n (see [6]), namely that

(6) $$\sum_{i=1}^{n} d_i \equiv D_n = n\log n + (2\gamma - 1)n + O(n^{1/3}),$$

where γ = Euler's constant, and

(7) $$d_n = O(n^\delta) \quad \text{for any } \delta > 0.$$

From (7) we see that

(8) $$\sum_{n=1}^{\infty} d_n n^{-(3/2)} < \infty.$$

We will show that when $0 < \mu < \infty$, $0 < \sigma^2 < \infty$,

(9) $$\sum_{n=1}^{\infty} d_n\varphi_n(k;\mu,\sigma^2) \sim \mu^{-1}\log k \quad \text{as } k \to \infty.$$

One then easily sees from (7) that

(10) $$\sum_{n=1}^{\infty} \frac{d_n}{\sqrt{n}}\varphi_n(k;\mu,\beta^2) = o(\log k).$$

Combining (4), (5), (8), (9) and (10) we obtain Theorem 2.

It hence remains only to prove (9). To this end we sum (9) by parts, obtaining

(11) $$\sum_{n=1}^{\infty} d_n\varphi_n = \sum_{n=1}^{\infty} D_n(\varphi_n - \varphi_{n+1}),$$

where the boundary term for the summation by parts is $\lim_{n\to\infty} D_n\varphi_n = 0$. From now on, write $\varphi_n(k;\mu,\sigma^2) = \varphi_n(k)$ for short.

Let $A_n = \log n + (2\gamma - 1)$.

Summing back by parts

(12) $$\sum_{n=1}^{\infty} A_n(\varphi_n - \varphi_{n+1}) = \sum_{n=1}^{\infty} a_n\varphi_n - \lim \varphi_n A_n,$$

where $a_n = A_{n+1} - A_n = \log n + C_n$,

$$\lim \varphi_n A_n = 0, \quad \text{and } C_n = O(1).$$

Thus

(13) $$\sum A_n(\varphi_n - \varphi_{n+1}) = \sum (\log n)\varphi_n(k) + \sum_{n=1}^{\infty} C_n\varphi_n(k).$$

But by the renewal density theorem (see [4])

(14) $$\sum_{n=1}^{\infty} \varphi_n(k) \to \frac{1}{\mu} \quad \text{as } k \to \infty,$$

and hence

(15) $$\sum C_n\varphi_n(k) = O(1) \quad \text{as } k \to \infty.$$

Let $S = \{n > 0: |k - n\mu| > k^{2/3}\}$ and $\bar{S}$ = compl. of S. Then

(16) $$\sum_{n\in S} (\log n)\varphi_n(k) < \infty.$$

Note that

17) $$|\log n - \log k/\mu| < (\text{const.})k^{-1/3} \quad \text{for } n \in \bar{S}.$$

Hence by decomposing $\sum_{n=1}^{\infty}(\log n)\varphi_n(k)$ into the sums over S and $\bar{S}$, and applying (14)–(17), a straight forward calculation shows that

(18) $$\sum_{n=1}^{\infty} A_n\{\varphi_n(k) - \varphi_{n+1}(k)\} \sim \mu^{-1}\log k, \quad \text{as } k \to \infty.$$

Finally (due to (6)), it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{1/3}|\varphi_n(k)-\varphi_{n+1}(k)| = O(1). \tag{19}$$

If we could remove the absolute value signs then we could sum by parts and make the desired conclusion. To this end, let

$$h(t) = \frac{1}{\sqrt{t}}\exp\left\{-\frac{(k-t\mu)^2}{at}\right\}, \quad t>0,$$

and observe that

$$\frac{d}{dt}h(t) = q(t)t^{-5/2}\exp\left\{-\frac{(k-t\mu)^2}{at}\right\}$$

where $q(\cdot)$ is a quadratic. Hence there exists an integer $0<N(k)<\infty$ such that $\varphi_n(k)-\varphi_{n+1}(k)$ is monotone on each of the intervals $(0, N(k)]$, $(N(k), \infty)$. Then, for example, on $(0, N]$ we have

$$\sum_{n=1}^{N} n^{1/3}|\varphi_n-\varphi_{n+1}| = \sum_{n=1}^{N} n^{1/3}(\varphi_n-\varphi_{n+1})$$
$$= O\Big\{\sum_{n=1}^{N} n^{-2/3}\varphi_n(k)\Big\}+O\big(N^{1/3}\varphi_N(k)\big) = o(1).$$

The sum on (N, ∞) is treated similarly. This proves the theorem.

4. The main result. From now on we shall only assume existence of a first moment. We also limit ourselves to non-negative random variables X_{ij}.

The proof consists of two parts. One part treats the remainder term by a method in which the principal tool is the Hardy–Littlewood maximal theorem. The other part expresses the main term in the answer in terms of the standard one-dimensional renewal function. It crucially uses the Wiener–Levy theorem, somewhat along the same lines that Wiener's theorem was used by Karlin [8] in the proof of the one-dimensional case.

THEOREM 3. *Let u_k be the renewal sequence associated with $\{X_{i,j}\geqslant 0;\ i\geqslant 0, j\geqslant 0\}$, and assume that $EX_{i,j}<\infty$, and variance $X_{ij}>0$. Then*

$$u_k = k^{-1}*v_k+(\gamma-1)v_k+w_k+\beta_k,$$

where

$$v_k = \sum_{n=0}^{\infty} P_n(k) = \textit{the one-dimensional renewal function},$$

$$w_k \to \frac{\log\mu}{\mu} \quad \text{as } k\to\infty,$$

$$\sum_{j=1}^{k}\beta_j^2 = O(k^{2/3}),$$

and $*$ denotes convolution.

Without further moment assumptions we can use the fact that $v_k\to\frac{1}{\mu}$ to conclude

COROLLARY. *Under the hypotheses of Theorem 3*

$$u_k = \mu^{-1}\log k+a_k+\beta_k,$$

where $a_k = o(\log k)$ and $\sum_{1}^{k}\beta_i^2 = O(k^{2/3})$.

Remark. Actually $w_k-\frac{\log\mu}{\mu}$ is the tail of an absolutely convergent series, that is

$$w_k-\frac{\log\mu}{\mu} = \sum_{j=k}^{\infty} a_j \quad \text{where } \sum|a_j|<\infty.$$

If one makes extra moment assumptions on $P(k)$, then one can conclude more about the a_j, v_k, and hence also $\frac{1}{k}*v_k$. For example if $\sum k^{\delta+1}P(k)<\infty$, then (i) $\sum|a_j|j^\delta<\infty$, and (ii) $\sum\left|v_k-\frac{1}{\mu}\right|k^\delta<\infty$.

Proof of Theorem 3. Let $h_k = \sum_{j=1}^{k} j^{-1}$: $h_0 = 0$; $H_n = \sum_{k=0}^{n} h_k$. Then $H_n = n\log n+\gamma n+o(\log n)$, where γ = Euler's constant, and according to (2.6)

$$D_n = H_n+(\gamma-1)n+B_n \text{ where } B_n = O(n^{1/3}).$$

Thus, summing by parts in (2.3) we get

$$u_k = \sum_{n=0}^{\infty} d_nP_n(k) = \sum_{n=0}^{\infty} D_n[P_n(k)-P_{n+1}(k)]$$
$$= \sum_{n=0}^{\infty} H_n[P_n(k)-P_{n+1}(k)]+(\gamma-1)\sum_{n=0}^{\infty} n[P_n(k)-P_{n+1}(k)]+$$
$$+\sum_{n=0}^{\infty} B_n[P_n(k)-P_{n+1}(k)]$$
$$\equiv t_k+\gamma_k+\beta_k \quad (H_0 = B_0 = 0).$$

(For fixed k, $P_n(k)\to 0$ exponentially fast, and hence all the above series converge.)

The behavior of γ_k is trivially determined. Namely

$$\gamma_k = (\gamma-1)\sum_{n=0}^{\infty}(n+1)[P_n(k)-P_{n+1}(k)]-(\gamma-1)\sum_{n=0}^{\infty}[P_n(k)-P_{n+1}(k)]$$

or

(1) $$\gamma_k = (\gamma-1)v_k,$$

where

$$v_k = \sum_{n=0}^{\infty} P_n(k) = \text{the one-dimensional renewal sequence.}$$

The main term. Consider $t_k = \sum_{n=0}^{\infty} H_n[P_n(k)-P_{n+1}(k)]$ and sum back by parts to get

(2) $$t_k = \sum_{n=0}^{\infty} h_n P_n(k).$$

Define

(3) $$t_k(r) = \sum_{n=0}^{\infty} h_n r^n P_n(k) = \sum_{n=1}^{\infty}\sum_{m=1}^{n}[r^{n-m}P_{n-m}(k)]*\left[\frac{r_m}{m}P_m(k)\right]$$

($*$ = convolution w.r. to k)

$$= \left[\sum_{n=0}^{\infty} r^n P_n(k)\right]*\left[\sum_{n=1}^{\infty}\frac{r^n}{n}P_n(k)\right]$$

$$= \left[\sum_{n=0}^{\infty} r^n P_n(k)\right]*\left[\sum_{n=1}^{\infty}\frac{r^n}{n}\delta_n(k)\right]+$$

$$+\left[\sum_{n=0}^{\infty} r^n P_n(k)\right]*\left[\sum\frac{r^n}{n}P_n(k)-\delta_n(k)\right]$$

and denote the last two terms by $\equiv s_k(r)+w_k(r)$, ($\delta_n(k)$ = the Kronecker delta).

Now let

$$f(\theta) = \sum_{k=0}^{\infty} P_1(k)e^{ik\theta} \quad\text{and}\quad \psi(r,\theta) = \sum_{k=0}^{\infty} w_k(r)e^{ik\theta}.$$

Then

(4) $$\psi(r,\theta) = \frac{\log\big(1-rf(\theta)\big)-\log(1-re^{i\theta})}{1-rf(\theta)}.$$

But

$$(1-re^{i\theta})\psi(r,\theta) = w_0(r)+\sum_{k=1}^{\infty}[w_k(r)-rw_{k-1}]e^{ik\theta},$$

and hence

(5) $$w_k(r)-rw_{k-1}(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\left(\frac{1-re^{i\theta}}{1-rf(\theta)}\right)\log\left(\frac{1-rf(\theta)}{1-re^{i\theta}}\right)d\theta.$$

Now note that

$$\frac{1}{|1-rf(\theta)|} = O\left(\frac{1}{\theta}\right), \quad\text{and}\quad \frac{1}{|1-re^{i\theta}|} = O\left(\frac{1}{\theta}\right);$$

also since $1-rf(\theta)$ and $1-re^{i\theta}$ always have non-negative real parts their arguments are bounded; and hence

$$\left|\log\frac{1}{1-rf(\theta)}\right|+\left|\log\frac{1}{1-re^{i\theta}}\right| = O\left(1+\log\frac{1}{|\theta|}\right)$$

uniformly in r. Furthermore, using the fact that $f(\theta) = 1+iu\theta+o(\theta)$, a rather straight forward calculation which we omit shows that

$$\left|\frac{1-re^{i\theta}}{1-rf(\theta)}\right| \text{ is uniformly bounded.}$$

Hence the integrand in (5) is uniformly bounded by an integrable function and by Lebesgue's theorem we may take the limit as $r\to 1$ inside the integral. Thus

(6) $$w_k(1-)-w_{k-1}(1-) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\frac{1-e^{i\theta}}{1-f(\theta)}\log\frac{1-f(\theta)}{1-e^{i\theta}}d\theta.$$

We will use a version of the Wiener–Levy theorem to show that

(7) $$\sum_{k=1}^{\infty}|w_k(1-)-w_{k-1}(1-)| < \infty,$$

and thus $w_k \equiv w_k(1-)\to$ constant. But $\sum w_k r^k \sim \log\mu/[\mu(1-r)]$, and hence

(8) $$w_k\to\log\mu/\mu.$$

Combining this with (2) and (3) we get

(9) $$t_k = \lim_{r\to 1} t_k(r) = \lim_{r\to 1} s_k(r)+w_k.$$

But

(10) $$\lim_{r\to 1} s_k(r) = \lim_{r\to 1}\left[\sum_{n=0}^{\infty} r^n P_n(k)\right]*\left[\sum_{m=1}^{\infty}\frac{r^m}{m}\delta_m(k)\right]$$

$$= \lim_{r\to 1}\sum_{m=1}^{k}\sum_{n=0}^{\infty}\frac{1}{m}r^{n+m}P_n(k-m) = \sum_{m=1}^{k} v_{k-m}\frac{1}{m} = (v_k)*\left(\frac{1}{k}\right),$$

and hence we can conclude that

(11) $$t_k = k^{-1}*v_k+w_k, \quad\text{where } w_k\to\mu^{-1}\log\mu.$$

Thus it remains to prove (7). To this end let

(12) $$g(\theta) = \frac{1-f(\theta)}{1-e^{i\theta}}.$$

Then we can write $g(\theta) = \sum_{k=0}^{\infty} \varrho_k e^{ik\theta}$ where $\varrho_k = \sum_{j=k}^{\infty} p_j$, and since by hypothesis $\sum_{k=0}^{\infty} \varrho_k = \sum_{k=0}^{\infty} kp_k = \mu < \infty$, $g(\theta)$ has an absolutely convergent Fourier series. Now if $\psi(\cdot)$ is a function which is analytic in a domain containing the range of $g(\theta)$, $-\pi < \theta \leqslant \pi$, and if $\psi[g(\theta)]$ can be defined as a single valued continuous function of θ, then by Theorem 6.8 of Arens and Calderón [1] with A = the algebra of functions with absolutely convergent Fourier series, we can conclude that $\psi(g(\theta))$ has an absolutely convergent Fourier series.

In the present case

(13) $$\psi[g(\theta)] = \frac{1}{g(\theta)} \log g(\theta).$$

From the first moment and aperiodicity hypotheses on f we can conclude that there is a $\delta > 0$ such that

$$|g(\theta)| \geqslant \delta, \quad -\pi < \theta \leqslant \pi,$$

and hence to prove (7) it suffices to show that $\log g(\theta)$ can be defined as a continuous function such that $\log g(o) = \log g(2\pi)$.

To this end let

(14) $$f(r, \theta) = \sum p_k r^k e^{ik\theta}, \quad \text{and} \quad g(r, \theta) = \frac{1-f(r, \theta)}{1-re^{i\theta}}.$$

Since $\operatorname{Re}(1-f(r, \theta)) > 0$ and $\operatorname{Re}(1-re^{i\theta}) > 0$ we see that $\log(1-f(r, \theta))$ and $\log(1-re^{i\theta})$ are single valued continuous functions; and hence also $\log g(r, \theta)$ is continuous, and $\log g(r, 0) = \log g(r, 2\pi)$. Hence, if we can show that

(15) $$\log g(r, \theta) \to \log g(\theta) \quad \text{as } r \to 1$$

uniformly in θ, then we are done. But

$$|\log g(\theta) - \log g(r, \theta)| = \left|\log\left\{1 + \frac{g(\theta)-g(r, \theta)}{g(r, \theta)}\right\}\right|.$$

Also $|g(r, \theta)| \geqslant \delta > 0$ for some $\delta > 0$, uniformly in $0 < r \leqslant 1$ and $-\pi < \theta < \pi$. Hence

$$\left|\frac{g(\theta)-g(r, \theta)}{g(r, \theta)}\right| \leqslant \frac{1}{\delta}|g(\theta)-g(r, \theta)|$$

$$= \frac{1}{\delta}\left|\sum_{k=0}^{\infty} \varrho_k (1-r^k) e^{ik\theta}\right| \leqslant \frac{1}{\delta}\sum_{k=0}^{\infty} \varrho_k (1-r^k) \to 0 \quad \text{as } r \to 1.$$

This proves (15), and hence (7).

The remainder term. Finally we consider the term

(16) $$\beta_k = \sum_{n=0}^{\infty} B_n [P_n(k) - P_{n+1}(k)] = \sum_{n=0}^{\infty} b_n P_n(k)$$

where $B_n = O(n^{1/3})$, and $b_n = B_n - B_{n-1}$ when $n \geqslant 1$, $b_0 = B_0 = 0$.

Let

$$B(z) = \sum_{n=0}^{\infty} b_n z^n = (1-z)\sum_{n=0}^{\infty} B_n z^n.$$

Then

(17) $$B(f(\theta)) = \sum_{n=0}^{\infty} b_n f^n(\theta) = \sum_{n=0}^{\infty} \beta_n e^{in\theta}$$

and

(18) $$B(f(r, \theta)) = \sum_{n=0}^{\infty} b_n f^n(r; \theta) = \sum_{n=0}^{\infty} \beta_n r^n e^{in\theta}.$$

Define

$$\lambda(z) = 1 - p + pz \quad 0 < p < 1.$$

We will use the following lemma:

Lemma. *There exists a p $(0 < p < 1)$ such that as $r \to 1$*

$$\int_{-\pi}^{\pi} |B[f(r, \theta)]|^2 d\theta = O\left\{\int_{-\pi}^{\pi} \left|B\left[\lambda\left(\exp\left\{-\frac{\mu(1-r)}{2p} + i\theta\right\}\right)\right]\right|^2 d\theta\right\} + O(1).$$

Proof. We write the inverse function of λ as

$$\lambda^{-1}(w) = 1 - t(1-w), \quad \text{where } t = 1/p.$$

Then we have $B[f] = B[\lambda(\lambda^{-1}[f])]$ and

20) $$B[f(r, \theta)] = B[\lambda(1-t[1-f(r, \theta)])].$$

Let $u(r, \theta) = \operatorname{Re} f(r, \theta)$ and $v(r, \theta) = \operatorname{Im} f(r, \theta)$. We note that

(21) $$u(r, \theta) = u(r, 0) + o(\theta) \quad v(r, \theta) = \mu_r \theta + o(\theta)$$

where $\mu_r = \sum_{n=0}^{\infty} nr^n p_n$, and where the terms $o(\theta)$ above and in the rest of this discussion are uniform for $0 \leqslant r \leqslant 1$ (due to the mean value theorem and the fact that $\frac{\partial u}{\partial \theta}$ and $\frac{\partial v}{\partial \theta}$ are uniformly continuous in r and θ); also that

(22) $$1 - u(r, 0) = \mu(1-r) + o(1-r).$$

We also claim that there is a $t_1 > 1$ and an $r_0 < 1$ such that for $r > r_0$

(23) $$u(r, 0) - u(r, \theta) \geqslant \tfrac{1}{2} t_1 \mu_r^2 \theta^2 + o(\theta^2).$$

To see this write (for θ small)

$$(24)\qquad u(r,0)-u(r,\theta) = \sum_{k=0}^{\infty} p_k r^k (1-\cos k\theta) \geqslant \sum_{k=0}^{\theta^{-1/3}} p_k r^k (1-\cos k\theta)$$

$$= \tfrac{1}{2}\sum_{k=0}^{\theta^{-1/3}} p_k r^k k^2 \theta^2 [1+o_0(1)] = [\theta^2+o(\theta^2)]\cdot \tfrac{1}{2}\sum_{k=0}^{\theta^{-1/3}} p_k r^k k^2.$$

Now if $\sum_{k=0}^{\infty} k^2 p_k \equiv \mu_2 < \infty$ then $\mu^2 < \mu_2$, and hence we can find an $r_0 < 1$ and $\theta_0 > 0$ such that

$$(25)\qquad \sum_{k=0}^{\theta^{-1/3}} p_k k^2 r^k > \mu^2 > \mu_r^2 \quad \text{for } r_0 < r \leqslant 1 \text{ and } \theta < \theta_0.$$

If $\sum k^2 p_k = \infty$ then $\sum_{k=0}^{\theta^{-1/3}} p_k k^2 r^k$ can be made arbitrarily large by taking θ sufficiently small and r close to 1, and hence (25) is trivially true. Since the variance of $\{p_k\}$ is assumed >0 the inequality $\mu^2 < \mu_2$ is strict, and hence we can insert the t_1 required in (23).

Thus since $\lambda^{-1}[f(r,\theta)] \neq 0$ for r near 1 and θ near 0 we can write

$$(26)\qquad \pi(r,\theta) = \log \lambda^{-1}[f(r,\theta)] = \log\{1-t[1-f(r,\theta)]\}$$

$$= \log\{1-t(1-u(r,0))-t(u(r,0)-u(r,\theta))+itv(r,\theta)\}.$$

Using the estimates (21), and (22), and expanding the log about 1, a somewhat lengthy calculation (which we spare the reader) shows that

$$(27)\qquad \begin{cases} \operatorname{Re}\pi(r,\theta) = -t\mu(1-r)-t[u(r,0)-u(r,\theta)]+ \\ \qquad +\tfrac{1}{2}t^2\mu_r^2\theta^2+o(1-r)+o(\theta^2)+O(1-r)o(\theta), \\ \operatorname{Im}\pi(r,\theta) = tv(r,\theta)\,[1+t\mu(1-r)+o(1-r)]+ \\ \qquad +o(\theta^2)+O(1-r+|\theta|)^3, \end{cases}$$

where all remainder terms are real and uniform in the variables not displayed. Applying (23) in (27), and taking t so that $1 < t < t_1$, we can in turn conclude that

$$\operatorname{Re}\pi(r,\theta) = -\tfrac{1}{2}t\mu(1-r)-\hat{u}(r,\theta), \qquad \operatorname{Im}\pi(r,\theta) = tv(r,\theta)+\hat{v}(r,\theta),$$

where $\hat{u}$ and $\hat{v}$ are real $\hat{u}$ is positive, and

$$(28)\qquad |\hat{v}(r,\theta)| \leqslant c\hat{u}(r,\theta)$$

for $1-r$ and θ sufficiently small; and where c is a positive constant and $\hat{u}(r,\theta)\to 0$ as $r\to 1$, $\theta\to 0$.

We may now write

$$(29)\qquad \int_{-\varepsilon}^{\varepsilon} |B[f(r,\theta)]|^2 d\theta = \int_{-\varepsilon}^{\varepsilon} |B[\lambda\{e^{\pi(r,\theta)}\}]|^2 d\theta$$

$$= \int_{-\varepsilon}^{\varepsilon} |B[\lambda\{e^{-\frac{1}{2}t\mu(1-r)-\hat{u}(r,\theta)+i[tv(r,\theta)+\hat{v}(r,\theta)]}\}|^2 d\theta.$$

We now make the change of variable

$$x = tv(r,\theta); \quad \theta = v^{-1}\left(r,\frac{x}{t}\right), \qquad d\theta = \frac{1}{t}\frac{d\theta}{dv}dx,$$

and note that $\left|\frac{d\theta}{dv}\right|$ is bounded since $0 < \mu < \infty$. Then

$$(30)\qquad \int_{-\varepsilon}^{\varepsilon} |B[f(r,\theta)]|^2 d\theta \leqslant K \int_{tv(r,-\varepsilon)}^{tv(r,\varepsilon)} |B[\lambda\{e^{-\frac{1}{2}t\mu(1-r)-\bar{u}(r,x)+i[x+\bar{v}(r,x)]}\}|^2 dx$$

where $\bar{u}(r,x) = \hat{u}\left(r, v^{-1}\left(r,\frac{x}{t}\right)\right)$, $\bar{v}(r,x) = \hat{v}\left(r, v^{-1}\left(r,\frac{x}{t}\right)\right)$, and where K can be chosen independent of r for $r \geqslant r_0$ close to 1. Finally, let

$$(31)\qquad \eta(r,z) = B[\lambda\{e^{-\frac{1}{2}t\mu(1-r)}z\}] \equiv \sum_{n=0}^{\infty} \eta_n(r) z^n \qquad |z| \leqslant 1$$

and

$$(32)\qquad \zeta(r,x) = e^{-\bar{u}(r,x)+i\bar{v}(r,x)}.$$

Note that (due to (21) if $|x| \leqslant x_0$ for x_0 sufficiently small, we still have

$$(33)\qquad |\bar{v}(r,x)| \leqslant \bar{c}\bar{u}(r,x), \qquad r \geqslant r_0.$$

for some positive $\bar{c}$.

We thus have

$$(34)\qquad \int_{-\varepsilon}^{\varepsilon} |B[f(r,\theta)]|^2 d\theta \leqslant K \int_{tv(r,-\varepsilon)}^{tv(r,\varepsilon)} |\eta[r,\zeta(r,x)e^{ix}]|^2 dx,$$

and it is to the right side of (32) that we apply the Hardy–Littlewood maximal theorem.

Let $\Omega_\sigma(x)$ denote the "Hardy–Littlewood domain" with vertex at e^{ix}, as defined in Zygmund [10], chapter IV, Section 7, formula (7.9) and the paragraph preceedings (7.9). From the geometry of the construction of Ω_σ, and from (34), it is clear that one, can choose ε sufficiently small and then σ sufficiently close to 1 so that

$$(35)\qquad \zeta(r,x)e^{ix} \in \Omega_\sigma(x)$$

for $tv(r,-\varepsilon) \leqslant x \leqslant tv(r,\varepsilon)$, $r \geqslant r_0$.

Hence

$$|\eta[r,\zeta(r,x)e^{ix}]| \leqslant \sup_{z\in\Omega_\sigma(x)} |\eta(r,z)| \equiv N_\sigma(r,x) \quad \text{(say)},$$

and

$$(36)\qquad \int_{tv(r,-\varepsilon)}^{tv(r,\varepsilon)} |\eta[r,\zeta(r,x)e^{ix}]|^2\,dx \leqslant \int_{-\pi}^{\pi} |N_\sigma(r,x)|^2\,dx \leqslant A \int_{-\pi}^{\pi} |\eta(r,e^{ix})|^2\,dx$$

where the constant A dependends on σ but *not* on r. The last inequality is a consequence of the Hardy–Littlewood theorem (see e.g. Theorem 7.10, chapter IV of [10]).

For $|\theta| \geqslant \varepsilon$, $|f(r,\theta)| \leqslant 1-\delta(\varepsilon)$, $\delta(\varepsilon) > 0$, and hence

$$(37)\qquad \int_{\varepsilon\leqslant|\theta|\leqslant\pi} |B[f(r,\theta)]|^2\,d\theta \leqslant \text{constant}.$$

Combining (34), (35) and (37) we obtain the lemma.

Returning to the proof of the theorem, applying the lemma and Bessel's equality, and recalling the definitions (18) and (31), we see that

$$(38)\qquad \sum \beta_n^2 r^{2n} \leqslant c \sum \eta_n^2(r)$$

for $r \geqslant$ some r_0 ($r_0 < 1$) and c = constant.

Write

$$B[\lambda(z)| = \sum \omega_n z^n.$$

Then

$$\eta_n(r) = \omega_n e^{-\frac{\mu}{2p}(1-r)n} = \omega_n r^{\frac{\mu}{2p}\left(\frac{1-r}{\log 1/r}\right)},$$

and by taking r_0 close enough to 1

$$(39)\qquad \eta_n^2(r) \leqslant \omega_n^2 r^{\mu n} \quad \text{for } r \geqslant r_0.$$

Thus

$$(40)\qquad \sum \beta_n^2 r^{2n} \leqslant c \sum \omega_n^2 r^{\mu n}, \qquad r \geqslant r_0.$$

We will show below that

$$(41)\qquad \omega_n = O(n^{-1/6})$$

and hence

$$(42)\qquad \sum_{n=1}^{\infty} \beta_n^2 r^n = O\Big(\sum_{n=1}^{\infty} n^{-1/3} (r^{\mu/2})^n\Big) \quad \text{as } r \to 1.$$

This in turn implies that

$$(43)\qquad \sum_{j=0}^{\infty} \beta_j^2 = O(n^{2/3})$$

(see section XIII.5 of Feller [4]).

Putting together (1), (11), and (43) implies the theorem.

It thus remains only to prove (41). To this end we note that

$$\omega_n = \text{coef. of } z^n \text{ in } B[\lambda(z)] = \sum_{n=0}^{\infty} b_n \lambda_n(k),$$

where $\lambda_n(k) = \binom{n}{k} p^k (1-p)^{n-k}$, and $b_n = B_n - B_{n-1}$, $B_n = O(n^{1/3})$; or

$$(44)\qquad \omega_n \leqslant \text{constant}.\ \sum_{n=0}^{\infty} n^{1/3} |\lambda_n(k) - \lambda_{n-1}(k)|.$$

Now $\lambda_n(k) - \lambda_{n-1}(k)$ changes sign only once; namely when $n = k/p$ and hence we may break the sum in (44) into two ranges: $n \leqslant \frac{k}{p}$ and $n > \frac{k}{p}$; and sum these by parts. One gets

$$\sum_{n=0}^{k/p} n^{1/3} |\lambda_n(k) - \lambda_{n-1}(k)| = \sum_{n=0}^{k/p} n^{-2/3} \lambda_n(k) + \lambda_{k/p}(k) k^{1/3}.$$

The sum in this expression is $O(k^{-2/3})m$ and $\lambda_k(k) k^{1/3} \sim$ const. $k^{-1/6}$. Similarly

$$\sum_{n=\frac{k}{p}+1}^{\infty} n^{1/3} |\lambda_n(k) - \lambda_{n-1}(k)| \sim \text{const. } k^{-1/6}.$$

This proves (41) and completes the proof of Theorem 3.

Proof of Remark. To prove (i) we repeat the argument using the Banach Algebra of sequences a_k such that $\sum_{-\infty}^{\infty} |a_k| |k|^s < \infty$. (See Essen [3] for details.) For a proof of (ii) see Essen [3] or Stone and Wainger [9].

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